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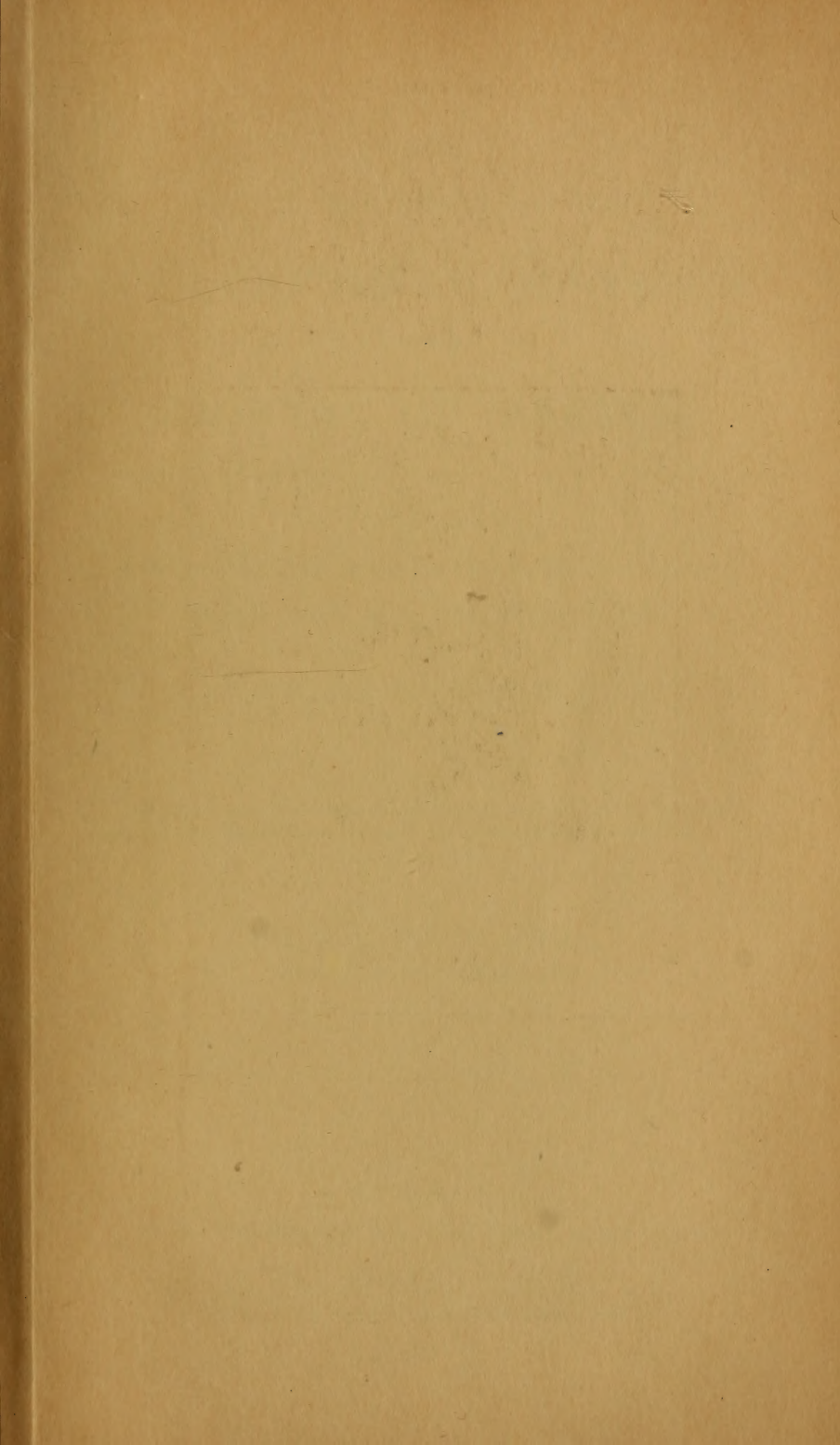
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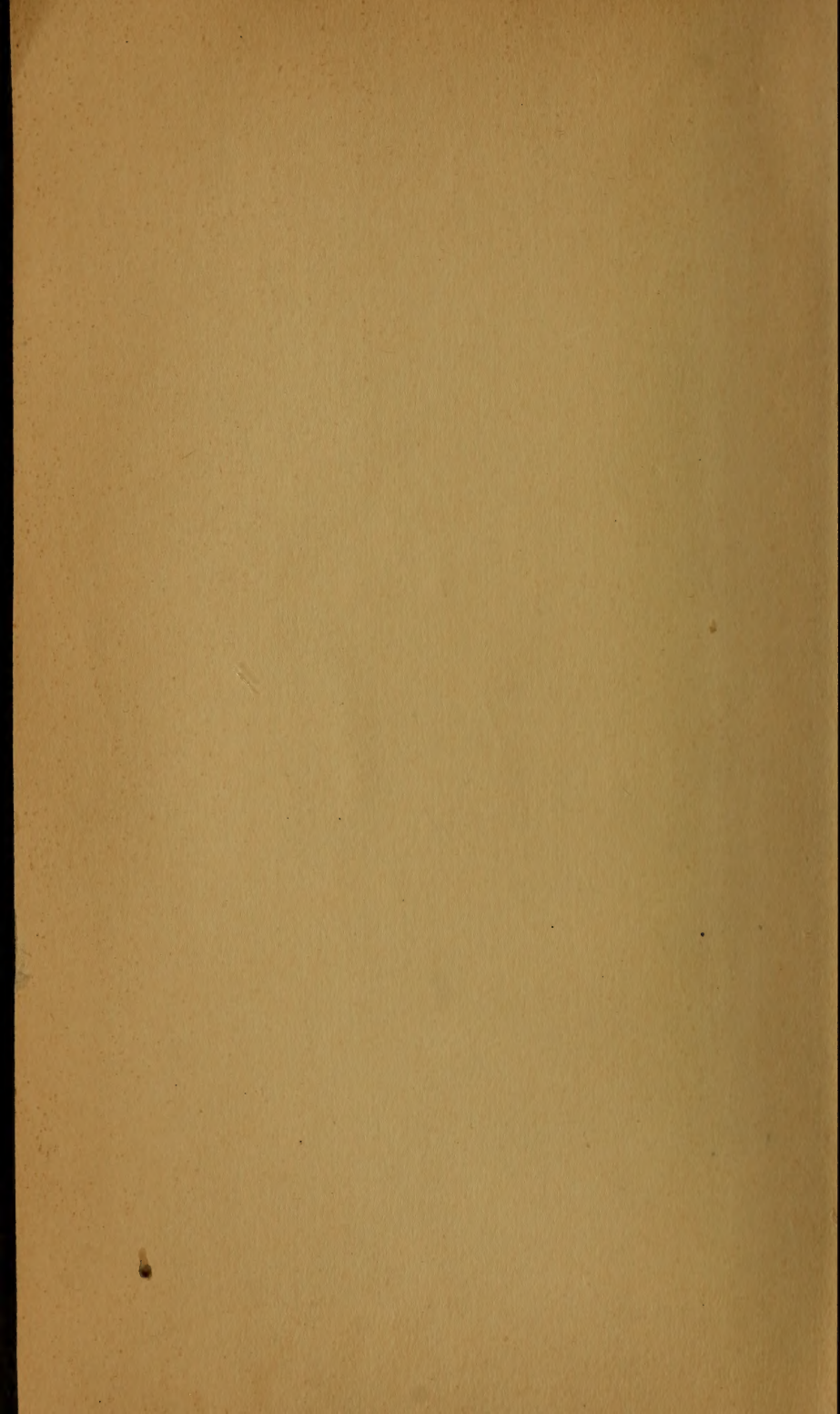


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AND

WILLIAM FRANCIS, F.L.S.

"Nec araneorum sane textus ideo melior quia ex se fila gignunt, nec noster
vilior quia ex alienis libamus ut apes." JUST. LIPS. *Polit. lib. i. cap. 1. Not.*

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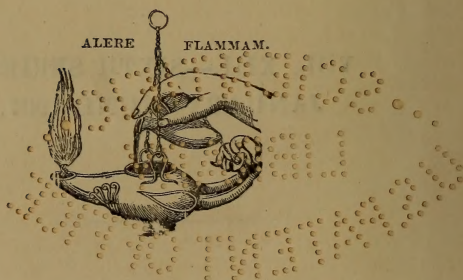
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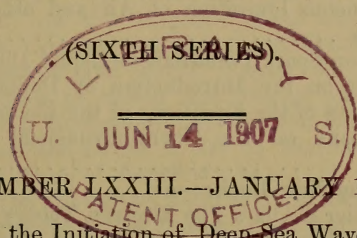
“Meditationis est perscrutari occulta; contemplationis est admirari
perspicua . . . Admiratio generat quæstionem, quæstio investigationem,
investigatio inventionem.”—*Hugo de S. Victore.*

—“Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condât,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu.”

J. B. Pinelli ad Mazonium.



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[SIXTH SERIES]

JANUARY 1907.

- I. *Initiation of Deep-Sea Waves of Three Classes*: (1) *from a Single Displacement*; (2) *from a Group of Equal and Similar Displacements*; (3) *by a Periodically Varying Surface-Pressure* *. (Continued from *Phil. Mag.*, Jan. 1906.) By Lord KELVIN.

- (1) *Disturbance due to an Initiational Form more convenient than that of §§ 3-31 of Previous Papers on Waves.* §§ 96-113.

§ 96. **T**HE investigations of §§ 5-31, including the "front and rear" of infinitely long free processions of waves in deep water, are all founded on initiational disturbances, according to the first of two typical forms described in §§ 3, 4. In this form the initial disturbance is everywhere elevation or everywhere depression, and its amount, at great distances from the origin varies inversely as the square root of the distance ρ , from a horizontal line at a small height h above the water-surface in the middle of the disturbance. In the present paper a new form of type-disturbance is derived indifferently from either the first or the second, of the forms of §§ 3, 4: from the first, by double differentiation with reference to time, t ; from the second, by single differentiation with reference to space, x .

§ 97 (being a repetition of §§ 1, 2, slightly modified with respect to notation). Consider a frictionless incompressible

* From the *Proc. Roy. Soc. Edin.* vol. xxvi. Communicated by the Author.

liquid, (called water for brevity,) in a straight canal, infinitely long and infinitely deep, with vertical sides. Let it be disturbed from its level by any change of pressure on the surface, uniform in every line perpendicular to the plane sides, and let it be left to itself under constant air pressure. It is required to find the displacement and velocity of every particle of water at any future time. Our initial condition will be fully specified by a given normal component of velocity, and a given normal component of displacement, at every point of the surface.

Taking O, any point at a distance h above the undisturbed water level, draw OX parallel to the length of the canal, and OZ vertically downwards. Let ξ, ζ be the displacement components, and $\dot{\xi}, \dot{\zeta}$ the velocity components, of any particle of the water whose undisturbed position is (x, z) . We suppose the disturbance infinitesimal; by which we mean that the change of distance between any two particles of water is infinitely small in comparison with their undisturbed distance; and the line joining them experiences changes of direction which are infinitely small in comparison with the radian. Water being assumed incompressible and frictionless, its motion, started primarily from rest by pressure applied to the free surface, is essentially irrotational. Hence we have

$$\xi = \frac{d}{dx} F(x, z, t); \quad \zeta = \frac{d}{dz} F(x, z, t); \quad \dot{\xi} = \frac{d}{dx} \dot{F}; \quad \dot{\zeta} = \frac{d}{dz} \dot{F}. \quad (132):$$

where $F(x, z, t)$, or F as we may write it for brevity when convenient, is a function which may be called the displacement-potential; and $\dot{F}(x, z, t)$ is what is commonly called the velocity-potential. Thus a knowledge of the function F , for all values of x, z, t , completely defines the displacement and the velocity of the fluid. And towards the determination of F we have, in virtue of the incompressibility of the fluid,

$$\frac{d^2 F}{dx^2} + \frac{d^2 F}{dz^2} = 0 \quad . \quad . \quad . \quad . \quad (133).$$

In virtue of this equation, the well-known primary theory of Gauss and Green shows that, if F is given for every point of the free surface of the water, and is zero at every point infinitely distant from it, the value of F is determinate throughout the fluid. The motion being infinitesimal, and the density being taken as unity, an application of fundamental hydrokinetics gives

$$\rho - \Pi = g(z - h + \zeta) - \frac{d^2 F}{dt^2} = g(z - h) + g \frac{dF}{dz} - \frac{d^2 F}{dt^2} \quad (134),$$

where g denotes gravity; Π the uniform atmospheric pressure on the free surface; and p the pressure at the point $(x, z + \zeta)$ within the fluid.

§ 98. Suppose now that $F(x, z, t)$ is a function which, besides satisfying (133), satisfies also the equation

$$g \frac{dF}{dz} = \frac{d^2 F}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (135);$$

we see by (134) that the corresponding fluid motion of which F is the displacement-potential (132), has constant pressure over every surface $(z + \zeta)$; that is to say, every surface which was level when the water was undisturbed. Thus our problem of finding any possible infinitesimal irrotational motion of the fluid, in which the free surface is under any constant pressure, is solved by finding solutions of (133) and (135).

§ 99. Now by differentiation we verify that, as found in § 3 above,

$$F = \frac{1}{\sqrt{z + \iota x}} \epsilon^{\frac{-gt^2}{4(z + \iota x)}} \quad . \quad . \quad . \quad . \quad . \quad (136)$$

satisfies (133) and (135). By changing ι into $-\iota$, and by integrations or differentiations performed on (136), according to

the symbol $\frac{d^{i+j+k}}{dt^i dx^j dz^k}$, where i, j, k are any integers positive

or negative, we can derive from (136) any number of imaginary solutions. And by addition of these, with constant coefficients, we can find any number of realised solutions. If, as in § 97, we regard any one of the formulas thus

obtained as a displacement-potential, then by taking $\frac{d}{dz}$ of it

we find ζ , the vertical component displacement, which we shall take as the most convenient expression in each case for the solutions with which we are concerned. Or we may, if we please, take any solution of (135) as representing, not a displacement-potential, but a velocity-potential, or a horizontal component of displacement or velocity, or a vertical component of displacement or velocity.

§ 100. Thus it was that in § 12 we took

$$\left. \begin{aligned} -\zeta &= \{R.S\} \frac{\sqrt{2}}{\sqrt{z + \iota x}} \epsilon^{\frac{-gt^2}{4(z + \iota x)}} \\ &= \phi(x, z, t) = \sqrt{\frac{2}{\rho}} \cos\left(\frac{gt^2 x}{4\rho^2} - \frac{1}{2} \chi\right) \epsilon^{\frac{-gt^2 z}{4\rho^2}} \end{aligned} \right\} \quad (137),$$

where $\rho = \sqrt{(z^2 + x^2)}$, and $\chi = \tan^{-1}(x/z)$,

and in all of §§ 1-31, this notation ϕ and $-\zeta$ was consistently used, with $-\zeta$ to denote, when positive, upward displacement of the water (represented by upward ordinates in the drawings). In the two curves of § 4, fig. 1, that which has its maximum over O represents (137), for $t=0$. The other curve of fig. 1, with positive and negative ordinates on the two sides of O, represents (137), with $-\{RD\}$ instead of $\{RS\}$. The symbols $\{RS\}$ and $\{RD\}$ were introduced in § 3 above; $\{RS\}$ to denote a realisation by taking half the sum of what is written after it with $\pm\iota$, and $\{RD\}$ to denote a realisation by taking $\frac{1}{2\iota}$ of the formula written after it minus $\frac{1}{2\iota}$ of the same formula with $+\iota$ changed into $-\iota$.

A new curve in which the ordinates are numerically equal to $\frac{1}{\sqrt{2}} \frac{d}{dx}$ of the ordinates of the second of the old curves of fig. 1, is now given in the accompanying diagram, fig. 33; and close above it the first of the old curves of fig. 1 is reproduced, with ordinates reduced in the ratio $2\sqrt{2}$ to 1, for the sake of comparison with the new curve. This new curve represents the more convenient initiational form referred to in the title of the present paper.

Its equation, found by taking $t=0$ in (139) or in (144) [most easily from the imaginary form of (139)], is as follows:

$$\psi(x, z, 0) = \frac{1}{2\sqrt{2}} \frac{\sqrt{(\rho+z)}}{\rho^3} (2z-\rho) \quad . \quad (138).$$

§ 101. The original derivation of the new particular solution, (which we shall call ψ), from the primary (136), as indicated in § 100, is shown by the following formula:

$$\left. \begin{aligned} \psi(x, z, t) &= \{RD\} \frac{d}{dx} \frac{-1}{\sqrt{(z+\iota x)}} e^{\frac{-gt^2}{4(z+\iota x)}} \\ &= \frac{1}{2\rho^{3/2}} \left\{ \cos\left(\frac{gt^2x}{4\rho^2} - \frac{3}{2}\chi\right) - \frac{1}{2} \frac{gt^2}{\rho} \cos\left(\frac{gt^2x}{4\rho^2} - \frac{5}{2}\chi\right) \right\} e^{\frac{-gt^2z}{4\rho^2}} \end{aligned} \right\} (139),$$

where $\rho = \sqrt{(z^2 + x^2)}$, and $\chi = \tan^{-1}(x/z)$.

An equivalent formula for the same derivation, which will be found more convenient in §§ 135-157 below, is as follows:

$$\psi(x, z, t) = \{RS\} \frac{1}{g} \frac{d^2}{dt^2} \frac{-1}{\sqrt{(z+\iota x)}} e^{\frac{-gt^2}{4(z+\iota x)}} = \frac{-1}{g\sqrt{2}} \frac{d^2}{dt^2} \phi(x, z, t) \quad (140).$$

Figure 33

Upper Curve $-\zeta = \frac{1}{2\sqrt{2}} \frac{\sqrt{(\rho+1)}}{\rho}$

Lower Curve $-\zeta = -\frac{1}{2\sqrt{2}} \frac{\sqrt{(\rho+1)}}{\rho^3} (2-\rho)$

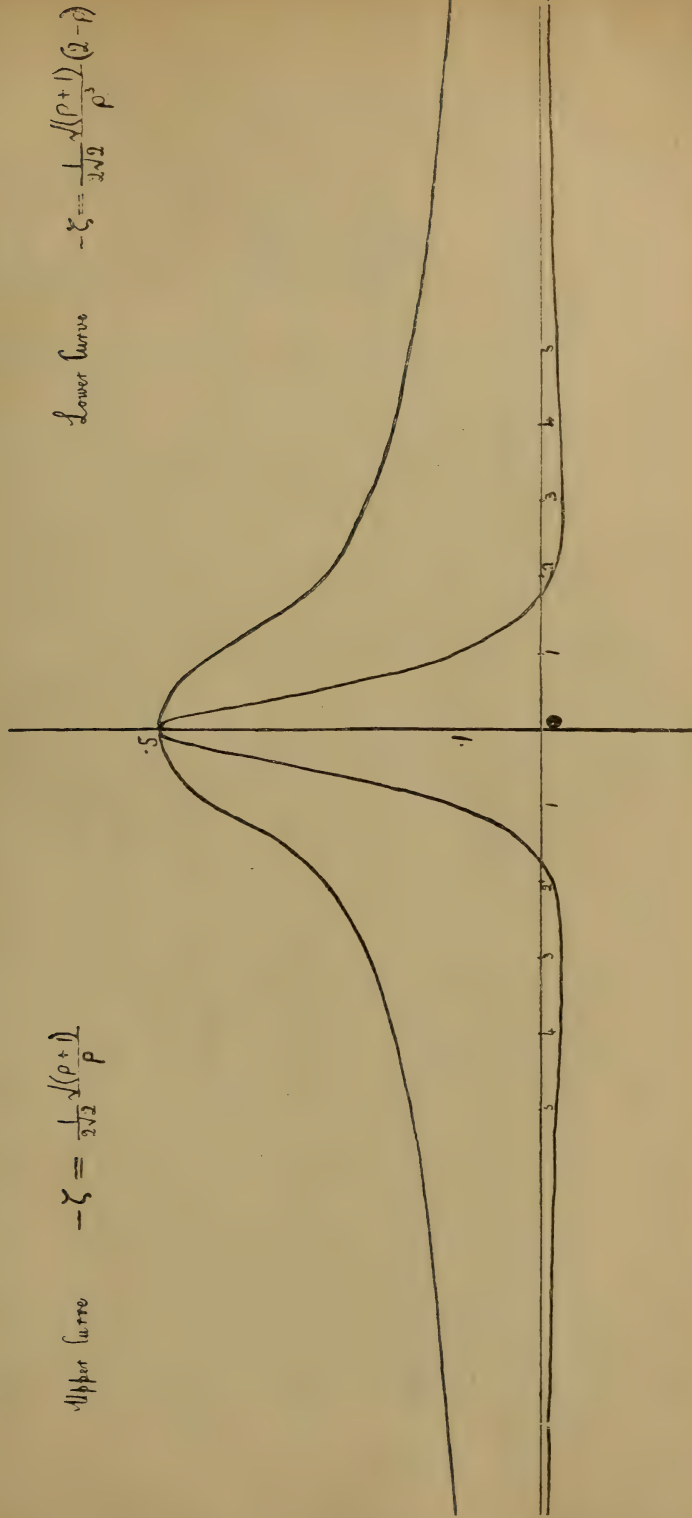
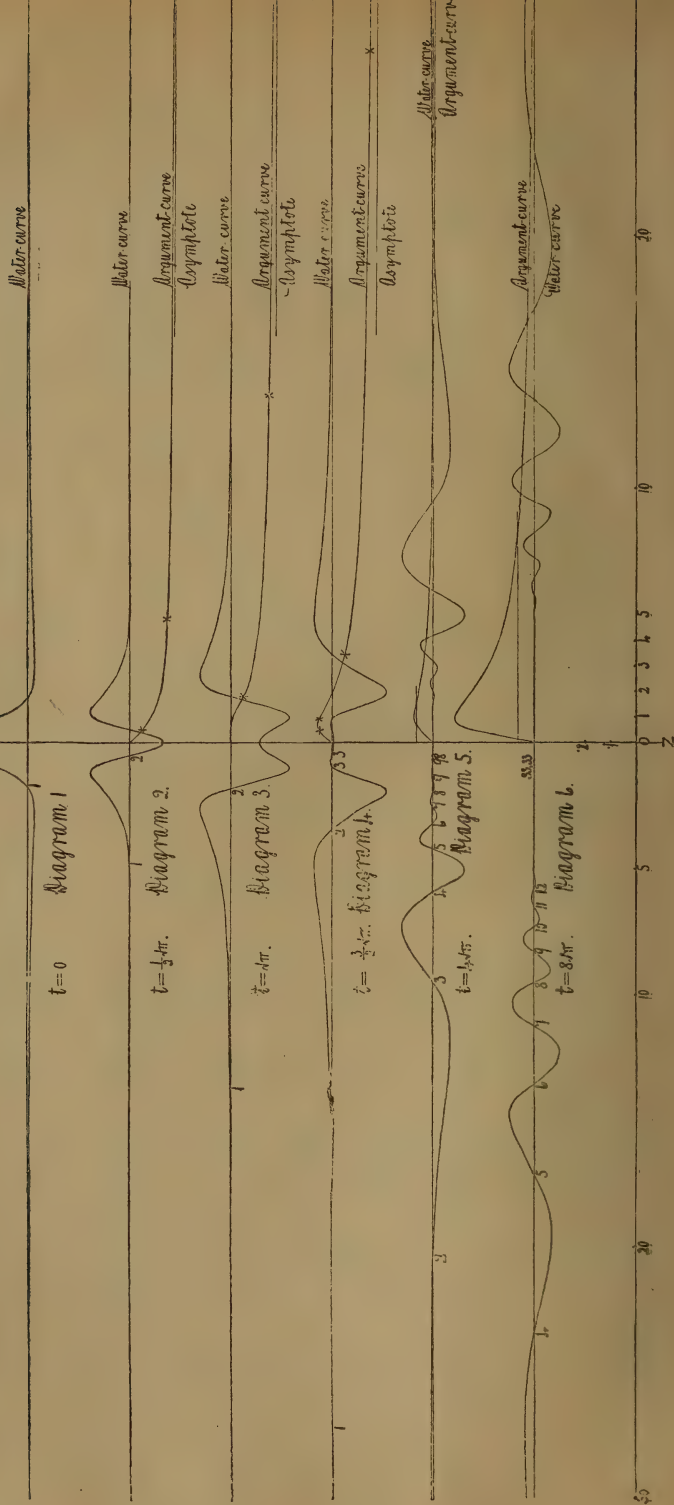


Figure 31.



The equivalence of (139) and (140) is easily proved by remarking that by (133) and (135),

$$\frac{dF}{dx} = \iota \frac{dF}{dz} = \frac{\iota}{g} \frac{d^2 F}{dt^2} \quad . \quad . \quad . \quad (141),$$

and therefore

$$\{RD\} \frac{d}{dx} \frac{-1}{\sqrt{(z+\iota x)}} e^{\frac{-gt^2}{4(z+\iota x)}} = \{RS\} \frac{1}{g} \frac{d^2}{dt^2} \frac{-1}{\sqrt{(z+\iota x)}} e^{\frac{-gt^2}{4(z+\iota x)}} \quad (142).$$

§ 102. Look now to fig. 33, and see within how narrow a space, say from $x = -2$ to $x = +2$, in the new curve, the main initial disturbance is confined, while in the old curve it spreads so far and wide that at $x = \pm 20$ it amounts to about .16 of the maximum disturbance in the middle, and according to the law of inverse proportion to square root of distance, which holds for large values of x for the old curve, at $x = 80$ it would still be as much as .1 of the maximum. The comparative narrowness of the initial disturbance represented by the new curve, and the ultimate law of decrease according to $x^{-\frac{3}{2}}$ (instead of $x^{-\frac{1}{2}}$ for the old curve) are great advantages of the new curve in the applications and illustrations of the theory to be given in §§ 135–157 below.

§ 103. Remark also that the total area of the old curve from $-\infty$ to $+\infty$ is infinitely great, while it is zero for the new curve. Remark also that the potential energy of the initial disturbance, being

$$\frac{1}{2}g \int_{-\infty}^{+\infty} dx [\xi(x, 1, 0)]^2 \quad . \quad . \quad . \quad (143),$$

is infinitely great for the old curve, while for the new it is finite.

§ 104. Equation (139) may be written in the following modified form, which is more convenient for some of our interpretations and graphic constructions,

$$\psi(x, z, t) = - \frac{\sqrt{\{x^2 + (\frac{1}{2}gt^2 - z)^2\}}}{2\rho^{3/2}} e^{\frac{-gt^2 z}{4\rho^2}} \cos A \quad (144),$$

$$\text{where } A = \frac{gt^2 x}{4\rho^2} - \frac{3}{2}\chi - \tan^{-1} \frac{gt^2 \sin \chi}{gt^2 \cos \chi - 2\rho} \quad (145).$$

§ 105. The main curves, which for brevity we shall call water-curves in the accompanying six diagrams of fig. 34, represent the surface displacements according to our new solution $\psi(x, z, t)$ for the six values of t respectively, 0, $\frac{1}{2}\sqrt{\pi}$,

$\sqrt{\pi}, \frac{3}{2}\sqrt{\pi}, 4\sqrt{\pi}, 8\sqrt{\pi}$. The formulas are simplified by taking $g=4$. This is merely equivalent to taking as our unit of length half the space descended in one second of time, by a body falling from rest under the influence of gravity. For simplification in the writing of formulas we take $z=1$ for the undisturbed level of the water-surface. The subsidiary curves, explained in § 107 below, are called argument-curves, as they represent the argument of the cosine in (144).

§ 106. One exceedingly curious and very interesting feature of these curves is the increasing number of values of x for which the displacement is zero as time advances, and the large figures, sixteen and sixty-four, which it reaches at the times, $4\sqrt{\pi}$ and $8\sqrt{\pi}$, of the last two diagrams. These zeros, for any value of t , are given by the equation

$$A = (2i + 1)\pi/2 \quad . \quad . \quad . \quad (146).$$

§ 107. Notwithstanding the highly complicated character of the function represented in (145), the zeros are easily found by tracing an argument-curve, with A as ordinate, and x abscissa (as shown on the x -positive halves of the six diagrams on two different scales chosen merely for illustration, not for measurement), and drawing parallels to the abscissa line at distances from it representing $-\frac{3}{2}\pi, -\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi$, &c. A parallel at distance $-\frac{1}{4}\pi$ is an asymptote to each of the argument-curves, and is shown in diagrams 2, 3, 4, on one scale of ordinates. The parallel corresponding to distance $\frac{1}{2}\pi$ is shown in the fifth and sixth diagrams, on the smaller scale of ordinates used in their argument-curves.

§ 108. The first diagram shows zeros at $x = \pm\sqrt{3}$, of which that at $x = -\sqrt{3}$ is marked 1. In the second diagram the argument-curve indicates zeros for the $-\frac{3}{2}\pi$ and $-\frac{1}{2}\pi$ parallels, which are seen distinctly on the water-curve. The zero corresponding to the $-\frac{1}{2}\pi$ parallel was formed at the origin at the time when $\frac{1}{2}gt^2$ was equal to z , that is, when t was $1/\sqrt{2}$, or $\cdot 707$. It is a coincidence of two zeros for x -positive and x -negative.

Diagram No. 3 shows that, shortly before its time, a maximum has come into existence in the argument-curve, which still indicates only two zeros. These are marked by crosses.

Diagram No. 4 shows that, in the interval between its time and the time of No. 3, two zeros of the water-curve for x -positive have come into existence. These and the corresponding zeros for x -negative are seen distinctly on the water-curve; and their indications for x -positive are marked by four crosses on the argument-curve.

Diagram No. 5 shows that, between its time and that of No. 4, twelve fresh zeros have come into existence on each side of OZ, one pair of which is indicated for example on the argument-curve by the parallel $\frac{1}{2}\pi$. Nine only out of all the sixteen zeros on either side are perceptible on the water-curve. The seven imperceptible zeros, on each side, all lie between $x=0$ and $x=\pm 1$.

Diagram No. 6 shows that, between its time and that of No. 5, forty-eight fresh zeros for x -positive have come into existence, one pair of which is indicated by the parallel $\frac{1}{2}\pi$. Fourteen only out of all the sixty-four zeros on each side are perceptible on the water-curve. Thirty-one of the fifty imperceptible zeros on each side lie between $x=0$ and $x=\pm 1$.

§ 109. After the time $1/\sqrt{2}$, the zeros originate in pairs on the two sides of the origin* (x -positive and x -negative): those on the positive side by the two intersections of one of the parallels corresponding to $(2i+1)\pi/2$ with the argument-curve. The maximum of the argument-curve travels slowly in the outward direction towards $x=1$ as time advances to infinity. At times $4\sqrt{\pi}$ and $8\sqrt{\pi}$, of diagrams 5 and 6, it has reached so close to $x=1$ that this point has been regarded as the actual position of the maximum, both for the purpose of drawing the curve, and for the determination of the total number of zeros.

§ 110. Each zero which originates according to an intersection on the outward side of the argument-curve travels outwards with increasing velocity to infinity, as time advances. Each of the others of the pairs of zeros, that is to say, each zero originating according to an intersection on the inward side of the argument-curve, travels very slowly inwards with velocity diminishing to nothing as time advances to infinity. Thus the motion of the water in the space between $x=-1$ and $x=+1$ becomes more and more nearly an increasing number of inward travelling waves, with lengths slowly diminishing to zero; and, as we see by the exponential factor in (144), with amplitudes and with slopes also slowly diminishing to zero: as time advances to infinity.

§ 111. The semi-period of one of these quasi standing waves is, as we find from (139), approximately equal to $\frac{2\pi\rho^2}{gta}$ when the time is so far advanced that $\frac{1}{2}gt^2$ is very great in

* If we continue the argument-curve to the side of the origin for x -negative, we must include large negative values of i in (146): but for simplicity we have confined the argument-curve to positive values of x .

comparison with ρ . Thus we see that the period is infinite at the origin. This agrees with the history of the whole motion at the origin, which, as we see by putting $x=0$ in (139), with $z=1$ and $g=4$, is expressed by the formula

$$-\zeta = \frac{1}{2} (1 - 2t^2) e^{-t^2} \quad . \quad . \quad . \quad (147).$$

The motion of the water in the space between $x=-1$ and $x=+1$ is of a very peculiar and interesting character. Towards a full understanding of it, it may be convenient to study the simplified approximate solution

$$-\zeta = -\frac{t^2}{\rho^{5/2}} \cos\left(\frac{t^2 x}{\rho^2} - \frac{5}{2} \tan^{-1} x\right) e^{-\frac{t^2}{\rho^2}} \quad . \quad (148),$$

which the realised part of (139) gives when $\frac{1}{2}gt^2$ is very large in comparison with ρ .

§ 112. The outward travelling zeros on the two sides, beyond the distances ± 1 from the origin, divide the water into consecutive parts, in each of which it is wholly elevated or depressed. These parts we may call half-waves. They travel outwards with ever-increasing length and propagational velocity. Each of the half-waves developed after $t = \sqrt{\pi}$, as it travels outward, increases at first to a maximum elevation or maximum depression, and after that diminishes to zero as time advances to infinity.

§ 113. It is interesting to trace the progress of each of the zeros in the intervals between the times of our six diagrams. This is facilitated by the numbers marked on several of the zeros in the different diagrams. Thus, confining our attention to the left-hand side of fig. 34, we see in diagram 1 a single zero numbered 1. The future zeros are to be numbered in the order of their coming into existence, 2 ; 3, 3 ; 4, 4 ; . . . ; 10, 10 ; . . . ; 33, 33 ; . . . all in pairs after zero 2. Thus diagram 2 shows zero 1 considerably advanced leftwards (that is, outwards); and zero 2 beginning its outward progress. Diagram 3 shows zeros 1 and 2 each advanced farther outwards, 1 farther than 2. Diagram 4 shows all the zeros which have come into existence at time $\frac{3}{2}\sqrt{\pi}$. These are zeros 1 and 2, both farther outwards than at time $\sqrt{\pi}$, and a pair, 3, 3, which have come into existence shortly before the time $\frac{3}{2}\sqrt{\pi}$. The outer of these two travels outwards and the inner inwards. Some time later 4, 4 come into existence between 3 and 3 : later still 5, 5 come into existence between 4 and 4.

In diagram 5, zero 1 has passed out of range leftwards :

but we see distinctly the outward zeros 2, 3, 4, 5, 6, 7, 8, 9, and indications of the inward zeros 9, 8. The whole train of zeros for time $4\sqrt{\pi}$, shown and ideally continued to the middle by numbers, is 1, 2, 3, 4, 5, 6, 7, 8, 9, 9, 8, 7, 6, 5, 4, 3 : sixteen in all.

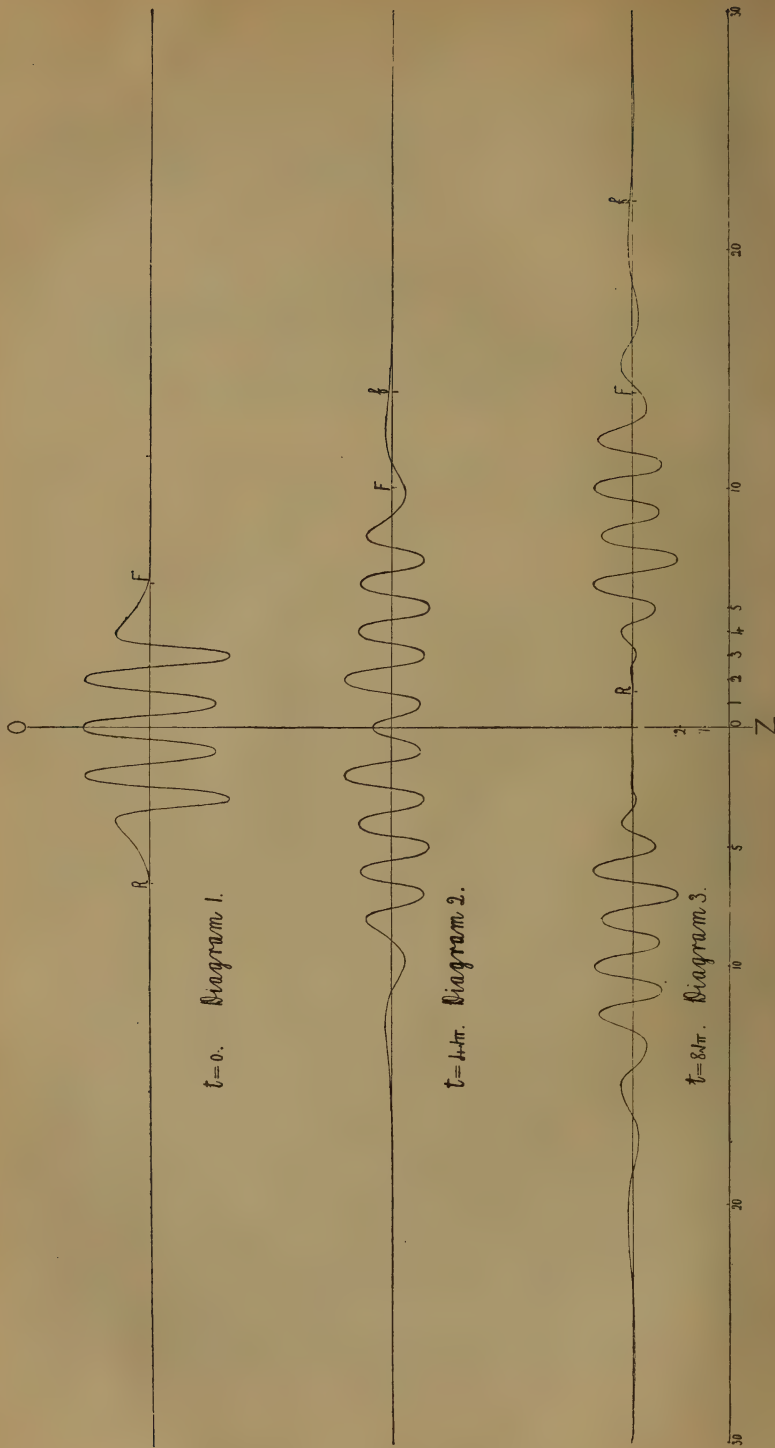
Zero 3 has passed out of the range of diagram 6, but we see in it distinctly the outward zeros 4, 5, ... 12, and an indication of the pair 33, 33, which has come into existence before the time $8\sqrt{\pi}$. The whole train of zeros for time $8\sqrt{\pi}$, indicated by numbers, is 1, 2, ... 32, 33, 33, 32, ... 4, 3 ; sixty-four in all.

(2) *Illustrations of the Indefinite Extension and Multiplication of a Group of Two-Dimensional, Deep-Sea Waves Initially Finite in Number.* §§ 114–117.

§ 114. The water is left at rest and free, after being initially displaced to a configuration of a finite number of sinusoidal mountains and valleys—five mountains and four valleys ; in the diagrams placed before the Society. The initial group of waves, shown in diagram 1, of fig. 35, is formed by placing side by side, at distances equal to c (taken as unity), nine of the curves of diagram 1, fig. 34, alternately positive and negative. Diagrams 2 and 3, of fig. 35, are made by corresponding superpositions of the curves of diagrams 5 and 6, of fig. 34. Thus what, according to the known law of deep-sea periodic waves (§ 19 above), would be definitely and precisely the wave-length, if the numbers of crests and hollows were infinitely great, would be 2 ; and as we are taking $g=4$, the period would be $\sqrt{\pi}$, and the propagational velocity would be $2/\sqrt{\pi}$.

§ 115. Immediately after the water is left free, the disturbance begins analysing itself into two groups of waves, seen travelling in contrary directions from the middle line of the diagram. The perceptible fronts of these two groups extend rightwards and leftwards from the end of the initial single static group, far beyond the “hypothetical fronts,” supposed to travel at half the wave-velocity, which (according to the dynamics of Osborne Reynolds and Rayleigh, in their important and interesting consideration of the work required to feed a uniform procession of water-waves) would be the actual fronts *if* the free groups remained uniform. How far this *if* is from being realised is illustrated by the diagrams of fig. 35, which show a great extension outwards in each direction far beyond distances travelled at half the “wave-velocity.” While there is this great extension of the fronts

Figure 35.



Initial group of five elevations and four depressions emerging as two groups travelling in opposite directions.

outward from the middle, we see that the two groups, after emergence from co-existence in the middle, travel with their rears leaving a widening space between them of water not perceptibly disturbed, but with very minute wavelets in ever-augmenting number following slower and slower in the rear of each group. The extreme perceptible rear travels at a speed closely corresponding to the "half wave-velocity," found by Stokes as exactly the group-velocity of his uniform succession of groups, produced by the interference of two co-existent infinite processions of sinusoidal waves, having slightly different wave-lengths.

§ 116. Our fairly uniform rear velocity is illustrated in diagrams 1 and 3, of fig. 35. In diagram 1, R indicates the perceptible rear of the component group commencing its rightward progress at $t=0$. In diagram 3, R shows the position reached at time $8\sqrt{\pi}$ (eight periods) by an ideal point travelling rightwards from the R of diagram 1 at a speed of half the wave-velocity. This R of diagram 3 corresponds to a fairly well-marked perceptible rear of the rightward travelling group.

Look now to $\bar{F}$, F , $\bar{f}$, f , in the three diagrams of fig. 35, and $\bar{f}$, f , in diagrams 2 and 3. In diagram 1, $\bar{F}$ marks a perceptible front for the rightward travelling component group. In diagrams 2 and 3, $\bar{F}$, $\bar{f}$ show ideal points travelling rightwards from it at speeds respectively, the half wave-velocity, and the wave-velocity. We see a manifest wave-disturbance far in advance of $\bar{F}$, $\bar{f}$; and very small but still perceptible wave-disturbance in front of $\bar{f}$, f . Thus the perceptible front travels at speed actually higher than the wave-velocity, and this perceptible front becomes more and more important relatively to the whole group with the advance of time, as we may judge from fig. 9 of § 20 above.

§§ 117. It is interesting to see by these diagrams how nearly the hypothetical group-velocity is found in the rears: while the fronts advance with much greater and with ever-increasing velocity. The more elaborate calculations and graphical constructions of §§ 20-29 above led to corresponding conclusions in respect to the front and rear of a procession, given initially as an infinitely great number of regular sinusoidal waves travelling in one direction. The diagrams, figs. 9 and 10, showed respectively, at twenty-five periods after a sinusoidal commencement, a front extending forward indefinitely, and a perceptible rear lagging scarcely two wave-lengths behind a point, travelling from the initial position of the rear at a speed of half the wave-velocity.

- (3) *The Initiation and continued Growth of a Train of Two-Dimensional Waves due to the Sudden Commencement of a Stationary, Sinusoidally Varying, Surface-Pressure.*
 §§ 118–158.

§ 118. A forcive consisting of a finite sinusoidally varying pressure is applied, and kept through all time applied, to the surface of the water within a finite practically limited space on each side of the middle line of the disturbance. In the beginning the water was everywhere at rest and its surface horizontal. The problem solved is, to find the elevation or depression of the water at any distance from the mid-line of the working forcive, and at any time after the forcive began to act.

§ 119. As a preliminary (§§ 119–126) let us consider the energy in a uniform procession of sinusoidal waves, in a straight canal, infinitely long and infinitely deep, with vertical sides. If the water is disturbed from rest by any pressure on its upper surface, and afterwards left to itself under constant air pressure, we know by elementary hydrokinetics that its motion will be irrotational throughout the whole volume of the water: and if, at any subsequent time, the surface is brought to rest, suddenly or gradually, all the water at every depth will come to rest at the instant when the whole surface is brought to rest. This, as we know from Green, is true even if the initial disturbance is so violent as to cause part of the water to break away in drops: and it would be true separately for each portion of the water detached from the main volume in the canal, as well as for the water remaining in the canal, if stoppage of surface motion is made for every detached portion before it falls back into the canal.

§ 120. Because the motion of the water is irrotational, we have

$$\xi = \frac{d\dot{F}}{dx}; \quad \zeta = \frac{d\dot{F}}{dz} \quad . \quad . \quad . \quad . \quad (149),$$

where $\dot{F}$ denotes the velocity-potential, F having been taken as the displacement-potential (§ 97 above). And by dynamics for infinitesimal motion, as in (64) of § 38 above,

$$p = - \frac{d}{dt} \dot{F}(x, z, t) + g(z - 1 + C) \quad . \quad . \quad (150).$$

To express the surface condition, let $z = 1$ be the undisturbed level: and let ζ_1 denote the vertical component displacement of a surface particle of the water, taken positive when

downwards; and let Π denote constant surface-pressure, and take $\frac{\Pi}{g}$ as the value of the arbitrary constant, C . Thus (150) gives, at the disturbed surface

$$0 = -\frac{d}{dt} \dot{F}(x, 1 + \xi_1, t) + g\xi_1 = -\frac{d}{dt} \dot{F}(x, 1, t) + g\xi_1 \quad (151).$$

The equality between the second and third members of this formula is due to the disturbance being infinitely small, which makes $\frac{d}{dt} \dot{F}(x, 1 + \xi_1, t) - \frac{d}{dt} \dot{F}(x, 1, t)$ an infinitely small quantity of the second order, negligible in comparison with $g\xi_1$, which is an infinitely small quantity of the first order.

§ 121. For a sinusoidal wave-disturbance of wave-length $2\pi/m$, travelling x -wards with velocity v , we have, as in (66) above,

$$\dot{F}(x, z, t) = -k\epsilon^{-m(z-1)} \sin m(x-vt) \quad . \quad . \quad (152).$$

For surface-equation (151) becomes

$$0 = kmv \cos m(x-vt) - g\xi_1 \quad . \quad . \quad (153).$$

This gives as the equation of the free surface

$$\xi_1 = h \cos m(x-vt) \quad . \quad . \quad . \quad (154),$$

where

$$h = \frac{kmv}{g} \quad . \quad . \quad . \quad (155).$$

Now by (149) and (152) with $z=1$, we find

$$\xi_1 = \frac{k}{v} \cos m(x-vt) \quad . \quad . \quad . \quad (156).$$

Comparison of this with (154) gives

$$v^2 = g/m = \lambda g/2\pi \quad . \quad . \quad . \quad (157).$$

§ 122. Let us now find A (activity), the rate of doing work by the pressure of the water on one side upon the water on the other side of a vertical plane (x). We have

$$A = \int_1^\infty dz p \dot{\xi} = \int_1^\infty dz \dot{\xi} \left[-\frac{d\dot{F}}{dt} + g(z-1+C) \right] \quad (158).$$

Eliminating from this $\dot{\xi}$ and $\dot{F}$ by (149) and (152), we find

$$A = -km \cos m(x-vt) \int_1^\infty dz \epsilon^{-m(z-1)} \left[-kmv \epsilon^{-m(z-1)} \cos m(x-vt) + g(z-1+C) \right] \quad . \quad (159).$$

Hence, performing the operations, $\int_1^x dz$, we find

$$A = -km \cos m(x-vt) \left[-\frac{kv}{2} \cos m(x-vt) + g \left(\frac{1}{m^2} + \frac{C}{m} \right) \right] \quad (160).$$

§ 123. Remarking now that $2\pi/mv$ is the periodic time of the wave, and denoting by W the total work per period, done by the water on the negative side of the plane (x) upon the water on the positive side, we have

$$W = \int_0^{2\pi/mv} dt \cdot A = \frac{\pi}{mv} \cdot \frac{1}{2} \cdot k^2 mv = \frac{1}{2} \pi k^2 \quad (161).$$

§ 124. We are going to compare this with the total energy, kinetic and potential, $K + P$, per wave-length. In the first place we shall find separately the kinetic energy, K , and the potential energy, P . We have (the density of the water being taken as unity)

$$K = \frac{1}{2} \int_0^\lambda dx \int_1^\infty dz (\dot{\xi}^2 + \dot{\zeta}^2) \quad \dots \quad (162);$$

$$P = \frac{1}{2} g \int_0^\lambda dx \zeta_1^2 \quad \dots \quad (163),$$

where ζ_1 denotes the surface displacement.

By (149) and (152) we find

$$\dot{\xi} = -mk\epsilon^{-m(z-1)} \cos m(x-vt) \quad \dots \quad (164);$$

$$\dot{\zeta} = mk\epsilon^{-m(z-1)} \sin m(x-vt) \quad \dots \quad (165);$$

$$\zeta_1 = \frac{k}{v} \cos m(x-vt) \quad \dots \quad (156) \text{ repeated.}$$

Hence,

$$K = \frac{1}{2} \int_0^\lambda dx \frac{m^2 k^2}{2m} = \frac{1}{4} m k^2 \lambda = \frac{1}{2} \pi k^2 \quad \dots \quad (166);$$

$$P = \frac{1}{2} g \frac{k^2 \lambda}{v^2} = \frac{1}{2} \pi k^2 \quad \dots \quad (167),$$

where v^2 is eliminated by (157).

§ 125. Thus we see that the kinetic energy per wave-length, and the potential energy per wave-length, are each equal to the work done per period by the water on the negative side, upon the water on the positive side, of any vertical plane perpendicular to the length and sides of the canal. Thus we arrive at the remarkable and well-known conclusion that in a regular procession of deep-sea waves, the

work done on any vertical plane is only half the total energy per wave-length. This is only half enough to feed a regular procession, advancing to infinity with abruptly ending front, travelling *with the wave-velocity* v . It is exactly enough to feed an ideal procession of regular periodic waves, coming abruptly to nothing at a front travelling *with half the "wave-velocity"* v ; which is Osborne Reynolds'* important contribution to the ideal doctrine of "group-velocity."

§ 126. The dynamical conclusion of § 125 is very important and interesting in the theory of two-dimensional ship-waves. It shows that the approximately regular periodic train of waves in the rear of a travelling forcive, investigated in §§ 48-54, and 65-79 above, cannot be as much as half the space travelled by the forcive, from the commencement of its motion; but that it would be exactly that half-space if some modifying pressure were so applied to the water-surface in the rear as to cause the waves to remain uniformly periodic to the end of the train; without, on the whole, either doing work on them, or taking work from them.

A corresponding statement is applicable to our present subject, as we shall see in §§ 156, 157 below.

§ 127. Go back to § 118; and first, instead of a sinusoidally varying pressure, imagine applied a series of impulsive pressures, each of which superimposes a certain velocity-potential upon that due to all the previous impulses; and let it be required to find the resulting velocity-potential at any time t , after some, or after all, of the impulses. Consider first a single impulse at time $t-q$; that is to say, at a time preceding the time t by an interval q . Let the velocity-potential at time t , due to that single impulse applied at the earlier time $t-q$, be denoted by

$$CV(x, z, q) \quad . \quad . \quad . \quad . \quad . \quad (168).$$

According to this notation the instantaneously generated velocity-potential is $CV(x, z, 0)$, and the value of this at the bounding surface of the water is $CV(x, 1, 0)$. Hence, by elementary hydrokinetics, if I denotes the impulsive surface-pressure, we have

$$I = -CV(x, 1, 0) \quad . \quad . \quad . \quad . \quad . \quad (169).$$

§ 128. Considering now successive impulses at times preceding the time t , by amounts $q_1, q_2, \dots q_i$; and denoting by $S(x, z, t)$ the sum of the resulting velocity-potentials at time t , we find

$$S(x, z, t) = C_1 V(x, z, q_1) + C_2 V(x, z, q_2) + \dots C_i V(x, z, q_i) \quad (170).$$

* 'Nature,' August 1877, and Brit. Assoc. Report, 1877.

Supposing now the impulses to be at infinitely short intervals of time, we translate the formula (170) into the language of the integral calculus as follows :

$$S(x, z, t) = \int_0^t dq f(t-q) V(x, z, q) \quad . \quad . \quad (171),$$

where $f(t-q)$ denotes an arbitrary function of $(t-q)$, according to which the surface-pressure, arbitrarily applied at time $(t-q)$, is as follows :

$$\Pi(t-q) = -f(t-q) V(x, 1, 0) \quad . \quad . \quad (172).$$

Hence the pressure applied to the surface at time t , denoted by $\Pi(x, 1, t)$, is as follows :

$$\Pi(x, 1, t) = -f(t) V(x, 1, 0) \quad . \quad . \quad (173).$$

§ 129. The solution (170) or (171) gives the velocity-potential throughout the liquid which follows determinately from the dynamical data described in §§ 127, 128. From it, by differentiations with reference to x and z , and integrations with respect to t , we can find the displacement components ξ, ζ of any particle of the liquid whose co-ordinates were x, z when the fluid was given at rest. But we can find them more directly, and with considerably less complication of integral signs, by direct application of the same plan of summing as that used in (170), (171). Thus if, instead of $V(x, z, q)$ in (171), we substitute $\frac{d}{dx} V(x, z, q)$, and again $\frac{d}{dz} V(x, z, q)$, we find $\dot{\xi}$ and $\dot{\zeta}$. And if we take

$$\int_0^t dq \frac{d}{dx} V(x, z, q) \quad \text{and} \quad \int_0^t dq \frac{d}{dz} V(x, z, q) \quad . \quad (174)$$

in place of $V(x, z, q)$ in (171), we find the two components ξ, ζ of the displacement of any particle of the fluid. Confining our attention to vertical displacements, and using (179) below, we thus find

$$\zeta(x, z, t) = \frac{1}{g} \int_0^t dq f(t-q) \frac{d}{dq} V(x, z, q) \quad . \quad (175).$$

§ 130. To illustrate the meaning of the notation and analytical expressions in (171), (173), (175), take the simplest possible example, $f(t-q)=1$. This makes Π the same for all values of t ; and (173) becomes

$$\Pi = -V(x, 1, 0) \quad . \quad . \quad . \quad (176);$$

and by integration (175) becomes

$$\zeta(x, z, t) = \frac{1}{g} \left[V(x, z, t) - V(x, z, 0) \right] \quad (177).$$

Putting now in this $z=1$, and using (176), we find

$$\zeta(x, 1, t) = \frac{1}{g} \left[V(x, 1, t) - V(x, 1, 0) \right] = \frac{1}{g} \left[V(x, 1, t) + \Pi \right] \quad (178).$$

The interpretation of this, as t increases from 0 to ∞ , is that the sudden application and continued maintenance of a pressure $-V(x, 1, 0)$ over the whole fluid surface, initially plane and level, produces a depression, ζ , which gradually increases from 0, at $t=0$, to its hydrostatic value Π/g , at $t=\infty$. The gradual subsidence of the difference from the static condition, as time advances from 0 to ∞ , is illustrated by the diagrams of fig. 34, for the case in which we choose for $V(x, 1, 0)$ the $\psi(x, 1, 0)$ of §§ 100–104 above.

§ 131. To understand thoroughly the meaning of $V(x, z, q)$ as defined in § 127; remark first that it is the velocity-potential of a possible motion of water, under the influence of gravity, with no surface-pressure, or with merely a pressure uniform over its infinite free surface. This is equivalent to saying that $V(x, z, q)$ fulfils the equations

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dz^2} = 0, \text{ and } g \frac{dV}{dz} = \frac{d^2 V}{dq^2} \quad (179).$$

Secondly, remark that at the instant $q=0$, there is no surface displacement; hence $V(x, z, q)$ is the velocity-potential at time q , due to an instantaneous impulsive pressure, $-V(x, 1, 0)$, applied to the surface of the fluid at rest and in equilibrium, at time $q=0$. Now, allowing negative values of q , think of a state of motion from which our actual condition of no displacement, and of velocity-potential equal to $V(x, z, 0)$, would be reached and passed through when q passes from negative to positive. It is clear that the values of $V(x, z, q)$ are equal for equal positive and negative values of q . Hence,

$$\text{when } q=0, \text{ we have } \frac{d}{dq} V(x, z, 0) = 0 \quad (180).$$

§ 132. Consideration of the $V(x, z, q)$, defined in § 127, which allows $V(x, 1, 0)$ to be any arbitrary function of x , but requires dV/dq to be zero when $q=0$, suggests an allied hydrokinetic problem:—to find W fulfilling (179) with W in place of V ; and, at time $q=0$, having $W=0$ and dW/dq

any arbitrary function of x . We assume, as is convenient for our present purpose, that for large values of x

$$V(x, z, 0) \doteq 0, \text{ and } W(x, z, 0) \doteq 0 \quad . \quad . \quad (181).$$

This implies that for all values of x and z , large or small, but for large values of q ,

$$V(x, z, q) \doteq 0, \text{ and } W(x, z, q) \doteq 0 \quad . \quad . \quad (182).$$

§ 133. In the V-problem the initiational condition is :—displacement zero and initiational velocity *virtually given* throughout the fluid as the determinate result of an arbitrarily distributed impulsive pressure on the surface.

In the W-problem the initiational condition is :—the fluid held at rest with its surface kept to any arbitrarily prescribed shape by fluid pressure, and then left free by sudden and permanent annulment of this pressure.

Without going into the question of a complete solution of this (V, W) problem for any arbitrary initiational data, we find a class of thoroughly convenient solutions in a formula originally given in the 'Proceedings of the Royal Society of Edinburgh,' January 1887; republished in the Phil. Mag., February 1887; and used in § 3 and § 99 above. We may now write that formula in the following comprehensive realised expression for V or W :—

$$\left. \begin{aligned} \{RS\} \text{ or } \{RD\} \frac{d^{i+j+k}}{dt^i dx^j dz^k} \frac{1}{\sqrt{(z+ux)}} e^{\frac{-gt^2}{4(z+ux)}} &= V(x, z, t), \text{ when } i \text{ is even;} \\ &= W(x, z, t), \text{ when } i \text{ is odd.} \end{aligned} \right\} \quad (183).$$

By using (179) we may, instead of (183), take the following as equally comprehensive :—

$$\left. \begin{aligned} \{RS\} \text{ or } \{RD\} \left(A + B \frac{d}{dx} \right) \frac{d^i}{dt^i} \frac{1}{\sqrt{(z+ux)}} e^{\frac{-gt^2}{4(z+ux)}} &= V(x, z, t), \text{ when } i \text{ is even;} \\ &= W(x, z, t), \text{ when } i \text{ is odd.} \end{aligned} \right\} \quad (183').$$

§ 134. Going back to (171) and (175), remark that integration by parts gives

$$\begin{aligned} \int_0^t dq f(t-q) \frac{d}{dq} V(x, z, t) &= f(0)V(x, z, t) - f(t)V(x, z, 0) \\ &\quad + \int_0^t dq f'(t-q) V(x, z, q) \quad . \quad (184). \end{aligned}$$

This shows that if by quadrature or otherwise we have calculated the velocity-potential $S(x, z, t)$, as given by (171), we can find the vertical component displacement of any

particle of the liquid by (175), without farther integration. The formula (184) also shows how by successive integrations by parts we can reduce

$$\int_0^t dq f(t-q) \frac{d^i}{dq^i} V(x, z, q) \quad . \quad . \quad . \quad (185)$$

to the primary integral $S(x, z, t)$, as expressed in (171).

§ 135. Going back now to §§ 128, 127, 118: to make the applied force a sinusoidally varying pressure put

$$f(t-q) = \frac{\cos}{\sin} \omega(t-q) \quad . \quad . \quad . \quad (186);$$

which, by (173), makes

$$\Pi(x, 1, t) = - \frac{\cos}{\sin} \omega t V(x, 1, 0) \quad . \quad . \quad (187).$$

And now let us arrange to fully work out our problem for two cases of surface distribution of pressure, corresponding to the two initiational forms ϕ, ψ , described in §§ 96-113 above. For this purpose take, with the notation of § 101,

$$V(x, z, t) = \phi(x, z, t);$$

$$\text{or } V(x, z, t) = \psi(x, z, t) = - \frac{1}{g\sqrt{2}} \frac{d^2}{dt^2} \phi(x, z, t) \quad (188).$$

For brevity we shall call these two cases case ϕ and case ψ . Thus, in these cases (171) and (175), expressing respectively the velocity-potential at, and the vertical component displacement of, any point of the fluid at any time, become

$$S_\phi(x, z, t) = \int_0^t dq \frac{\cos}{\sin} \omega(t-q) \phi(x, z, q);$$

$$S_\psi(x, z, t) = \int_0^t dq \frac{\cos}{\sin} \omega(t-q) \psi(x, z, q) \quad . \quad (189);$$

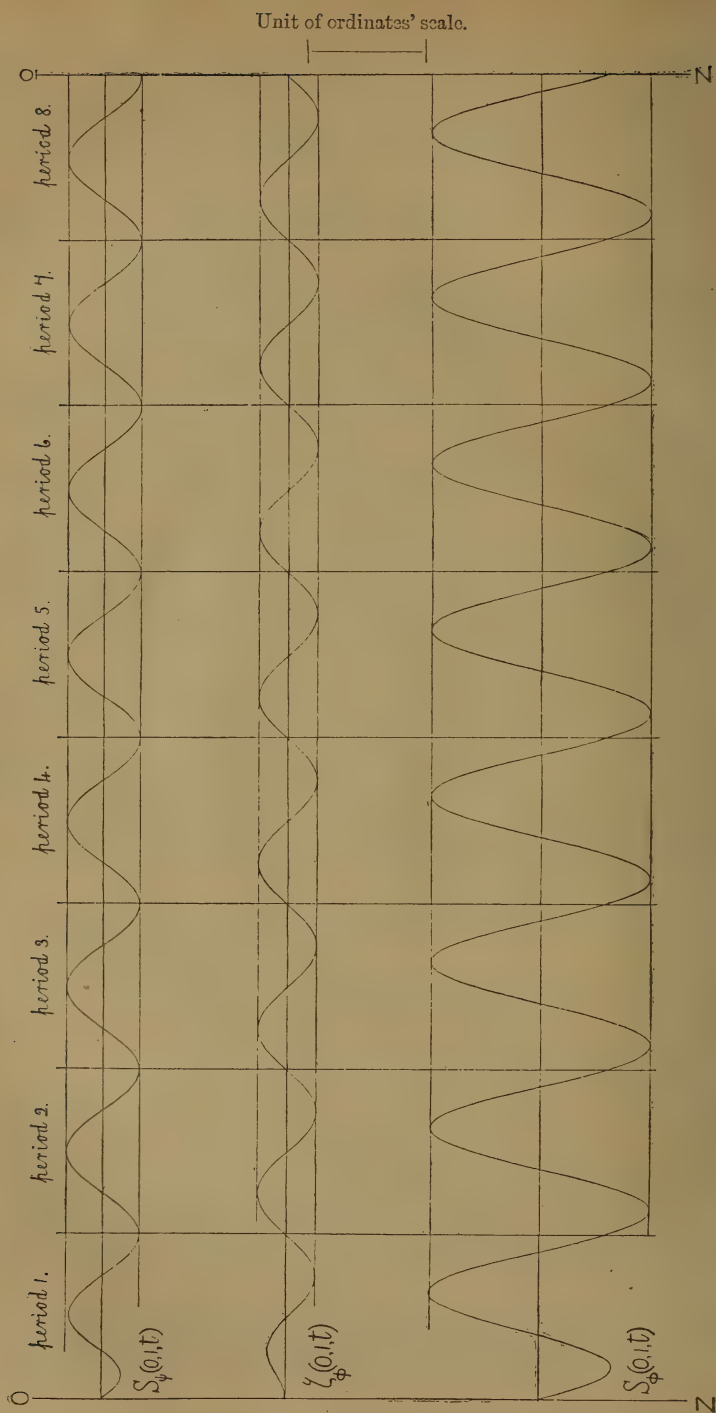
$$\zeta_\phi(x, z, t) = \frac{1}{g} \int_0^t dq \frac{\cos}{\sin} \omega(t-q) \frac{d}{dq} \phi(x, z, q);$$

$$\zeta_\psi(x, z, t) = \frac{1}{g} \int_0^t dq \frac{\cos}{\sin} \omega(t-q) \psi(x, z, q) \quad . \quad (190).$$

§ 136. The illustrations in figs. 36, 37, 38 are time-curves in which the ordinates have been calculated by continuous quadrature from one or other of the four formulas (189), (190).

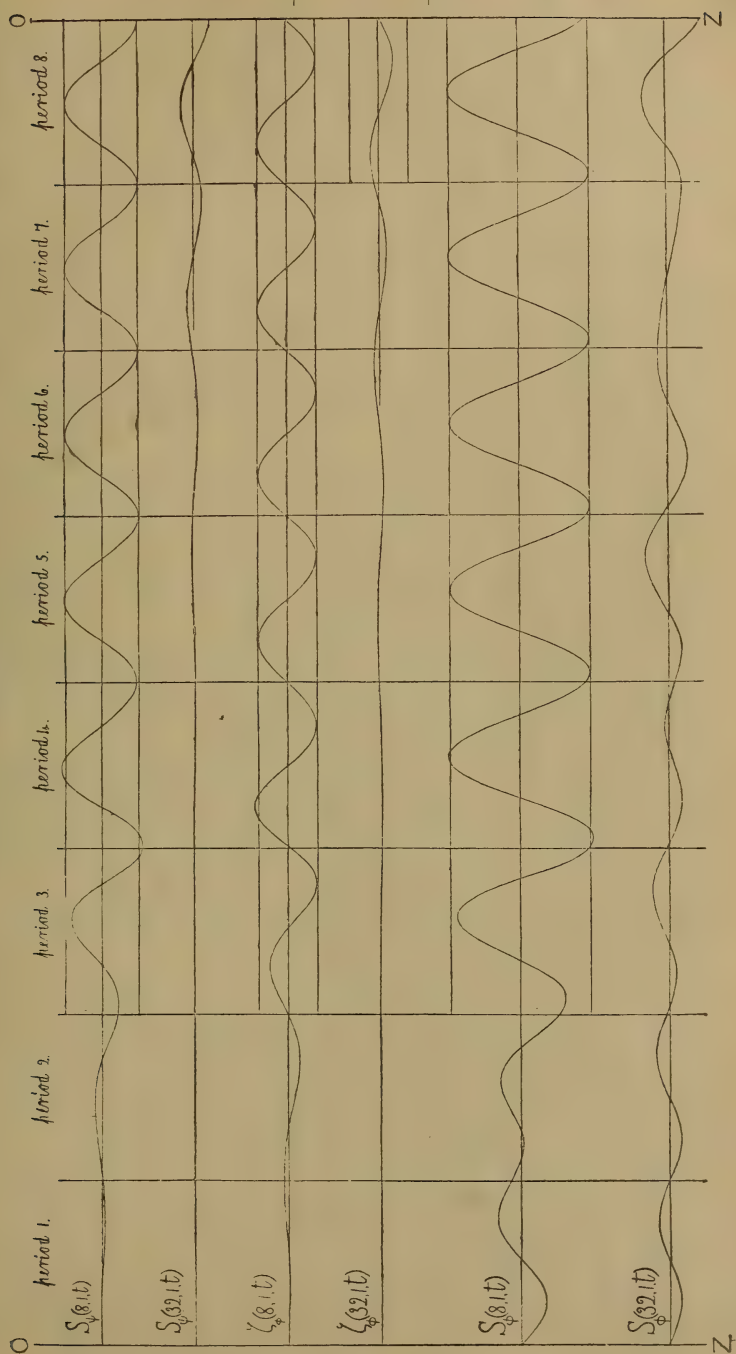
§ 137. The curves in fig. 39, being space curves in which the ordinates are vertical component displacements of the water-surface, are therefore pictures of the water-surface

FIGURE 36.



Unit of ordinates' scale.

FIGURE 37.



(greatly exaggerated in respect to slopes of course), and may be shortly named water-surface curves. Their ordinates have been calculated by an analytical method described in § 151 below. They cannot be calculated continuously for successive values of x by the method of continuous quadratures; if that were the method employed, the value of the ordinate for each value of x would need to be calculated by an independent quadrature $\left(\int_0^t dq\right)$ from 0 to the particular

value of t for which the water-surface is represented by the curve. The values of t chosen for fig. 39 are respectively $i\tau$, $(i+1/8)\tau$, $(i+2/8)\tau$, $(i+3/8)\tau$, $(i+4/8)\tau$, where i is any very large integer, and τ denotes $2\pi/\omega$, the period of the varying surface-pressure to which the fluid motion considered is due.

In all our illustrations we have taken $\omega=\sqrt{\pi}$, which makes $\tau=2\sqrt{\pi}$, and, with $g=4$ as in § 105, makes the wave-length $\lambda=8$.

§ 138. In figs. 36 and 37, all the curves correspond to $\cos \omega(t-q)$ in the formulas. In fig. 38, all the curves correspond to $\sin \omega(t-q)$ in the formulas.

In fig. 39, the inscriptions of times correspond to $\cos \omega(t-q)$ in the formulas. The same curves, with the inscriptions altered to $(i+2/8)\tau$, $(i+3/8)\tau$, $(i+4/8)\tau$, $(i+5/8)\tau$, $(i+6/8)\tau$, correspond to $\sin \omega(t-q)$ in the formulas.

§ 139. In fig. 36, representing velocity-potentials and a surface displacement, none of the curves shows any perceptible deviation from sinusoidality except within period 1. Towards the end of period 1 the numbers found by the quadratures show deviations from sinusoidality diminishing to about 1/10 per cent., and imperceptible in the drawings. This proves that sinusoidality is exact within 1/10 per cent. through all time after the end of the first period.

It is interesting to see, in period 1, how nearly the rise from the initial zero follows the same law for $S_\phi(0, 1, t)$ and $S_\psi(0, 1, t)$: notwithstanding the vast difference in the law of initiating surface-pressure, represented by (188), for these two cases. In fig. 36, the initiating surface-pressure commences suddenly at its negative maximum value, $-\sqrt{2}$ for case ϕ , and $-.5$ for case ψ , of which the former is 2.83 times the latter. The semi-amplitudes of the subsequent variations of velocity-potential shown in the first and third curves are .954 for case ϕ and .318 for case ψ , of which the former is 3.00 times the latter.

§ 140. The first, and third, and fifth, curves of fig. 37 show, at a distance of one wave-length from the origin, the

Unit of ordinates' scale.

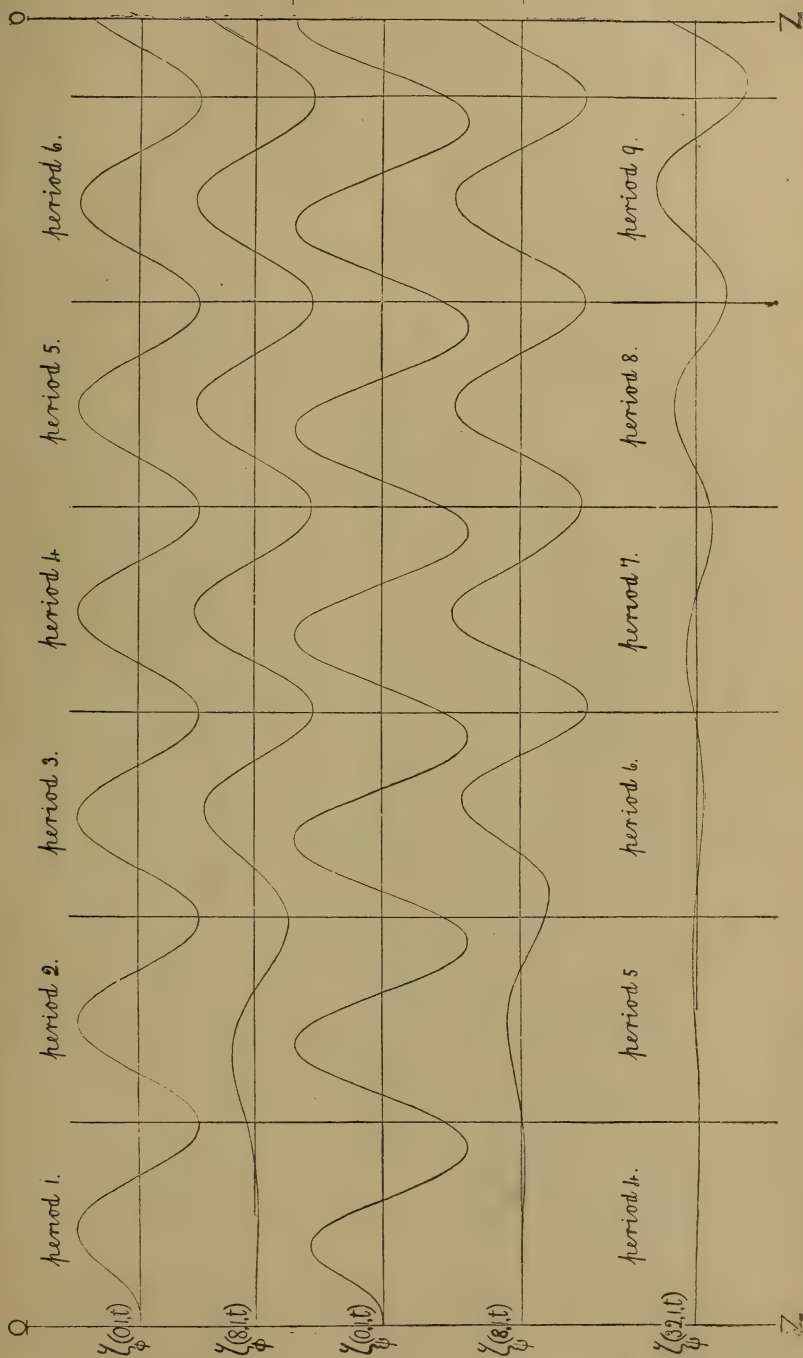
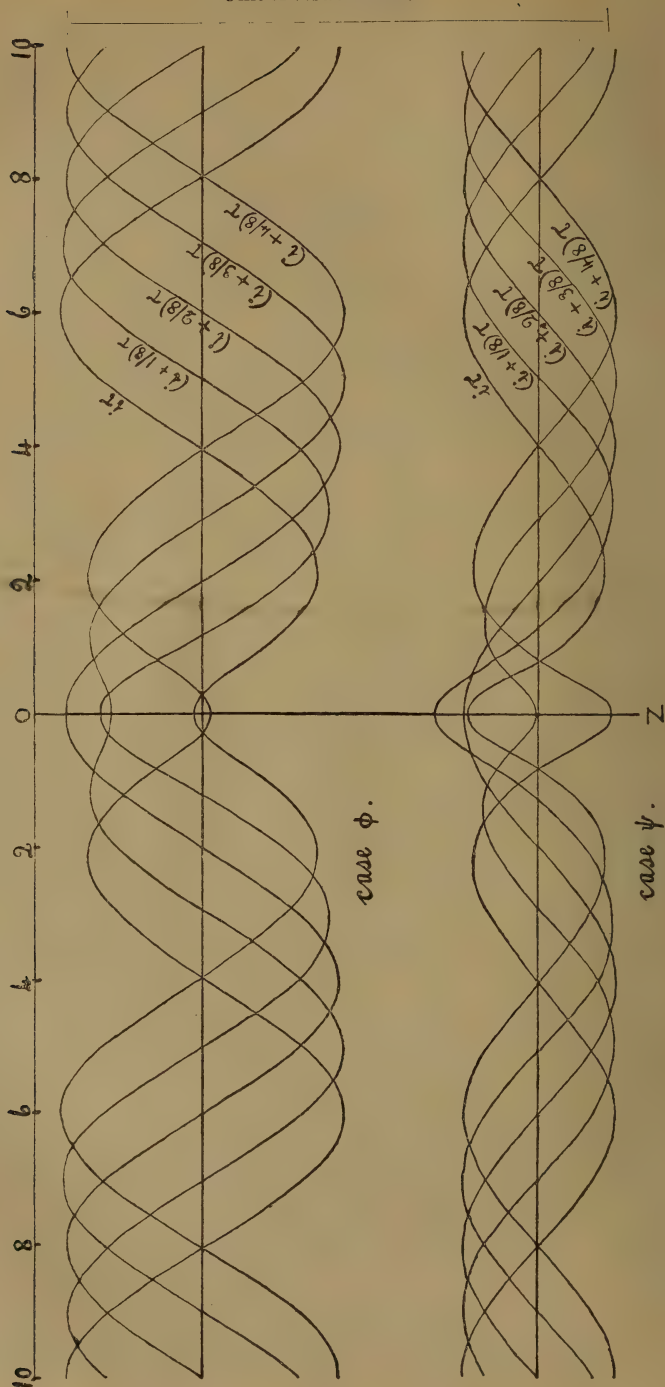


FIGURE 38.

Unit of vertical scale.

FIGURE 39.



complete history of velocity-potential and of surface displacement through all time from the beginning of application of pressure to the surface. The very approximately accurate sinusoidality of each of these three curves through periods 6, 7, 8, shows that the continuation through endless time is in each case sinusoidal.

In remarkable contrast with the initial agreement between $S\psi(0, 1, t)$ and $S\phi(0, 1, t)$, to which we alluded in § 139, we find very instructively a remarkable contrast between $S\psi(8, 1, t)$ and $S\phi(8, 1, t)$ throughout the whole of the first period. Remembering that in a liquid of unit density the pressure is equal to minus the rate of augmentation of the velocity-potential per unit of time, and remarking that the displacement $\xi\phi(0, 1, t)$ is, as is shown in its curve, very nearly zero throughout the first period, and that $\xi\psi(0, 1, t)$ is certainly still more nearly zero throughout the first period, though we have no curve to represent it, we see that the negatives of the tangents of the slopes in the curves for $S\psi(8, 1, t)$ and $S\phi(8, 1, t)$ represent very nearly the values of the applied surface-pressures during the whole of the first period*. Look now to fig. 33; see how near to zero is $\psi(8, 1, 0)$, and how far from zero is $\phi(8, 1, 0)$; and we see dynamically how it is that $S\psi(8, 1, t)$ is very nearly zero throughout the first period, and $S\phi(8, 1, t)$ is very far from zero, and is somewhat near to being sinusoidal.

§ 141. We have also a very instructive comparison between $\xi\phi(8, 1, t)$ and $S\phi(8, 1, t)$. In the ϕ case, for values of x as large as 8, or larger, we approach somewhat nearly to the case of a sinusoidally varying uniform surface-pressure over an infinite plane area of water, in which there would be no surface displacement, and the pressure at and below the surface would be at every instant equal to the applied surface-pressure plus the gravitational augmentation of pressure below the surface. Thus we see why it is that, with a great periodic variation of applied surface-pressure, at $x=8$, there is scarcely any rise and fall of the surface level there, until after a period and a half from the beginning of the motion, as shown in the curves for $\xi\phi(8, 1, t)$.

§ 142. The second, fourth, and sixth, curves of fig. 37 represent the arrival of three classes of disturbance, $S\psi$, $\xi\phi$, $S\phi$, at $x=32$, four wave-lengths from the origin. If the front of the disturbance travelled at exactly the wave-velocity, the disturbances of the different kinds would all commence suddenly at the end of period 4. In the cases of $S\psi(32, 1, t)$

* Remember that downward ordinates in all the curves of figs. 36, 37, 38, 39, correspond to positive values of the quantities represented.

and $\xi_\phi(32, 1, t)$ the diagram shows that they are quite imperceptible at the end of period 4, and begin to be considerable at the end of period 8, which would be the exact time of arrival if there was a definite "group-velocity" equal to half the wave-velocity. The largeness of $S_\phi(32, 1, t)$, approximately uniform throughout the first four periods, is explained in § 141. Its gradual augmentation through periods 5, 6, 7, 8, depends on the wave propagation of disturbances from the origin, as shown for $S_\psi(32, 1, t)$ and $\xi_\phi(32, 1, t)$ in the second and fourth curves.

§ 143. The $\xi_\phi(0, 1, t)$ curve of fig. 38 may be compared with the curve of the same designation in fig. 36. They differ because of a quarter period difference in the phase of commencement of the disturbing pressure, which commences suddenly at its maximum for all the curves of fig. 36, and commences at zero for all the curves of fig. 38. If the S_ψ , S_ϕ curves for initiating pressure commencing at zero were drawn, they would differ from the first and third curves of fig. 36 in being at the commencement tangential to the line of abscissas, instead of being inclined to it in the positive direction, as shown in fig. 36. The ξ curves are all initially tangential to the line of abscissas, but the tangency is only of the first order in fig. 36, while it is of the second order in fig. 38.

§ 144. The third and fourth curves * of fig. 38 show the whole history for the points, $x=0$, and $x=\lambda$, of the surface displacement expressed by the formulas

$$\xi_\psi(x, 1, t) = \frac{1}{g} \int_0^t dq \sin \omega(t-q) \frac{d}{dq} \psi(x, 1, q) \quad (191),$$

which expresses the surface displacement due to surface-pressure expressed by

$$\Pi(x, 1, t) = -\sin \omega t \psi(x, 1, 0) \quad . \quad . \quad (192).$$

The fifth curve of fig. 38 shows the history, after period 3, to almost half a period after period 9, of the disturbance at the place $x=32$. The disturbance has not yet become sinusoidal, but would certainly become almost exactly sinusoidal after a few more periods.

§ 145. In fig. 39, two sets of five curves show, for case ϕ and case ψ , the periodically varying water-surface on each side of the middle, at any long enough time after the beginning of the motion, to give a regular regime of sinusoidal

* The scale of ordinates of the third, fourth, and fifth curves of fig. 38 is double that of the first and second, indicated on the figure.

vibration as far as two or three wave-lengths on each side of the middle. The third curve in each case is a curve of sines. The first curve represents the surface at the beginning of a period from $i\tau$ to $(i+1)\tau$. The fifth curve, being the first curve inverted, represents the water-surface at the middle of the period. The other two curves may be described as components of the first and third, according to the following formula :

$$\zeta(x, 1, t) = P \sin \omega t - Q \cos \omega t \quad . \quad . \quad (193),$$

where

$$P = -A \cos 2\pi x/\lambda \quad . \quad . \quad . \quad (194),$$

and Q is a continuous transcendental function of x , having equal values for $\pm x$, expressed by (195) for positive or negative values of x , exceeding a wave-length.

For x positive, $Q = -A \sin 2\pi x/\lambda$; for x negative, $Q = +A \sin 2\pi x/\lambda$ (195),

where A denotes the semi-amplitude of the vibration, at any time long enough after the beginning, and place far enough from the middle of the disturbance, to have very approximately sinusoidal motion. The determination of the transcendental function Q , and the calculation of A , for both P and Q , will be virtually worked out in § 151 below.

§ 146. We have now an exceedingly interesting and suggestive analysis of the circumstances represented in fig. 39. Consider separately the two motions corresponding to $P \sin \omega t$ alone, and to $-Q \cos \omega t$ alone. The motion $P \sin \omega t$, if at any instant given from $x = -\infty$ to $x = +\infty$, would continue for ever, as an infinite series of standing waves, without any surface-pressure. Hence our application of surface-pressure is only required for the Q -motion: and if this motion be at any instant given from $x = -\infty$ to $x = +\infty$, it will go on for ever, provided the pressure $-\cos \omega t \frac{\phi}{\psi}(x, 1, 0)$ is applied and kept applied to the surface.

§ 147. The plan of § 146 may be generalised as follows:—Displace the water according to the formula (193) with P omitted, and with Q any arbitrary function of x for moderately great positive or negative values of x , gradually changing into the formula (195) for positive and negative values outside any arbitrarily chosen length MON (MO not necessarily equal to ON). Find mathematically the sinusoidally varying surface-pressure, $F(x) \cos \omega t$, required to cause the motion to continue according to this law. Superimpose, upon the motion thus guided by surface-pressure,

the motion $-A \cos 2\pi x/\lambda \sin \omega t$, which needs no surface-pressure. In the motion thus compounded, we have equal sinusoidal waves travelling outwards in the two directions beyond MN (semi-amplitude A): and, in the space MN, we have a varying water-surface found by superimposing on the motion $P \sin \omega t$ an arbitrary shape of surface, varying sinusoidally according to the formula $-Q \cos \omega t$.

§ 148. A curiously interesting dynamical consideration is now forced upon us. The P-component of motion needs, as we have seen, no surface-pressure. The Q-component of motion is kept correct by the surface-pressure $F(x) \cos \omega t$, which, in a period, does no total of work on the Q-motion; but work must be done to supply energy for the two trains of waves travelling outwards in the two directions. Hence this work is done by the activity of the surface-pressure upon the P-component of the motion.

§ 149. Another curious question is forced upon us. Our solution of §§ 135-145 has given us determinately and unambiguously, in every variety of the cases considered, the motion of every particle of the water throughout the space occupied. The synthetic method of quadratures which we have used could lead to no other motion at any instant due to the applied surface-pressure; but now, in § 147, we have considered a Q-motion alone, kept correct by the applied surface-pressure. Would this motion be unstable? and, if unstable, would it in a sufficiently long time subside into the motion expressed in the determinate solution of §§ 135-145? The answer is Yes and No. At any instant, say at $t=0$, let the whole motion be the Q-component alone of § 148. Let now the surface-pressure, $F(x) \cos \omega t$, be suddenly commenced and continued for ever after. It will, according to §§ 135-145, produce determinately a certain compound motion (P, Q) which will be superimposed upon the motion existing at time $t=0$; and this last-mentioned motion, given with its infinite amount of energy distributed from $x=-\infty$ to $x=+\infty$, and left with no surface-pressure, would clearly never come approximately to quiescence, through any range of distance from 0 on the two sides. Thus we see that, though the Q-motion alone of § 148 is essentially unstable, the condition of the fluid does not subside into the determinate solution of §§ 135-145. It would so subside, if it were given initially only through any finite space however great, on each side of 0. In fact, any given distribution of disturbance through any finite space however great on each side of 0, left to itself without any application of surface-pressure, becomes dissipated away to infinity on the two

sides; and leaves, as illustrated in §§ 96-113, an ever-broadening space on each side of 0, through which the motion becomes smaller and smaller as time advances.

§ 150. It remains only to look into some of the analytical details concerned in the practical working out of our solutions (189), (190). Taking $\cos \omega(t-q)$ in the formulas, and taking case ϕ , we find by (190)

$$S_{\phi}(x, z, t) = P \cos \omega t + Q \sin \omega t \quad . \quad . \quad (196);$$

$$\left. \begin{aligned} \text{where } P &= \int_0^t dq \cos \omega q \phi(x, z, q); \\ \text{and } Q &= \int_0^t dq \sin \omega q \phi(x, z, q) \end{aligned} \right\} \quad (197).$$

When P and Q have been thus found by quadratures, for all values of t , and any particular value of x , by integration by parts on the plan of § 134, we readily find, without farther quadratures, or integrations, expressions for the seven other formulas included in (189), (190).

§ 151. Let us first find P and Q for $t = \infty$. Using the exponential form for ϕ , given by (137), we find

$$\left. \begin{aligned} P &= \{RS\} \sqrt{\frac{8m}{g}} \int_0^{\infty} dq \cos \omega q e^{-mq^2}; \\ \text{and } Q &= \{RS\} \sqrt{\frac{8m}{g}} \int_0^{\infty} dq \sin \omega q e^{-mq^2} \end{aligned} \right\} \quad (198),$$

$$\text{where } m = \frac{1}{4} g/(z + \iota x).$$

Hence, according to an evaluation given by Laplace in 1810 *, we find, taking $g=4$,

$$P = \{RS\} \sqrt{\frac{\pi}{2}} e^{\frac{-\omega^2}{4m}} \quad . \quad . \quad . \quad (199).$$

The definite integral for Q is a transcendent function of ω and m , not expressible finitely in terms of trigonometrical functions or exponentials. By using the series for $\sin \omega q$ in terms of $(\omega q)^{2i+1}$, and evaluating $\int_0^{\infty} dq q^{2i+1} e^{-q^2}$ by integrations by parts, we find the following convergent series for the evaluation of B, for $t = \infty$; and $g=4$:—

* *Mémoires de l'Institut*, 1810. See Gregory's 'Examples,' p. 480.

$$Q = \{RS\} \left[\frac{1}{\sqrt{2}} \left[\frac{\omega}{\sqrt{m}} - \frac{1}{2 \cdot 1 \cdot 3} \left(\frac{\omega}{\sqrt{m}} \right)^3 + \frac{1}{2^2 \cdot 1 \cdot 3 \cdot 5} \left(\frac{\omega}{\sqrt{m}} \right)^5 - \frac{1}{2^3 \cdot 1 \cdot 3 \cdot 5 \cdot 7} \left(\frac{\omega}{\sqrt{m}} \right)^7 + \text{etc.} \right] \right. \\ \left. = \frac{1}{\sqrt{2}} \left[\omega \sqrt{\rho} \cos \frac{\chi}{2} - \frac{(\omega \sqrt{\rho})^3}{2 \cdot 1 \cdot 3} \cos \frac{3}{2} \chi + \frac{(\omega \sqrt{\rho})^5}{2^2 \cdot 1 \cdot 3 \cdot 5} \cos \frac{5}{2} \chi - \frac{(\omega \sqrt{\rho})^7}{2^3 \cdot 1 \cdot 3 \cdot 5 \cdot 7} \cos \frac{7}{2} \chi + \text{etc.} \right] \right] \quad (200),$$

where, as in §§ 100–113 above,

$$\rho = \sqrt{(z^2 + x^2)}, \text{ and } \chi = \tan^{-1}(x/z).$$

This series converges for every value of $\omega\sqrt{\rho}$ however great. But for values of $\omega\sqrt{\rho}$ greater than 4, it diverges to large alternately positive and negative terms before it begins to converge. The largest value of $\omega\sqrt{\rho}$ for which we have used it is $\omega\sqrt{\rho} = 5.03$, corresponding to $x = 8$, and requiring, for the accuracy we desire, twenty-one terms of the series. But for this value of $\omega\sqrt{\rho}$ and for all larger values, we have used the ultimately divergent series (208), found in expressing analytically, not merely for $t = \infty$ as in (198), (199), (200), but for all positive values of t great and small, the growth to its final condition when $t = \infty$, of the disturbance produced by our periodically varying application of pressure to the surface of the water initially ($t = 0$) at rest. The curve for $i\tau$ in fig. 39 has been actually calculated by (200) for values of x up to 8, and by the ultimately divergent series for values of x from 5 to 10. The agreement between those of the values which were calculated both by (200) and by the ultimately divergent series (208), was quite satisfactory: so also was the agreement between values of Q found by quadratures for $x = 1$ and $x = 8$, with values found by (200) for $x = 1$ and by (208) for $x = 8$. It is also satisfactory that the values of P found by quadratures, for $x = 1$, and $x = 8$, agreed well with their exact values given by (199), for $t = \infty$.

§ 152. Going back now to the expressions (197) for P and Q , we see that, by an obvious analytical method of treatment, we can reduce them, and therefore (§ 150) all our other formulas, to expressions in terms of a function defined as follows:—

$$E(\sigma) = \int_0^\sigma d\sigma \epsilon^{-\sigma^2} \quad . \quad . \quad . \quad (201),$$

a function well known to mathematicians* through the last

* The beautiful mathematical discovery $\int_0^\infty d\sigma \epsilon^{-\sigma^2} = \frac{1}{2} \sqrt{\pi}$, seems to have been made by Euler about 1730.

hundred and fifty or two hundred years, in the mathematical theory of Astronomical Refraction, and in the theory of Probabilities. I have taken *E* as an abbreviation of Glaisher's* notation "Erfc," signifying what he calls "Error Function Complement," which he uses in connection with his name "Error Function," defined by

$$\text{Erf}(\sigma) = \int_{\sigma}^{\infty} d\sigma e^{-\sigma^2} = \frac{1}{2}\sqrt{\pi} - \text{Erfc}(\sigma) \quad (202).$$

Using the imaginary expression for ϕ in § 137, we find

$$P = \{\text{RS}\} \sqrt{\frac{2m}{g}} \epsilon^{\frac{-\omega^2}{4m}} \int_0^t dq \left[\epsilon^{-(\sqrt{mt} - \iota \frac{\omega}{2\sqrt{m}})^2} + \epsilon^{-(\sqrt{mt} + \iota \frac{\omega}{2\sqrt{m}})^2} \right] \quad (203);$$

$$Q = \{\text{RS}\} \frac{1}{\iota} \sqrt{\frac{2m}{g}} \epsilon^{\frac{-\omega^2}{4m}} \int_0^t dq \left[\epsilon^{-(\sqrt{mt} - \iota \frac{\omega}{2\sqrt{m}})^2} - \epsilon^{-(\sqrt{mt} + \iota \frac{\omega}{2\sqrt{m}})^2} \right] \quad (204),$$

where $m = \frac{1}{4} g/(z + \iota x)$, as in § 151.

Taking advantage now of the notation (201), we reduce these two expressions to the following:—

$$P = \{\text{RS}\} \sqrt{\frac{2}{g}} \epsilon^{\frac{-\omega^2}{4m}} \left[E\left(\sqrt{mt} - \iota \frac{\omega}{2\sqrt{m}}\right) + E\left(\sqrt{mt} + \iota \frac{\omega}{2\sqrt{m}}\right) \right] \quad (205);$$

$$Q = \{\text{RS}\} \frac{1}{\iota} \sqrt{\frac{2}{g}} \epsilon^{\frac{-\omega^2}{4m}} \left[E\left(\sqrt{mt} - \iota \frac{\omega}{2\sqrt{m}}\right) - E\left(\sqrt{mt} + \iota \frac{\omega}{2\sqrt{m}}\right) + 2E\left(\iota \frac{\omega}{2\sqrt{m}}\right) \right] \quad (206).$$

§ 153. Remark first in passing that, when $\sqrt{mt}$ is infinitely great in comparison with $\omega/2\sqrt{m}$, these two expressions agree with the expressions, (198), for *P* and *Q* with $t = \infty$, which we used in connection with the explanation of fig. 39.

§ 154. And now, with a view to finding *P* and *Q* for any chosen values of x , z , t , we have the following known series†:—

$$E(\sigma) = \sigma - \frac{\sigma^3}{3} + \frac{1}{1.2} \frac{\sigma^5}{5} - \frac{1}{1.2.3} \frac{\sigma^7}{7} + \dots \quad (207),$$

$$E(\sigma) = \frac{1}{2}\sqrt{\pi} - \frac{\epsilon^{-\sigma^2}}{2\sigma} \left[1 - \frac{1}{2\sigma^2} + \frac{1.3}{(2\sigma^2)^2} - \frac{1.3.5}{(2\sigma^2)^3} + \dots \right] \quad (208).$$

* Phil. Mag., October 1871.

† See Glaisher, "On a Class of Definite Integrals," Phil. Mag., October 1871; and Burgess, "On the Definite Integral $\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt$," Trans. Roy. Soc. Edin., 1898.

The series (207) converges for all values of σ , great or small, real or imaginary: (208) converges in its first i terms, if $2\sigma^2 > 2i - 3$ (modulus understood if σ^2 is imaginary), and after that it diverges, the true value being intermediate between the sum of the convergent terms and this sum with the first term of the divergent series added. The proper rule of procedure to find the result with any desired degree of accuracy, is to first calculate by the ultimately divergent series, and see whether or not it gives the result accurately enough. If it does not, use the convergent series (207), which, by sufficient expenditure of arithmetical labour, will certainly give the result with any degree of accuracy resolved upon.

§ 155. As a guide, not only for numerical calculation, but for judging the character of the desired result without calculation, it is convenient to find the moduluses of the three complex arguments of the function E , in (205), and (206). They are as follows:—

$$\begin{aligned} \text{mod}\left(\sqrt{mt} + i\frac{\omega}{2\sqrt{m}}\right) &= \sqrt{\left(\frac{t^2g}{4\rho} + \frac{t\omega x}{\rho} + \frac{\omega^2\rho}{g}\right)} \\ &\doteq t\sqrt{\frac{g}{4x}} + \omega\sqrt{\frac{x}{g}}, \text{ when } x \doteq \infty \quad (209); \end{aligned}$$

$$\begin{aligned} \text{mod}\left(\sqrt{mt} + i\frac{\omega}{2\sqrt{m}}\right) &= \sqrt{\left(\frac{t^2g}{4\rho} - \frac{t\omega x}{\rho} + \frac{\omega^2\rho}{g}\right)} \\ &\doteq t\sqrt{\frac{g}{4x}} - \omega\sqrt{\frac{x}{g}}, \text{ when } x \doteq \infty \quad (210); \end{aligned}$$

$$\text{mod}\left(i\frac{\omega}{2\sqrt{m}}\right) = \omega\sqrt{\frac{\rho}{g}} \quad \dots \quad (211).$$

§ 156. The very interesting questions regarding the front of the procession of waves in either direction, of which we have found illustrations in figs. 36, 37, 38, and which we had under consideration in §§ 11–31, 114–117 above, are now answerable in a thoroughly satisfactory mathematical manner, by aid of the formulas (205), (206), (209), (210), (211). When, in the arguments of E , in (205), and (206), $\sqrt{mt}$ is very great in comparison with $\omega/2\sqrt{m}$, the two added terms in (205) are approximately equal, and (206) is reduced approximately to its last term; and all the solutions (189), (190), become approximately sinusoidal, in respect to t .

This is the case when $t\sqrt{\frac{g}{4x}}$ is very great in comparison with unity, and in comparison with $\omega\sqrt{\frac{x}{g}}$, as we see by

looking at the moduluses shown in (209), (210), (211). This allows us to neglect ω in the arguments of E in (205), (206), and makes P and Q constant relatively to t .

§ 157. When t is small or large, and x not so small as to give preponderance to the first terms of the moduluses (209), (210), we have in (205), (206), (189), (190) a full representation of the whole circumstances of the wave-front, extending from $x=\infty$ back to the largest value of x that

allows preponderance of $t\sqrt{\frac{g}{4x}}$ over $\omega\sqrt{\frac{x}{g}}$, in the moduluses, (209), (210). Let, for example,

$$t\sqrt{\frac{g}{4x}} = \omega\sqrt{\frac{x}{g}} \quad . \quad . \quad . \quad . \quad (212).$$

This gives

$$x = \frac{gt}{g\omega} = \text{half the wave-velocity} \quad . \quad . \quad (213).$$

The moving point thus defined is what in my first paper to the Royal Society of Edinburgh (January 1887), "On the front and Rear of a Free Procession of Waves in Deep Water," I called the "Mid-Front," defined in (45) of that paper, which agrees with our present (213). The following passage was the conclusion of that old paper:—"The rear of a wholly free procession of waves may be quite readily studied after the constitution of the front has been fully investigated, by superimposing an annulling surface-pressure upon the originating pressure represented by (12) above [this is a case of (173) of our present paper], after the originating pressure has been continued so long as to produce a procession of any desired number of waves." The instruction thus given with reference to the relation between front and rear has been virtually followed, though with some differences of detail, in §§ 20-24 of my second Royal Society paper, on the same subject, and under the same title (June 20, 1904). That second paper contained a first instalment of the "calculations and graphic representations" promised in the old first; the present paper contains, in figs. 35, 36, 37, 38, a further instalment of such illustrations.

§ 158. Throughout my work, §§ 96-157, I have had most valuable assistance from Mr. George Green, not only in the very long and laborious calculations and drawings, which have been wholly made by him, but also in many interesting and difficult questions which occurred in the fundamental mathematics of the subject.

We hope to apply before long the method of § 128 to calculate, by aid of the formula $\int_0^t dq \psi(x-vq, z, t-q)$, the initiation and continued growth of Canal Ship-waves, due to the sudden commencement and continued application of a moving, steady surface-pressure, $\psi(x, 1, 0)$. We hope also to apply (139) of the present paper to the fulfilment of my old promise (§ 30, June 20, 1904, R.S.E.) to deal with the beautifully varying procession seen circling outwards from the place of a stone thrown into deep water.

II. *On the Change of Elastic Constants of Ferromagnetic Substances by Magnetization.* By K. HONDA and T. TERADA, *Lecturers in Physics in the Tokyo Imperial University*.*

[Plates I.-V.]

§ 1. INTRODUCTION.

THE elastic constant of a substance in a magnetic field is here defined as the ratio of the stress applied to the strain caused thereby, the magnetic force constantly acting on the substance. The change of elasticity is then the difference of this quantity, when the magnetic field is on, and when it is off. In the case of ferromagnetic substances, for which the hysteresis is considerable, the change of elasticity so defined may, or may not, coincide with that of the same ratio when the stress is first applied and then the field. However, little attention seems to have hitherto been paid to this point. Thus, in many† of the experiments hitherto made regarding the change of elastic constants by magnetization, the right order of properly applying the stress and the field was inverted. In some experiments‡ the effect of tension on the magnetic elongation was observed, and the change of the modulus of elasticity was deduced on the assumption that the effect was due to the change of elasticity

* Communicated by the Authors.

† Guillemin, *C. R.* xxii. pp. 264 & 432 (1846); J. S. Stevens & H. G. Dorsey, *Phys. Rev.* (2) ix. p. 116 (1899); G. Wiedemann, *Electricität*, iii. p. 796; C. Barus, *Amer. Journ.* xxxiv. p. 175 (1887), *Phys. Rev.* xiii. p. 257 (1901); Day, 'Electrician,' xxxix. p. 480 (1897); J. S. Stevens, *Phys. Rev.* (3) x. p. 111 (1900); K. Honda & S. Shimizu, *Journ. Sc. Coll., Arts.* 12 & 13, vol. xvi. (1902).

‡ Bock, *Wied. Ann.* liv. p. 442 (1895), *Phil. Mag.* (5) xxxix. p. 548 (1895); K. Tangl, *Ann. de Phys.* vi. p. 34 (1901); K. Honda & S. Shimizu, *loc. cit.*

by magnetization. The acoustical method* used by several experimenters is also an indirect one, and unsuitable for any accurate determinations. In H. Rensing's experiment†, Kundt's tube was used; but the field in which the bar was placed was far from being uniform.

Our present experiments were undertaken with the main object of measuring the change of elastic constants of the same specimen by the direct as well as the indirect method and comparing the results thus obtained: the specimen to be tested being placed in a nearly uniform field.

If ξ be the length or the angle of twist of the specimen, S the applied tension or couple respectively, and H the magnetic field, we have the relation

$$\frac{\partial}{\partial H} \left(\frac{\partial \xi}{\partial S} \right) = \frac{\partial}{\partial S} \left(\frac{\partial \xi}{\partial H} \right), \quad \dots \quad (1)$$

provided the change is independent of the order of applying the field and the stress. Thence we have

$$\left(\frac{\partial \xi}{\partial S} \right)_H - \left(\frac{\partial \xi}{\partial S} \right)_0 = \frac{\partial (\xi_H - \xi_0)}{\partial S} \dots \quad (2)$$

If M be the modulus of elasticity or that of the rigidity respectively,

$$\frac{\partial \xi}{\partial S} = \frac{c}{M},$$

where c is a constant depending on the dimensions of the specimen. Hence we have finally

$$\begin{aligned} \frac{1}{M_H} - \frac{1}{M_0} &= \frac{1}{c} \frac{\partial (\xi_H - \xi_0)}{\partial S} \\ &= \frac{\frac{\partial (\xi_H - \xi_0)}{\partial S}}{M_0 \frac{\partial \xi_0}{\partial S}}, \end{aligned}$$

$$\text{or} \quad \frac{M_H - M_0}{M_H} = \frac{\delta M}{M_H} = - \frac{\frac{\partial (\xi_H - \xi_0)}{\partial S}}{\frac{\partial \xi_0}{\partial S}} \dots \quad (3)$$

Thus the change of elasticity may be calculated from the effect of tension on the magnetic elongation; and that of

* Wartmann, *Ann. de Chim. et de Phys.* xxiv. p. 360 (1848); Trèves, *C. R.* lxvii. p. 321 (1868); *Archives des Sc. Nat.* N. S. xxxiii. p. 74 (1868); Maurain, *C. R.* cxxi. p. 248 (1895).

† Rensing, *Ann. der Phys.* xiv. p. 363 (1904).

rigidity, from the change of twist caused by magnetization, provided relation (1) holds.

The validity of (3) depends upon relation (1), which is usually employed in the theory of magnetostriction. It was thought, therefore, that it would be interesting to test experimentally how far the relation actually holds. Accordingly the present investigation was carried out in the following order:—

I. Experiments on the change of elasticity:

- (a) Measurement of the change of elasticity by the elongation method.
- (b) Measurement of the modulus of elasticity.
- (c) Comparison of the results with those of the flexure method.

II. Experiments on the change of rigidity:

- (a) Measurements of the change of rigidity by the oscillation method.
- (b) Ditto by Barus's method.
- (c) Ditto by the ordinary method.

The specimens tested in the present experiments were Swedish iron, tungsten-steel, nickel, and nickel-steels of different percentages. They were generally tested in the form of wires. For the change of rigidity, the rod was also used in the case of nickel and iron.

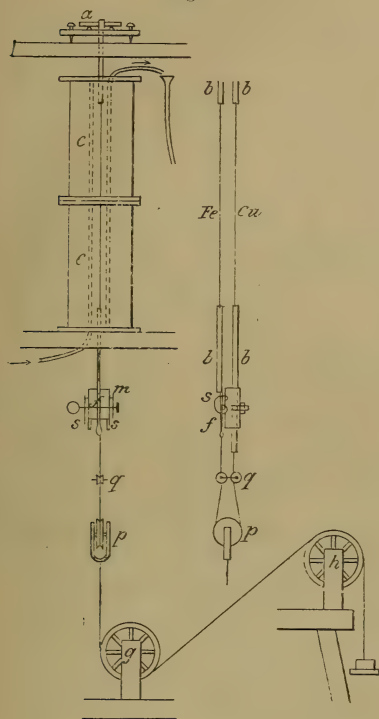
§ 2. EXPERIMENTS ON THE CHANGE OF ELASTICITY BY MAGNETIZATION.

(a) *Measurement of the Change of Elasticity by the Elongation Method.*

Since the elongation due to the change of elasticity is a small fraction of the total elongation, it was desirable to devise some differential method. For this purpose, the elastic elongation at no field was compensated by that of a non-magnetic wire, and the differential elongation due to magnetic field was observed. As shown in the annexed figure 1, the magnetic wire to be tested and the compensating copper wire were hung side by side in the vertical magnetizing coil *cc*. To both ends of the wires were brazed brass rods *bb*, whose elongation could be neglected. The upper rods were hung separately on two horizontal brass beams *aa'* resting on pairs of knife-edges, the distance between which could be adjusted at will. To the lower rod attached to the copper wire, a light carriage *m* for a mirror system was clamped. Two fine spiral springs *ss* of german-silver attached to the sides of the carriage supported horizontally the axis of the

light mirror with a suitable counterpoising. The lower rod for the magnetic wire carried a flexible band f made of a

Fig. 1.



turbid in any way, the rotation could be observed in the usual manner by means of a vertical scale and a telescope.

To increase the sensitiveness of the arrangement, a wire about 74 cm. long was used; for want of a sufficiently long coil, two coils of the same dimensions were used, placing the one above another. The length of each coil was 39.90 cm., the internal diameter 5.80 cm., and $4\pi n = 393.5$. The air gap between the two coils was only 1.7 cm., so that the heterogeneity of the field in this space was inconsiderable. The wire to be tested then lay nearly in a uniform field.

The compensating copper wire was chosen of such dimensions as to produce an elongation nearly equal to that of the specimen. The final compensation was always made by properly adjusting the distance between the two knife-edges supporting the horizontal beams, from which the wires were suspended. A certain weight was added to the pan, or

light mirror with a suitable counterpoising. The lower rod for the magnetic wire carried a flexible band f made of a bundle of very fine copper wires, which pressed the axis of the mirror lightly upon the plane vertical side of the carriage. To the lower ends of these rods, a flexible silk cord was attached; a pulley p hanging in this cord was pulled vertically downward by a weight. The friction-wheel q served to adjust the distance between the two vertical wires. Instead of hanging the weight directly, a system of pulleys gh was used in order to avoid any injurious shock accompanying the loading and unloading of the wires. When the elongations of the two wires due to tension were exactly equal, there was of course no rotation of the axis of the mirror. If, however, the compensation were dis-

removed from it, and the distances of the knife-edges were adjusted, until neither the addition nor removal produced any rotation of the mirror. Since the elastic elongation was not exactly a linear function of the tension applied, a compensated system for a given initial and additional weight caused a rotation of the mirror for a different initial weight and the same additional weight. Hence the compensation was made anew for every initial weight, 1 kilogram always being used for the applied weight. It was, however, a matter of no great ease to obtain an exact compensation. Hence the slight deflexion in no field was always recorded, and taken into account in the calculation of the deflexion due to the change of elasticity by magnetization. To prevent shocks in putting on the additional weight, the face of the initial weight was covered by a mat of cotton-wool. In this way, the reading of the deflexion was greatly facilitated.

In the above arrangement, the rotation of the mirror caused by a change of temperature is very small, as it is affected only by the relative expansion of the specimen and the compensating wire. But to avoid the effect of heating as much as possible, the coil was waterjacketed.

To increase the sensitiveness of the arrangement, a long scale-distance of 7.23 m. was used. The scale was graduated on a ground glass, and illuminated from behind by a mantled gas-flame. The telescope was placed in front of the apparatus and near the rotating mirror; consequently the line joining the centres of the scale and the mirror deviated slightly from the line of sight of the telescope. In this way, the scale at such a long distance could be read to a tenth of a millimetre with the observing telescope.

If the specimen undergoes a differential elongation δl , a rotation $\delta\phi$ of the axis is produced, such that $\delta\phi = \delta l / 2r$, where r is the radius of the axis. If the scale-distance be d and the reading on the scale corresponding to the elongation δl be n , we have

$$\delta\phi = \frac{n}{2d}; \quad \text{hence} \quad \delta l = \frac{rn}{d}.$$

In our case, $2r = 1.61$ mm., so that a deflexion amounting to one division of the scale in the field of the telescope corresponds to an elongation of 4.47×10^{-8} per unit length of the specimen. The sensitiveness of the apparatus was therefore about ten times as great as that of the apparatus used by many previous investigators.

The magnetizing current supplied by a set of secondary

batteries was measured by a Siemens-Halske ammeter, which was occasionally compared with a Kelvin ampere-balance.

The present arrangement also enabled us to measure the magnetic change of length of the specimen under constant tension by applying a magnetic field to the specimen and observing the deflexion of the mirror corresponding to different fields.

Our procedure in the experiment was usually as follows:—The wire to be tested was first annealed for about four hours in a charcoal-fire, and then gradually cooled. To avoid surface oxidation during the process of annealing, the wire was buried in a fine powder of charcoal, and sealed in an iron tube, which was placed in a furnace. In this way a small quantity of carbon might enter into the wire, but the oxidation was quite inappreciable. Variation in the diameter of the wire caused by the oxidation was avoided in the above manner.

To either end of the wire thus annealed was brazed the brass rod before mentioned, and the whole was suspended in position as shown in the foregoing figure. Then the mirror and pulley-systems were placed in position. For avoiding a continuous slight vibration of the mirror, as well as for the rapid damping of the vibration due to loading or unloading the wire, it was necessary that a glass rod fixed to a stand should come lightly in contact with the bifilar cords near the friction-wheel *q*. The addition or removal of a weight from the pan was occasionally observed to cause a lateral displacement of the image in the mirror; this was, however, easily effaced by adjusting the clamp of the carriage to the rod.

The magnetic change of length under constant tension was first measured. Since the hysteresis effect of tension on the length change was found to be considerable, the tension was varied cyclically between zero and its maximum value before beginning the experiment, in order to remove any initial effects.

According to the results of our experiments, the ascending branch in the curve of elongation due to tension lies a little above the descending one. In the following tables the length changes under constant tension are results for the stage of increasing weights, taken after several cycles of loading and unloading. The usual processes of observation were followed. The wire was first loaded with the smallest weight, which was generally 1 kilogram; the demagnetization by reversals was then carefully effected. A series of gradually increasing currents was passed through the magnetizing coil, and the corresponding deflexions of the scale were noted. The wire

was then completely demagnetized, and the zero position of the scale tested. Another weight of 1 kilogram was added and demagnetized, and the same processes were repeated. In this way a set of observations corresponding to the different weights was taken.

The change of elasticity was next measured. We applied an initial load to the pan, demagnetized the wire by reversals, added and removed a weight of 1 kilogram, and adjusted the compensation until the deflexion due to the additional weight was as small as possible. The demagnetization was then carefully effected. The small deflexion due to the additional weight for no field was recorded. Then we applied a magnetizing field and observed the deflexion due to the additional weight. After the demagnetization the deflexion due to the same load for no field was again tested. The difference between the two deflexions for no field and for the magnetic field gives the change of elasticity due to magnetization, provided we know the total elongation caused by the additional weight of 1 kilogram. To avoid any accidental error in observing these deflexions, the deflexions for several cycles of loading and unloading were, in each case, recorded, and their mean was taken. The initial weight was next increased, and the same procedure repeated. The compensation was always readjusted for each different initial weight.

Our object was partly to test the validity of equation (3), and since the change of elasticity was considerably affected by tension, the initial and the additional tensions were so chosen as to exactly correspond to those in the experiments of the magnetic change of length under constant tension.

In the calculation of the percentage change of elasticity due to magnetization, the values of elasticity were taken from the results of our experiments specially undertaken for this purpose.

The specimens tested and their dimensions were as follows:—

Substances.	Length.	Diameter.
Pure nickel	73·25 cm.	0·928 mm.
Commercial nickel	71·90	0·930
Swedish iron	71·87	0·904
Tungsten steel	73·76	0·886
28·74 per cent. nickel-steel...	73·80	0·964
50·72 " " " ...	73·60	0·880
70·32 " " " ...	73·25	0·892

The experimental results are given in the following pages.

NICKEL.

(i.) *Pure Nickel.* Pl. I. figs. 1 & 2.

The change of length $\frac{\delta l}{l}$ by magnetization for different tensions T and external magnetic fields H are given in the following table and fig. 1.

TABLE I.

 $t = 9^{\circ}8.$

T=1540 gr./mm. ²		T=2283 gr./mm. ²		T=3021 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6.$	H.	$\frac{\delta l}{l} \times 10^6.$	H.	$\frac{\delta l}{l} \times 10^6.$
2.1	+ 0.02	4.1	- 0.18	4.3	- 0.11
5.8	- 0.59	7.7	- 0.97	8.5	- 0.73
9.4	- 2.09	11.3	- 2.26	12.6	- 1.67
14.0	- 4.37	15.6	- 3.69	16.6	- 2.81
24.2	- 11.94	24.4	- 7.40	24.4	- 5.27
33.4	- 20.28	33.0	- 12.01	32.8	- 8.25
49.1	- 28.25	49.8	- 23.96	48.4	- 16.11
74.2	- 33.1	74.6	- 32.2	72.7	- 28.96
94.3	- 35.7	93.7	- 35.4	93.4	- 33.9
143.2	- 39.7	142.3	- 40.1	142.2	- 39.7
203	- 42.5	201	- 43.4	201	- 43.5
276	- 44.1	276	- 45.5	275	- 46.0
322	- 44.6	322	- 46.0	322	- 46.8
389	- 44.9	388	- 46.5	388	- 47.3

T=3762 gr./mm. ²		T=4498 gr./mm. ²		T=5240 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6.$	H.	$\frac{\delta l}{l} \times 10^6.$	H.	$\frac{\delta l}{l} \times 10^6.$
4.1	- 0.07	4.3	- 0.04	3.8	- 0.00
7.8	- 0.42	8.8	- 0.38	8.5	- 0.18
12.5	- 1.30	15.8	- 1.45	13.6	- 0.64
16.8	- 2.28	24.4	- 3.22	18.0	- 1.27
24.4	- 4.27	33.4	- 5.18	24.2	- 2.37
33.6	- 6.76	40.8	- 6.98	34.5	- 4.33
49.6	- 12.23	49.8	- 9.39	49.6	- 7.65
73.6	- 24.23	73.8	- 17.77	73.8	- 13.84
93.6	- 31.7	93.1	- 26.8	93.1	- 21.40
142.2	- 39.3	142.3	- 37.5	142.3	- 35.2
201	- 43.8	201	- 42.9	203	- 41.6
275	- 46.8	275	- 46.5	276	- 45.7
321	- 47.8	321	- 47.7	323	- 47.2
388	- 48.5	387	- 48.7	390	- 48.4

As may be seen from fig. 1, nickel contracts by magnetization in a way similar to that given by many earlier experimenters; in weak fields the contraction rapidly increases with the increasing field, but gradually tends to an asymptotic value, as the field attains a higher value. The increased tension decreases the amount of contraction for low fields, while for higher fields it increases the contraction. Thus there exists a certain field in which the effect of the additional tension is null. This neutral point for two successive tensions shifts toward the higher field, as the tension increases. In their general course, the curves are quite similar to those already obtained by Mr. S. Shimizu and one of us*.

The change of elasticity of nickel is given in the table below and in Pl. I. fig. 2.

TABLE II.

$$\Delta T = \pm 743 \text{ gr./mm.}^2 \quad t = 10^{\circ} 0 \text{ C.}$$

T=1540 gr./mm. ²		T=3021 gr./mm. ²		T=4498 gr./mm. ²	
H.	$\frac{\delta E}{E_{II}} \times 10^3.$	H.	$\frac{\delta E}{E_{II}} \times 10^3.$	H.	$\frac{\delta E}{E_{II}} \times 10^3.$
4.5	- 0.05	9.3	- 0.54	11.6	- 0.40
10.4	- 1.59	19.5	- 1.67	25.0	- 1.24
25.0	- 8.41	34.1	- 3.98	51.0	- 3.31
31.3	- 12.15	49.9	- 8.41	70.3	- 6.11
37.7	- 14.40	64.3	- 12.22	99.6	- 9.90
50.3	- 7.10	71.1	- 11.06	106.3	- 8.18
71.5	- 3.10	90.9	- 6.19	129.4	- 5.43
118.2	- 0.42	131.8	- 2.43	191.9	- 3.21
196.7	+ 0.98	210	- 1.01	...	...
287	+ 2.20	288	- 0.61	277	- 1.72
406	+ 2.19	397	+ 0.27	390	- 0.83

In the above table ΔT denotes the weight applied or removed for studying the elongation of the specimen.

The coefficient of elasticity first decreases as the field increases, attains a minimum value, whence it begins to increase, at first rather abruptly and then gradually. The effect of tension is to drive the field corresponding to the minimum elasticity towards the higher field, and also to

* K. Honda & S. Shimizu, Journ. Sc. Coll. vol. xvi. Art. 9 (1902).

depress the absolute amount of decrease. For a weak tension the elasticity increases by magnetization in fields higher than a certain field at which the change is zero. This field is displaced toward the higher field by the increasing tension.

The change of elasticity calculated from the effect of tensions upon the magnetic change of length is as follows:—

TABLE III.
 $\Delta T = 743 \text{ gr./mm.}^2$

T=1540 gr./mm. ²		T=3021 gr./mm. ²		T=4498 gr./mm. ²	
H.	$\frac{\delta E}{E_H} \times 10^2$.	H.	$\frac{\delta E}{E_H} \times 10^2$.	H.	$\frac{\delta E}{E_H} \times 10^2$.
20	- 7.4	...	...	...	...
35	-19.1	30	- 3.1	40	- 2.4
40	-17.4	60	-14.3	...	...
60	- 6.5	70	-14.3	80	-12.4
100	- 0.7	90	- 6.1	100	-13.1
200	+ 2.4	150	0.0	130	- 7.6
300	+ 3.6	250	+ 1.2	200	- 3.8
380	+ 3.5	350	+ 2.5	300	- 1.6

Curves are also given in fig. 2 in dotted lines. Comparing the above results with those directly observed, we notice at once that the general course of the change of elasticity is quite similar for both results, the fields in which the change is maximum coinciding in a remarkable way. The amount of the change is generally greater in the present case than in the previous one.

(ii.) *Commercial Nickel.* Pl. I. figs. 3 & 4.

The change of length by magnetization is generally similar to that in pure nickel. But the effect of tension is much smaller, and the field in which the effect of an additional tension is null does not differ so much for different tensions as in the former specimen. The results are given in Table IV. and in fig. 3.

TABLE IV.

 $t=9^{\circ}5\text{ C.}$

T=1325 gr./mm. ²		T=1962 gr./mm. ²		T=2600 gr./mm. ²		T=3239 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6$	H.	$\frac{\delta l}{l} \times 10^6$	H.	$\frac{\delta l}{l} \times 10^6$	H.	$\frac{\delta l}{l} \times 10^6$
2.3	-0.09	2.5	-0.09	2.6	-0.09	2.3	-0.04
6.9	-0.92	7.8	-0.78	8.8	-0.90	8.3	-0.54
13.7	-3.04	14.5	-2.50	16.2	-2.54	15.4	-1.90
22.4	-6.20	22.5	-5.32	23.1	-4.63	23.1	-4.02
38.2	-11.62	38.1	-10.72	39.0	-9.84	38.9	-9.02
49.1	-14.62	49.3	-14.02	50.4	-13.22	50.2	-12.32
71.9	-19.16	71.5	-18.76	71.5	-18.44	71.5	-17.60
120.4	-25.20	120.2	-25.44	120.4	-25.40	120.5	-25.46
191.6	-29.46	191.5	-30.3	192.0	-31.0	192.0	-31.4
261	-31.7	258	-32.7	262	-33.3	258	-34.2
328	-32.9	328	-34.0	331	-34.8	330	-35.9
390	-33.7	390	-34.8	392	-35.7	391	-36.8

The change of elasticity is as follows. Curves are also given in fig. 4 with full lines.

TABLE V.

 $\Delta T = \pm 638\text{ gr./mm.}^2$ $t=8^{\circ}9\text{ C.}$

T=1326 gr./mm. ²		T=2600 gr./mm. ²		T=3874 gr./mm. ²	
H.	$\frac{\delta E}{E_H} \times 10^2$	H.	$\frac{\delta E}{E_H} \times 10^2$	H.	$\frac{\delta E}{E_H} \times 10^2$
7.6	-0.65	6.7	-0.41	6.8	0.00
14.0	-1.24	10.9	-0.70	11.1	-0.53
22.9	-1.60	19.8	-1.34	24.6	-1.33
37.7	-1.90	32.6	-2.36	39.1	-2.26
52.1	-1.96	52.3	-2.87	51.6	-2.92
75.3	-1.48	77.8	-2.29	76.7	-3.26
123.2	-0.40	108.0	-1.72	105.7	-2.66
193.5	+0.45	194.0	-0.14	165.3	-1.33
271	+1.07	262	+0.70	260	-0.09
...	...	330	+0.99	331	+0.49
383	+1.42	397	+1.14	398	+0.78

Here the general aspect of the change is similar to that of the pure nickel. The amount of the change is, however, far less than in the previous specimen. The effect of tension is also to drive the maximum of the change towards higher fields, but not in such a decided way as in the case of pure nickel. Moreover, the increased tension increases the amount of the maximum change, up to the highest tension employed. These remarkable differences between the pure and the commercial nickel are probably due to the imperfect annealing

rather than to the impurity of the latter specimen, as it was slightly annealed with a gas-flame.

The change of elasticity calculated from the tension effect on the magnetic change of length somewhat resembles in its aspect that for pure nickel, but the amount of the change is much smaller. The calculated values are given in Table VI. and in fig. 4 with dotted lines.

TABLE VI.— $\Delta T = \pm 638$ gr./mm.²

	T=1326 gr./mm. ²	T=2600 gr./mm. ²
H.	$\frac{\delta E}{E_H} \times 10^2$.	$\frac{\delta E}{E_H} \times 10^2$.
10	-2.23	-0.96
20	-2.97	-1.75
40	-3.26	-2.42
70	-1.19	2.23
100	+0.44	0
200	+1.78	+1.75
300	+2.97	...

Thus, in pure as well as in commercial nickel, we find a sensible deviation from relation (3).

SWEDISH IRON AND TUNGSTEN-STEEL.

(i.) *Swedish Iron.* Pl. I. figs. 5 & 6.

The change of length due to magnetization is given in Table VII. and in fig. 5.

TABLE VII. $t = 9^{\circ} \cdot 2$ C.

T=1627 gr./mm. ²		T=2410 gr./mm. ²		T=3190 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.
3.8	-0.02	3.8	-0.09	3.5	-0.10
8.1	+0.94	8.3	+0.64	8.6	+0.42
12.4	+1.44	12.8	+1.00	14.1	+0.66
17.5	+1.68	16.8	+1.10	17.7	+0.72
23.3	+1.78	22.3	+1.18	23.4	+0.74
35.3	+1.80	33.8	+1.18	33.4	+0.68
54.2	+1.60	52.5	+0.98	53.8	+0.44
78.2	+1.26	74.8	+0.62	78.0	+0.04
117.2	+0.56	113.3	-0.08	113.6	-0.60
185.0	-0.82	186.3	-1.54	187.8	-2.12
250	-2.10	252	-2.80	253	-3.40
318	-3.30	320	-3.94	322	-4.56
384	-4.34	385	-4.22	388	-5.56

Table VII. (*continued*).

T=3974 gr./mm. ²		T=4754 gr./mm. ²		T=5535 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$	H.	$\frac{\delta l}{l} \times 10^6$.
3.3	-0.08	3.5	-0.10	3.2	-0.06
7.8	+0.20	8.4	+0.04	8.2	-0.09
13.4	+0.38	14.0	+0.12	12.4	-0.05
18.0	+0.42	17.4	+0.13	17.1	-0.05
23.3	+0.42	22.5	+0.11	22.3	-0.08
34.9	+0.34	33.9	+0.06	33.8	-0.21
51.2	+0.12	51.6	-0.18	52.0	-0.52
74.8	-0.29	74.8	-0.64	74.8	-0.94
112.9	-1.08	113.3	-1.44	112.9	-1.74
185.0	-2.54	186.0	-2.94	185.8	-3.24
249	-3.76	251	-4.20	250	-4.48
317	-4.90	317	-5.32	317	-5.62
382	-5.84	382	-6.30	380	-6.58

For small tensions, the change is an elongation in low fields, attains a maximum in a certain field, passes zero, and afterwards is changed to a contraction. The maximum elongation decreases with the increasing tension, driving the zero towards the weak field; and beyond a certain tension, the length of the wire steadily contracts with the increase of the field. It seems also that the field corresponding to the maximum elongation slowly moves towards the weaker fields as the tension is increased. These results are in complete agreement with those obtained by former experimenters.

The change of elasticity for this sample was unexpectedly small, its amount being not very much greater than the limit of experimental error. The values obtained are given in Table VIII. and in fig. 6 with a full line.

TABLE VIII.

$$\Delta T = 784 \text{ gr./mm.}^2 \quad t = 11^{\circ}.5 \text{ C.}$$

T=1627 gr./mm. ²	
H.	$\frac{\delta E}{E_H} \times 10^2$.
37.1	+0.02
120.4	+0.20
404	+0.28

Thus $\frac{\delta E}{E}$ is as a whole very small, and increases with the field tending to an asymptotic value.

On the other hand, the calculated values of $\frac{\delta E}{E}$ from the magnetic elongation are decidedly greater than those obtained by our direct method, as may be seen from Table IX. and fig. 6 in dotted lines. The difference is remarkable as well as interesting.

TABLE IX.
 $\Delta T = \pm 784 \text{ gr./mm.}^2$

	T=1627 gr./mm. ²	T=3190 gr./mm. ²	T=4754 gr./mm. ²
H.	$\frac{\delta E}{E_H} \times 10^2.$	$\frac{\delta E}{E_H} \times 10^2.$	$\frac{\delta E}{E_H} \times 10^2.$
10	0.9	0.6	0.4
20	1.4	0.8	0.5
40	1.6	0.9	0.8
70	1.7	1.0	0.9
100	1.7	1.1	0.8
200	1.7	1.1	0.8
300	1.4	1.1	0.8
350	1.4	1.0	0.8

(ii.) *Tungsten-Steel*. Pl. I. figs. 7 & 8.

The elongation by magnetization increases at first rapidly, then slowly, and afterwards gradually decreases. Tension decreases elongation regularly. The results are tabulated as follows:—

TABLE X. $t = 12^\circ.5 \text{ C.}$

T=1693 gr./mm. ²		T=2509 gr./mm. ²		T=3322 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6.$	H.	$\frac{\delta l}{l} \times 10^6.$	H.	$\frac{\delta l}{l} \times 10^6.$
10.0	0.03	9.0	0.03	9.5	0.04
13.8	0.09	13.1	0.08	13.5	0.09
19.4	0.60	19.5	0.64	19.5	0.57
24.9	1.39	25.2	1.46	25.2	1.32
42.2	2.72	42.8	2.62	42.7	2.39
64.5	3.26	65.5	3.16	64.9	2.88
96.2	3.56	97.4	3.48	97.0	3.15
167.2	3.72	168.8	3.60	168.0	3.24
246	3.63	246	3.48	245	3.13
330	3.48	333	3.26	331	2.92
399	3.28	402	3.11	400	2.78

Table X. (*continued*).

T = 4136 gr./mm. ² .		T = 5762 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.
8.2	0.04	8.9	0.03
13.6	0.10	13.1	0.07
19.1	0.52	19.3	0.44
24.8	1.15	25.0	1.05
42.2	2.17	42.3	1.92
64.7	2.64	64.0	2.26
97.4	2.87	94.9	2.46
168.6	2.96	166.0	2.51
246	2.82	244	2.34
332	2.61	334	2.10
401	2.47	397	1.93

As in the case of Swedish iron, the change of elasticity is very small, gradually increasing with the field. Its amount does not much exceed the limit of experimental error, as given in Table XI. :—

TABLE XI.

$$\Delta T = \pm 816 \text{ gr./mm.}^2 \quad t = 10^{\circ} 8 \text{ C.}$$

T = 1692 gr./mm. ²		T = 4947 gr./mm. ²	
H.	$\frac{\delta E}{E_H} \times 10^2$.	H.	$\frac{\delta E}{E_H} \times 10^2$.
245	0.08	210	0.50
380	0.10	400	0.39

The calculated values of $\frac{\delta E}{E}$ are decidedly greater than those above given, as seen in Table XII. and the dotted curves in fig. 8.

TABLE XII.
 $\Delta T = \pm 816 \text{ gr./mm.}^2$

	T=1692 gr./mm. ²	T=4947 gr./mm. ²
H.	$\frac{\delta E}{E_H} \times 10^2$.	$\frac{\delta E}{E_H} \times 10^2$.
50	...	0.4
100	0.2	0.5
200	0.3	0.6
300	0.5	0.6
400	0.5	0.7

Thus, in the case of iron and steel, relation (3) does not hold, even approximately, when the tension is small. But as the tension is increased the discrepancy becomes less.

NICKEL-STEELS.

(i.) 28.74 per cent. Nickel-Steel. Pl. I. figs. 9 & 10.

The specimen first elongates rapidly and then almost proportionally with the field. The effect of tension is generally to depress the change of length. The general feature of the change may be seen in the following table and in fig. 9.

TABLE XIII. $t = 10^{\circ} 3$.

T=1427 gr./mm. ²		T=2115 gr./mm. ²		T=2798 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.
0.2	-0.01	0.3	-0.04	0.2	-0.01
0.6	-0.05	0.6	-0.09	0.6	-0.06
1.2	+0.13	1.1	+0.07	1.3	+0.13
2.0	+0.27	2.1	+0.26	2.2	+0.26
4.1	+0.52	4.2	+0.51	5.4	+0.49
8.9	+0.80	12.8	+0.84	15.3	+0.71
26.1	+1.11	25.4	+1.04	26.0	+0.85
43.9	+1.29	43.3	+1.22	43.4	+1.04
68.3	+1.52	68.1	+1.47	68.5	+1.30
113.6	+1.98	113.6	+1.93	114.1	+1.74
207	+2.87	206	+2.78	208	+2.66
303	+3.77	302	+3.71	305	+3.57
404	+4.67	404	+4.66	405	+4.50

Table XIII. (*continued*).

T=3487 gr./mm. ²		T=4170 gr./mm. ²		T=4856 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.
0.4	-0.02	0.2	0.00	0.2	0.00
0.8	0.00	0.6	-0.04	0.5	-0.02
1.5	+0.11	1.3	+0.05	1.3	+0.05
3.5	+0.26	4.2	+0.20	3.0	+0.13
8.5	+0.42	9.3	+0.32	8.0	+0.23
14.7	+0.50	15.2	+0.37	13.9	+0.29
25.6	+0.60	26.2	+0.48	25.8	+0.39
43.0	+0.77	44.2	+0.65	44.1	+0.56
68.1	+1.01	68.3	+0.88	68.7	+0.80
110.5	+1.40	113.9	+1.31	114.1	+1.24
204	+2.28	207	+2.17	207	+2.10
299	+3.15	304	+3.03	305	+3.01
397	+4.03	403	+3.93	405	+3.92

The observed values of $\frac{\delta E}{E}$ are generally small, but are rather greater than those for iron. The elasticity increases by magnetization in a manner similar to that of iron. An inconspicuous maximum is, however, observed in high fields. Increased tension decreases the amount of the change; this makes the maximum conspicuous. The following table and the full lines in fig. 10 show the results of our experiment.

TABLE XIV.

$$\Delta T = \pm 688 \text{ gr./mm.}^2 \quad t = 10^{-2}.$$

T=1427 gr./mm. ²		T=2798 gr./mm. ²		T=4170 gr./mm. ²	
H.	$\frac{\delta E}{E_H} \times 10^2$.	H.	$\frac{\delta E}{E_H} \times 10^2$.	H.	$\frac{\delta E}{E_H} \times 10^2$.
4.1	0.07	5.8	0.07	6.2	0.03
14.4	0.34	14.7	0.25	26.4	0.12
48.3	0.41	44.5	0.31	49.0	0.14
113.7	0.53	113.7	0.41	112.8	0.15
237	0.51	239	0.44	238	0.04
391	0.48	397	0.46	418	0.00

The values of $\frac{\delta E}{E}$ as calculated from the tension effect on the change of length differ from the above values rather considerably for slight tension; but they approach each other for high tension, as may be seen in Table XV. and the dotted curves in fig. 10. Here the change of elongation due to tension being very small, the calculated values of $\frac{\delta E}{E}$ are not very certain.

TABLE XV.

$$\Delta T = \pm 688 \text{ gr./mm.}^2$$

T = 1427 gr./mm. ² T = 2793 gr./mm. ² T = 4170 gr./mm. ²			
H.	$\frac{\delta E}{E_{II}} \times 10^2$.	$\frac{\delta E}{E_{II}} \times 10^2$.	$\frac{\delta E}{E_{II}} \times 10^2$.
20	0.17	0.56	...
50	...	0.68	0.17
100	0.17	0.73	0.17
200	0.19	0.82	0.15
300	0.22	0.90	0.08
400	0.24	0.97	0.06

(ii.) 50.72 per cent. *Nickel-Steel*. . Pl. II. figs. 11, 12, 13, & 14.

The change of length due to magnetization differs widely for different tensions. For slighter tensions, the wire generally elongates by magnetization, except at very weak fields, where it slightly contracts, a fact not observed in previous experiments*. The amount of the elongation increases, rapidly at first, and tends soon to an asymptotic value, as the field is increased. The increased tension depresses the elongation in a remarkable degree, so that beyond a certain tension, the change of length by magnetization changes its sign and becomes a contraction. The effect of tension, however, decreases uniformly with the increase of the tensions. The results are given in Table XVI. and graphically in figs. 11 and 12.

* H. Nagaoka and K. Honda, Jour. Sc. Coll. xix. Art. 11 (1903)
K. Honda and S. Shimizu, *ibid.* xx. Art. 6 (1905).

TABLE XVI.

 $t=10^{\circ}0\text{ C.}$

T=890 gr./mm. ²		T=1713 gr./mm. ²		T=2538 gr./mm. ²		T=3360 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6$	H.	$\frac{\delta l}{l} \times 10^6$	H.	$\frac{\delta l}{l} \times 10^6$	H.	$\frac{\delta l}{l} \times 10^6$
0.2	-0.09	0.3	-0.13	0.2	-0.03	0.2	-0.01
0.6	-0.44	0.6	-0.30	0.7	-0.11	0.7	-0.01
1.0	-0.52	1.1	-0.13	1.2	+0.13	1.4	+0.12
1.4	-0.06	1.8	+0.50	2.4	+0.83	3.2	+0.43
1.8	+0.66	3.4	+1.64	4.8	+1.57	...	...
3.8	+2.48	7.4	+3.67	8.6	+2.38	7.8	0.94
6.6	+4.37	14.3	+5.60	14.9	+3.19	15.8	1.39
15.6	+8.25	22.3	+6.90	22.7	+3.76	22.6	1.62
22.6	+9.85	35.4	+7.96	35.8	+4.23	35.3	1.85
35.6	+11.62	52.0	+8.51	52.6	+4.54	52.3	2.01
53.0	+12.49	70.4	+8.84	70.8	+4.76	70.4	2.12
70.8	+12.91	108.2	+9.43	108.2	+4.96	108.2	2.25
108.2	+13.41	177.4	+9.70	178.5	+5.15	178.0	2.36
241	+13.98	240	+9.87	242	+5.26	241	2.42
306	+14.10	302	+9.97	306	+5.33	304	2.46
390	+14.20	386	+10.08	391	+5.40	387	2.53
T=4184 gr./mm. ²		T=5003 gr./mm. ²		T=5828 gr./mm. ²		T=6652 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6$	H.	$\frac{\delta l}{l} \times 10^6$	H.	$\frac{\delta l}{l} \times 10^6$	H.	$\frac{\delta l}{l} \times 10^6$
0.2	-0.02	0.2	-0.01	0.2	-0.01	0.2	-0.01
0.3	-0.07	0.7	-0.09	0.6	-0.04	...	...
1.2	-0.02	1.4	-0.11	1.5	-0.10	1.0	-0.07
3.4	+0.11	4.5	-0.18	4.5	-0.20	4.6	-0.29
6.9	+0.26	...	...	8.0	-0.29	...	...
11.9	+0.44	10.8	-0.18	15.8	-0.40	12.7	-0.52
17.8	+0.55	22.6	-0.16	35.7	-0.50	22.6	-0.64
22.8	+0.61	35.9	-0.16	52.7	-0.55	53.1	-0.83
35.8	+0.73	70.8	-0.18	108.7	-0.67	83.5	-0.95
71.2	+0.83	178.0	-0.24	178.0	-0.73	134.6	-1.05
180.0	+0.93	262	-0.25	241	-0.78	224	-1.13
307	+1.00	328	-0.24	306	-0.78	304	-1.14
393	+1.07	390	-0.22	391	-0.77	388	-1.14

The observed values of the change of elasticity are as follows. Curves are also given in fig. 14 in full lines.

TABLE XVII.

$$\Delta T = \pm 823 \text{ gr./mm.}^2 \quad t = 11^{\circ} \cdot 3 \text{ C.}$$

T=890 gr./mm. ² T=1713 gr./mm. ² T=3360 gr./mm. ² T=5003 gr./mm. ²							
H.	$\frac{\delta E}{E_H} \times 10^2.$	H.	$\frac{\delta E}{E_H} \times 10^2.$	H.	$\frac{\delta E}{E_H} \times 10^2.$	H.	$\frac{\delta E}{E_H} \times 10^2.$
0.8	0.09	0.7	0.05	1.6	0.20	1.8	0.00
3.9	0.96	4.1	1.08	6.2	0.64	6.5	0.23
13.2	3.67	13.4	2.82	12.8	0.99	13.1	0.39
26.0	5.55	24.6	3.76	24.5	1.18	24.0	0.56
41.3	6.89	39.2	4.72	40.7	1.42	40.2	0.68
60.5	7.56	67.5	5.19	62.4	1.51	62.4	0.86
93.8	8.05	103.3	5.40	103.3	1.66	104.3	0.99
153.5	8.26	187.6	5.57	159.5	1.76	161.0	1.03
271	8.39	276	5.68	276	1.81	277	1.07
338	8.65	386	5.68	386	1.89	387	1.13

The elasticity increases by magnetization, the increase tending soon to an asymptotic value as the field increases. The effect of tension is to decrease it decidedly. At $T=5003 \text{ gr./mm.}^2$, the change is reduced to about one-sixth of that at $T=890$.

The calculated values of $\frac{\delta E}{E}$ are generally greater than the above values, and the difference becomes smaller, as the tension is increased. With a tension of about 5 kg., this difference almost vanishes. The results are given in Table XVIII. and in fig. 14 with dotted lines.

TABLE XVIII.

$$\Delta T = \pm 823 \text{ gr./mm.}^2$$

T=890 gr./mm. ² T=1713 gr./mm. ² T=3360 gr./mm. ² T=5003 gr./mm. ²				
H.	$\frac{\delta E}{E_H} \times 10^2.$	$\frac{\delta E}{E_H} \times 10^2.$	$\frac{\delta E}{E_H} \times 10^2.$	$\frac{\delta E}{E_H} \times 10^2.$
10	3.26	3.48	1.31	0.37
30	5.65	6.18	2.10	0.68
100	6.75	7.61	2.52	0.91
200	6.75	8.07	2.67	1.07
300	...	...	...	1.07
400	6.86	8.19	2.80	0.97

(iii.) 70.32 per cent. *Nickel-Steel*. Pl. II. figs. 15, 16, 17, & 18.

The change of length by magnetization resembles that for the former alloy, though the amount of the change is much smaller. Tension always decreases the elongation, the effect being smaller at greater tensions. Though the tension was increased up to 5686 gr./mm.², we could not observe any indication that greater tensions would change the elongation into contraction. The elongation seems to tend gradually to zero with the increase of the load. These results are seen in Table XIX. and figs. 15, 16, and 17.

TABLE XIX.

 $t = 9^{\circ}.2$.

T=806 gr./mm. ²		T=1671 gr./mm. ²		T=2476 gr./mm. ²		T=3277 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.
0.3	-0.18	0.3	-0.18	0.2	-0.06	0.3	-0.09
0.6	-0.48	0.6	-0.45	0.6	-0.28	0.6	-0.21
1.0	-0.80	0.8	-0.60	1.0	-0.20	1.0	-0.11
1.4	+0.06	1.2	-0.09	1.4	+0.15	1.6	+0.11
1.8	+1.23	1.8	+0.61	...	...	3.0	+0.47
2.9	+2.59	3.0	+1.41	2.7	+0.70	4.9	+0.83
6.8	+5.08	4.9	+2.28	6.1	+1.60	10.0	+1.37
14.3	+6.70	17.0	+4.10	14.7	+2.46	17.2	+1.74
33.0	+7.68	34.1	+4.73	33.2	+3.01	33.2	+2.04
48.9	+7.90	44.3	+4.86	49.8	+3.16	49.8	+2.19
71.2	+8.03	66.6	+4.99	71.2	+3.25	71.2	+2.26
109.2	+8.12	107.8	+5.06	109.2	+3.33	108.7	+2.32
193.0	+8.21	185.8	+5.14	193.0	+3.42	191.8	+2.40
287	+8.25	283	+5.20	287	+3.45	286	+2.44
391	+8.35	364	+5.27	392	+3.49	389	+2.48

T=4081 gr./mm. ²		T=4882 gr./mm. ²		T=5686 gr./mm. ²	
H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.	H.	$\frac{\delta l}{l} \times 10^6$.
0.2	-0.01	0.2	-0.03	0.2	-0.01
0.6	-0.17	0.6	-0.13	0.7	-0.09
1.1	-0.05	0.9	-0.08	1.0	-0.04
1.9	+0.09	1.3	-0.04	1.9	+0.05
2.5	+0.23	2.7	+0.18	3.6	+0.20
4.3	+0.48	7.2	+0.53	7.3	+0.40
7.3	+0.82	16.0	+0.84	16.2	+0.64
15.9	+1.21	33.4	+1.08	33.2	+0.90
33.5	+1.49	49.5	+1.19	49.8	+0.98
71.2	+1.69	71.2	+1.26	71.5	+1.05
109.2	+1.76	108.7	+1.32	109.2	+1.11
192.5	+1.85	192.8	+1.38	192.5	+1.16
288	+1.88	288	+1.42	287	+1.10
392	+1.93	394	+1.44	391	+1.24

The observed change of elasticity is generally similar to that for 50·72 per cent. Ni, but is less in amount, as shown in Table XX. and in full lines in fig. 18.

TABLE XX.
 $\Delta T = \pm 805 \text{ gr./mm.}^2$ $t = 9^{\circ} \cdot 2 \text{ C.}$

T=806 gr./mm. ²		T=1671 gr./mm. ²		T=3277 gr./mm. ²		T=4882 gr./mm. ²	
H.	$\frac{\delta E}{E_H} \times 10^2.$	H.	$\frac{\delta E}{E_H} \times 10^2.$	H.	$\frac{\delta E}{E_H} \times 10^2.$	H.	$\frac{\delta E}{E_H} \times 10^2.$
0·7	-0·14	0·7	-0·08	0·7	+0·18	0·6	-0·01
...	...	3·0	-1·17	2·6	0·40	0·8	+0·05
6·6	+2·45	5·3	1·92	4·7	0·53	3·8	0·30
9·7	3·57	9·6	25·2	12·0	1·07	12·0	0·45
19·9	4·13	41·1	3·33	26·1	1·35	25·1	0·60
41·8	4·51	68·9	3·38	53·4	1·42	52·7	0·68
70·8	4·71	136·3	3·45	98·8	1·55	98·8	0·70
177·3	4·92	326	3·72	193·0	1·50	192·6	0·70
260	4·92	309	3·69	285	1·57	284	0·78
386	4·92	388	3·74	393	1·53	393	0·70

The calculated values of $\frac{\delta E}{E}$ are generally greater than the above values, the difference being less for greater tensions, as seen in Table XXI. and fig. 18 with dotted lines. The difference almost vanishes for a tension of 3 kg.

TABLE XXI.
 $\Delta T = \pm 805 \text{ gr./mm.}^2$

T = 806 gr./mm. ²		T = 1671 gr./mm. ²		T = 3277 gr./mm. ²		T = 4882 gr./mm. ²	
H.	$\frac{\delta E}{E_H} \times 10^2.$	$\frac{\delta E}{E_H} \times 10^2.$	$\frac{\delta E}{E_H} \times 10^2.$	$\frac{\delta E}{E_H} \times 10^2.$	$\frac{\delta E}{E_H} \times 10^2.$	$\frac{\delta E}{E_H} \times 10^2.$	$\frac{\delta E}{E_H} \times 10^2.$
5	4·09	2·40	0·75	0·32			
10	6·31	2·12	1·00	0·45			
100	7·13	4·25	1·45	0·55			
200	7·13	4·28	1·37	0·55			
300	7·15	4·29	1·45	0·53			

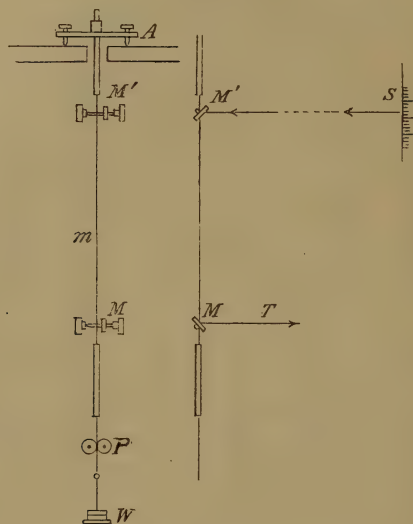
From the above results for nickel-steels, we may conclude that, for small tensions, the change of elasticity is considerably less than that calculated from the tension effect on the magnetic change of length. But the difference between the two becomes less and less as the tension is increased, almost vanishing for great tensions.

(b) *Measurement of the Coefficient of Elasticity.*

Since it was necessary, in the calculation of $\frac{\delta E}{E}$, to know the coefficient of elasticity in no field, a special arrangement was devised for this purpose.

Referring to fig. 2, m is the specimen, at the ends of which brass rods of moderate diameter are brazed. The upper rod

Fig. 2.



is clamped to the tripod A resting on a rigid wooden frame. To the lower rod is attached a flexible cord, which, passing through the interspace between the grooves of two friction-wheels P fixed to a stand, carries the scale-pan for the weight W. Near the upper and lower ends the specimen touches the horizontal axes, carrying two mirrors M and M' respectively. The axes are so pivoted at their ends that they rotate with almost no friction. S is a vertical glass scale illuminated from behind; its image formed by successive reflexions in M and M' is observed by a telescope T. If the load on the pan causes the elongations $\delta l'$ and δl respectively at M' and M, the rotations of the axes are given by

$$\phi' = \frac{\delta l'}{r} \quad \text{and} \quad \phi = \frac{\delta l}{r},$$

where r is the common thickness of the two axes. Again, if n' and n be the deflexions of the scale due to separate rotations ϕ' and ϕ respectively, we have

$$\phi' = \frac{n'}{2d} \quad \text{and} \quad \phi = \frac{n}{2(d+a)},$$

where $d = M'S$ and $a = MM'$. The actual deflexion observed is $s = n - n'$.

$$\begin{aligned} \text{Now} \quad \delta l - \delta l' &= r(\phi - \phi') \\ &= \frac{rs}{2d} \left\{ 1 - \frac{a}{d} \left[1 + \left(\frac{n'}{s} - \frac{a}{d} \right) \left(1 - \frac{a}{d} \right) \right] \right\}. \end{aligned}$$

In our experiments

$$d = 624.5 \text{ cm.}, \quad a = 65.83 \text{ cm.}, \quad r = 0.795 \text{ mm.},$$

and $\frac{a}{d} = 0.1054.$

Since n' was very small compared with s , and $\frac{n'}{s}$ appears as a small correction, it was sufficient for our purposes to determine n' roughly only once for a specimen throughout the sets of observations. This was easily done by directing the telescope to M' and reading off the deflexion. Usually n' did not exceed 7 divisions of the scale, while s was greater than 90 divisions. —

To prevent shocks by loading, the face of the initial weight was covered with a mat of cotton wool. The wire to be tested was surrounded by a tube of paper to prevent disturbance due to air-currents.

Denoting the weight, by which the elongation $\delta l - \delta l'$ is produced, by P , we have for the coefficient of elasticity

$$E = \frac{\frac{P}{A}}{\frac{\delta l - \delta l'}{a}},$$

where A is the section of the wire. P in our experiments was 100 grams, which gave a deflexion of more than 90 divisions of the scale. To test the working of the present apparatus, weights of 10, 50 and 100 grams were used with the following results :—

TABLE XXII.

$t = 12^\circ \text{ C.}$

Weight.	Tungsten-Steel.	Nickel.	28.74 per cent. Ni.
gr.	div.	div.	div.
10	9.7	9.1	9.2
50	48.1	47.3	46.3
100	96.2	96.0	92.9

Thus the proportionality is nearly satisfied up to 100 grams; this also shows that the friction at different parts of the arrangement is negligibly small.

The coefficient of elasticity differed slightly for different initial tensions, and moreover underwent more or less hysteresis due to the previous load. In our experiments, the weights were varied from 1 to 7 kilograms, and then again down to 1 kilogram. The results are tabulated in the following table:—

TABLE XXIII.

 $t = 12^{\circ} \text{C.}$

Pure Nickel.		Commercial Nickel.		Swedish Iron.		Tungsten-Steel.	
gr./mm. ²	$E \times 10^{-12}$	gr./mm. ²	$E \times 10^{-12}$	gr./mm. ²	$E \times 10^{-12}$	gr./mm. ²	$E \times 10^{-12}$
1599	1.708	1376	1.818	1689	1.802	1758	1.928
3080	1.760	2651	1.972	3254	1.883	3386	1.938
4557	1.835	3923	2.049	4815	1.916	5011	1.938
6040	1.854	5199	2.125	6380	1.947	6640	1.955
7523	1.867	6477	2.152	7947	1.955	8272	1.949
9003	1.887	7750	2.193	9511	1.962	9900	1.951
10480	1.902	9023	2.212	11700	1.968	11523	1.945
7523	1.888	...	...	7947	1.935	8272	1.937
4557	1.851	5199	2.146	4815	1.919	5011	1.923
1599	1.739	1376	1.851	1689	1.788	1758	1.914

28.74 per cent. Ni.		50.72 per cent. Ni.		70.32 per cent. Ni.	
gr./mm. ²	$E \times 10^{-12}$	gr./mm. ²	$E \times 10^{-12}$	gr./mm. ²	$E \times 10^{-12}$
1480	1.570	1780	1.376	1500	1.866
2850	1.619	...	...	...	...
4220	1.643	5070	1.551	4260	1.982
5600	1.667	...	...	...	...
6970	1.677	8370	1.621	7030	1.996
8340	1.693	...	...	...	...
9710	1.702	11650	1.619	9800	1.992
6970	1.675	8370	1.567	7030	1.936
4220	1.666	5070	1.548	4260	1.961
1480	1.591	1780	1.377	1500	1.864

The mean curves corresponding to these numbers are given in Pl. II. figs. 19, 20, 21, 22, 23, 24 and 25. They are drawn on a very exaggerated scale. They show distinct hysteresis; in three of them, the ascending branch lies below the descending

one, while in the others the case is reversed. In commercial nickel, the increase of elasticity is the greatest; in tungsten steel it is the least. In 50·72 per cent. Ni and 70·32 per cent. Ni, the elasticity seems to attain a vague maximum at a high tension.

In the calculation of $\frac{\delta E}{E}$ for different tensions, the values of E for corresponding tensions were employed. This was absolutely necessary, since for different tensions, E changed more than 10 per cent. in many of the specimens.

(c) *Comparison of the Results with those obtained by the Flexure Method.*

In a previous paper by Messrs. S. Shimizu, S. Kusakabe, and one of us, the change of elasticity of magnetic bars as determined by the flexure method is given. It is highly interesting to compare the results with those obtained by our elongation method.

In the flexure method, we observed a slight change of flexure by magnetization when there was no suspended weight; this perhaps arises from the initial bending of the specimen due to its own weight. In subsequent experiments of the kind, it was thought preferable to subtract this initial change of flexure from that of flexure due to a suspended load. We have therefore recalculated the former results, as shown in Table XXIV. and in Pl. III. figs. 26, 27, and 28.

TABLE XXIV.

<i>Soft Iron.</i>				<i>Nickel.</i>			
T=500 gr.		T=2540 gr.		T=183 gr.		T=727 gr.	
H.	$\frac{\delta E}{E_H} \times 10^2.$	H.	$\frac{\delta E}{E_H} \times 10^2.$	H.	$\frac{\delta E}{E_H} \times 10^2.$	H.	$\frac{\delta E}{E_H} \times 10^2.$
18·8	-0·03	20·1	+0·26	10·5	-0·22	10·0	-0·19
26·1	+0·56	26·9	+0·55	16·9	-0·79	16·7	-0·72
29·0	+0·77	33·9	+0·76	24·8	-0·96	21·9	-0·95
36·2	+0·89	52·9	+0·91	34·5	-1·52	26·4	-0·98
59·0	+1·01	69·4	+0·97	48·1	-0·53	39·6	-0·75
133·4	+1·04	129·0	+0·99	68·2	+0·11	68·2	+0·08
231	+1·04	212	+1·00	...	...	117·3	+1·35
291	+1·04	290	+1·02	240	+2·64	218	+2·57
383	+1·04	398	+1·07	376	+3·01	377	+3·16
				496	+3·29	492	+3·25

Table XXIV. (*continued*).

<i>Tungsten-Steel.</i>					
T=439 gr.		T=1020 gr.		T=1810 gr.	
H.	$\frac{\delta E}{E_H} \times 10^2.$	H.	$\frac{\delta E}{E_H} \times 10^2.$	H.	$\frac{\delta E}{E_H} \times 10^2.$
14.0	-0.01	15.1	-0.01	18.2	-0.04
19.4	-0.04	18.2	-0.03	20.5	-0.08
23.1	-0.17	22.4	-0.07	23.0	-0.07
25.4	-0.08	25.6	-0.03	38.4	+1.24
36.6	+1.61	30.2	+0.80	56.5	+1.48
68.1	+1.89	41.1	+1.34	75.7	+1.58
91.9	+2.02	68.5	+1.57	144.3	+1.61
270	+2.12	224	+1.88	230	+1.66
387	+1.99	348	+1.81	361	+1.69
486	+2.10	472	+1.78	460	+1.69

In the former results, the value of $\frac{\delta E}{E}$ depended considerably on the suspended weight; but in our recalculated results it was only slightly affected by the weight, except in the case of nickel in weak fields, where its initial decrease was considerably reduced.

For Swedish iron and tungsten-steel, the results somewhat resemble those of the present experiments deduced from the magnetic elongation. But they are several times greater as compared with the results of the direct experiment. In nickel, the course of the curve is considerably different from that of the present experiments; its form is, however, very interesting, if we consider it in connexion with that of the curve $\frac{\delta K}{K}$ to H. It is here to be noticed that as regards magnetic quality, the specimen in the present experiments was very different from that used in the former experiments, so that the quantitative comparison of the two results is of little value.

§ 3. EXPERIMENTS ON THE CHANGE OF RIGIDITY BY MAGNETIZATION.

(a) *Measurement of the Change of Rigidity by the Oscillation Method.*

One of our methods consisted in giving torsional oscillation to the wires magnetized with different fields and calculating

the coefficient of rigidity for different fields from the periods of oscillation. The wires tested were the same as those used in the above experiment for the change of elasticity. The wire was hung vertically in the field of the magnetizing coils used in the previous experiment. Its upper end was brazed to a rigid brass rod, which was clamped to the frame above. To the lower end a similar rod was brazed, to which the oscillating weights were fixed. In order to sufficiently diminish the disturbance due to the resistance of air, Foucault currents &c., the period of oscillation was made very long by using a weight with considerable moment of inertia. A rectangular brass bar ($42.80 \times 0.895 \times 1.946$ cm.³) was fixed horizontally to the lower end of the lower rod, at its middle point. Two equal cylindrical weights (each 657 gr.) made of lead could be fixed symmetrically on the bar at any desired distance from the middle. Besides, two equal cylindrical weights (1671 and 1640 gr.) of lead could be put on the bar with their axes coinciding with that of the rod. Suitable combinations of these different weights enabled us to adjust the periods of oscillations for different tensions to convenient values. A light mirror was fixed to the lower rod. The image of a horizontal scale placed in front of the arrangement was observed with a telescope in the usual manner.

To start the torsional oscillation of the wire, a lead tube was directed to one end of the horizontal bar. The tube going to the observer was held by his hand; the oscillation was started by blowing through the tube, while observing through the telescope. The amplitude of the oscillation could be increased or diminished at will, by blowing with suitable force on the bar in a suitable phase of its oscillation. In this way, the equality of the amplitude could easily be effected within 1 mm. of the scale. It was usually 5 cm. with a scale distance of 1.46 m., which corresponds to an amplitude of about 1° .

It was necessary to protect the oscillating system from disturbance due to air-currents.

Our procedure was as follows:—The wire to be tested was first demagnetized by reversals, the zero of the scale set to the position of equilibrium; a field was applied, and then oscillation started. Care was taken to adjust the amplitude for different fields so as to make them as equal as possible, in order that in the comparison of periods for different fields, the effects of amplitudes might be neglected. One of the observers signalled at each complete oscillation, while the

other recorded the time by a mean-time chronometer. According to the usual method of time-observation, 60 consecutive oscillations were employed for the determination of its period. The first and the last 10 were signalled and recorded; we thus obtained 10 sets of time records for 50 complete oscillations. The mean of these observations gave the period of oscillation, which usually ranged from 10 to 20 seconds. Its value may be considered accurate to $\frac{1}{1000}$ of a second. These processes were repeated for a series of successively increasing fields, the demagnetization being of course effected before each experiment. After a series of observations was taken, the experiment at no field was again repeated, and we usually found the result fairly unchanged.

Since the magnetizing coil was waterjacketed, the heating-effect was inappreciable. The magnetizing current was measured both before and after each experiment, and the mean was taken. The current remained fairly constant during each experiment, except in a few cases.

Though the period of oscillation was long and the oscillating weights were moderately distant from the lower end of the coil, it was necessary to determine experimentally the retarding effect of the Foucault currents, which was liable to be produced in the system vibrating in the magnetic field. For this purpose, a copper wire of the same size as the specimen was oscillated in a number of fields and the corresponding periods were determined. We found that for the periods used in our experiments, the effect was negligibly small. Logarithmic decrements for different fields were also determined; but generally no appreciable change was observed. In some of the magnetic wires, the damping of the oscillation was found to become slightly faster as the field was increased. But the change of the logarithmic decrement was too small to sensibly affect the period of oscillation, when referred to the equation

$$T = T_0 \left(1 + \frac{\kappa^2 T_0^2}{8\pi^2} \right),$$

where T is the actual period of oscillation, T_0 that for no damping, and κ the logarithmic decrement. This agrees with the results* by Ignaz Klemenčič and H. Tomlinson for iron.

* Ignaz Klemenčič, *Wien. Ber.* lxxviii. 7 Nov. 1878, p. 8; H. Tomlinson, *Proc. Roy. Soc.* xl. p. 447; *Phil. Trans.* clxxix. p. 1 (1888).

The values of rigidity at no field were calculated from the period of oscillation for the bar only, whose moment of inertia was known from its dimensions.

In the oscillation method the right order of applying the twist and the field was followed. Since the oscillation was very slow, the value of the rigidity thus determined may properly be considered to be the statical value. The results of experiments are as follows :—

Nickel. Pl. III. fig. 29.

TABLE XXV.

$$K_0 = 0.96 \times 10^{12}, \quad t = 14^{\circ}.7 \text{ C.}$$

T=1158 gr./mm. ²		T=3410 gr./mm. ²		T=6215 gr./mm. ²		T=9075 gr./mm. ²	
H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$
3.9	-0.21	3.9	-0.10	4.1	-0.00	...	...
10.5	-1.61	10.2	-0.25	12.0	-0.10	11.8	-0.01
23.7	-4.21	24.1	-1.51	24.1	-0.34	24.1	-0.08
46.9	-6.80	48.2	-5.82	47.9	-1.87	47.7	-0.67
78.2	-6.80	77.8	-8.65	78.0	-5.11	77.6	-2.04
117.0	-5.92	116.2	-8.55	116.5	-9.14	123.7	-5.06
190	-3.33	189	-6.58	190	-9.22	196	-8.57
276	-0.43	297	-2.86	276	-6.93	287	-8.59
406	+2.12	390	-0.62	401	-3.21	382	-6.35

Among ferromagnetic metals tested, nickel undergoes the greatest change of rigidity and shows the most remarkable feature as regards the effect of tension. For small tensions, the rigidity at first decreases rapidly and after passing through a minimum increases slowly with the field. In a certain field the change is zero, beyond which there is an increase. As the tension is increased, the change becomes less for low fields, but greater for higher fields; the maximum change occurs in a higher field, and its amount increases up to a certain tension, and then slowly decreases. The point of no change shifts toward higher fields, as the tension increases.

Swedish Iron and Tungsten-Steel. Pl. III. figs. 30 & 31.

TABLE XXVI.

*Swedish Iron.**Tungsten-Steel.* $K_0 = 0.756 \times 10^{12}$. $t = 14^{\circ}.6$ C. $K_0 = 0.608 \times 10^{12}$. $t = 14^{\circ}.0$ C.

T=3113 gr./mm. ²		T=3240 gr./mm. ²	
H.	$\frac{\delta K}{K_H} \times 10^2$.	H.	$\frac{\delta K}{K_H} \times 10^2$.
24.5	-0.03	24.2	-0.03
74.4	+0.15	72.1	+0.00
146.2	+0.40	162.7	+0.02
280	+0.28	...	...
385	+0.25	403	+0.12

The rigidity of Swedish iron slightly decreases in weak fields, but generally increases by magnetization. The amount of increase becomes greater and then slowly decreases with the field. The change of rigidity of tungsten-steel is very small; it slightly decreases at first, and then steadily increases with the field. Both of these specimens are characterized by the smallness of the change of rigidity as well as that of elasticity. By a similar method, H. Tomlinson* found that for a moderate field the change of rigidity of an annealed iron wire is insensibly small.

Nickel-Steels. Pl. III. figs. 32, 33, & 34.

TABLE XXVII.

28.74 per cent. Ni. $K_0 = 0.613 \times 10^{12}$. $t = 15^{\circ}.5$ C.

T=928 gr./mm. ²		T=1806 gr./mm. ²		T=4096 gr./mm. ²	
H.	$\frac{\delta K}{K_H} \times 10^2$.	H.	$\frac{\delta K}{K_H} \times 10^2$.	H.	$\frac{\delta K}{K_H} \times 10^2$.
...	...	3.8	0.03	6.2	0.06
12.2	0.29	12.0	0.25	11.8	0.13
48.0	0.43	47.9	0.34	48.1	0.29
117.7	0.53	116.9	0.42	116.3	0.34
206	0.54	204	0.44	204	0.35
303	0.53	301	0.44	300	0.33
409	0.48	406	0.46	403	0.38

* H. Tomlinson, *loc. cit.*

Table XXVII. (*continued*).50.72 per cent. Ni. $K_0 = 0.365 \times 10^{12}$. $t = 14^{\circ}3$ C.

T=1114 gr./mm. ² T=3277 gr./mm. ²				T=5974 gr./mm. ² T=8725 gr./mm. ²			
H.	$\frac{\delta K}{K_H} \times 10^2$.	H.	$\frac{\delta K}{K_H} \times 10^2$.	H.	$\frac{\delta K}{K_H} \times 10^2$.	H.	$\frac{\delta K}{K_H} \times 10^2$.
4.0	-0.60	3.8	-0.23	3.8	-0.37	3.8	-0.22
7.9	-0.23	7.8	+0.28	7.7	-0.28	8.0	-0.59
16.4	+1.24	15.9	+1.48	15.7	+0.46	15.8	+0.51
32.9	+3.17	33.0	+3.03	32.6	+1.41	31.2	+1.05
58.0	+4.73	56.8	+4.14	56.8	+2.31	54.9	+1.75
121.0	+6.01	117.3	+5.44	117.3	+3.46	116.6	+2.58
196	+6.73	205	+6.06	202	+4.07	204	+3.24
283	+7.12	278	+6.57	288	+4.24	290	+3.42
387	+7.16	379	+6.92	386	+4.64	394	+3.60

70.32 per cent. Ni. $K_0 = 0.640 \times 10^{12}$. $t = 16^{\circ}5$ C.

T=1086 gr./mm. ² T=3198 gr./mm. ²				T=5828 gr./mm. ² T=8510 gr./mm. ²			
H.	$\frac{\delta K}{K_H} \times 10^2$.	H.	$\frac{\delta K}{K_H} \times 10^2$.	H.	$\frac{\delta K}{K_H} \times 10^2$.	H.	$\frac{\delta K}{K_H} \times 10^2$.
3.3	-1.18	1.9	-0.02	2.4	-0.23	2.3	-0.60
5.2	-0.86	3.3	+0.25	5.9	+0.09	6.0	-0.40
7.8	-0.24	6.2	+0.97	...	...	...	...
15.6	+1.72	12.9	+1.99	11.7	+0.64	11.3	-0.12
24.1	+3.00	...	...	24.2	+1.50	24.5	+0.44
48.0	+4.64	46.3	+4.17	47.7	+2.22	48.6	+1.00
77.6	+5.41	90.5	+4.97	78.4	+2.70	86.5	+1.44
116.2	+5.96	163.2	+5.50	167.0	+3.27	...	...
204	+6.33	238	+5.67	241	+3.49	208	+2.01
300	+6.43	316	+5.79	301	+3.55	306	+2.14
394	+6.55	401	+5.86	410	+3.64	414	+2.08

In general features, the change of rigidity is similar for the three specimens. Magnetization generally increases the rigidity; the change increases rapidly with the field and soon tends to asymptotic values. The effect of tension is invariably to diminish the change of rigidity. The last two specimens, which in weak fields display a slight decrease of rigidity, resemble each other not only in their general behaviour, but also in the amount of their change. In 28.74 per cent. Ni, the amount of the change is much less than that for the others. It is to be noticed that the general aspect of the change of rigidity resembles that of the change of elasticity.

(b) *Measurement of the Change of Rigidity by Barus's method.*

The same specimens, on the other hand, were tested by the differential method used by Barus*, in which the wires were first twisted and then magnetized. In our experiments, instead of the magnetic wire, a copper one was used for the compensation of the twist at no field, in order to avoid any ambiguity caused by the magnetization of the compensating wire. The specimens were cut from the previous samples into about one-third of their lengths. The copper wire was so chosen that its total twist for a given couple was nearly equal to that of the magnetic specimen to be tested. The two wires were connected in the same line by a copper rod of moderate diameter with a small mirror attached to it, and hung vertically from a torsion circle. The lower end of the specimen was also rigidly connected to another copper rod. To this rod, a rigid pin was perpendicularly fixed, the ends of which slid in the vertical grooves cut on the inside of a flat hollow cylinder at the centre of the lower torsion circle. In this way, the tension applied to the pan hanging on the lower end of the lower copper rod was transmitted independently of the twist. The upper and lower ends of this connected system could therefore be twisted to any desired amount by means of the torsion circles, while the tension was constantly acting on the wire. A magnetizing coil, whose length was 30 cm. and $4\pi n = 379.7$, was placed co-axially with the lower wire, which was, in our case, the specimen to be tested. The coil had a small resistance of 0.6Ω , so that the effect of heating was negligibly small, though the water-jacketed arrangement was dispensed with. A long scale distance of 6.797 m. was used to increase the sensitiveness of the arrangement. The sensitiveness was such that a deflexion of 1 scale-division corresponded to a change of total twist of $15''.2$.

To stop the vibration of the mirror, a short brass wire was fixed horizontally to the vertical copper rod a little below the reflecting mirror and bent downward. Just below it, a small mercury cup was placed, into which the wire dipped.

Our method of observation was as follows:—First the torsion circles were adjusted so that the magnetic field produced no deflexion of the mirror. The readings of both circles for this position corresponded to those of the untwisted state of the wire. Next the lower circle was twisted by θ

* Barus, *loc. cit.*

and then the upper circle was twisted in the opposite sense, until the mirror regained its original position for no field. Let the angle through which the upper circle was twisted be denoted by θ' ; θ' was nearly equal to θ . After demagnetization by reversals, the mirror usually turned through a small angle; but the amount of the rotation was usually so small that we could leave it out of account. Different fields were successively applied and corresponding deflexions were read off. Next the twist was increased and the procedure repeated.

The experiments were made also for different tensions. Before increasing the tension, the wire was untwisted to the original state of no twist, after passing through a cyclic twist of gradually decreasing amplitude about the position of no twist. This procedure was always necessary to restore the wire to the state of no twist, since the simple untwisting usually left a residual deflexion.

If the couples required to give unit twist to the two wires be τ and τ' respectively, we have

$$\tau\theta = \tau'\theta';$$

if the angular deflexion of the mirror due to magnetization be $\delta\theta$, and $\delta\tau$ the increment of τ , we have

$$\tau_{II} = \tau + \delta\tau \quad \text{and} \quad (\tau + \delta\tau)(\theta - \delta\theta) = \tau'(\theta' + \delta\theta)$$

or

$$\tau_{II}(\theta - \delta\theta) = \tau'(\theta' + \delta\theta) \quad \text{and} \quad \delta\tau(\theta - \delta\theta) = \delta\theta(\tau' + \tau);$$

hence by division, we get

$$\frac{\delta\tau}{\tau_{II}} = \frac{\delta\theta(\tau + \tau')}{\tau'(\theta' + \delta\theta)} = \frac{\delta\theta\left(1 + \frac{\tau}{\tau'}\right)}{\theta' + \delta\theta},$$

$$\text{or} \quad \frac{\delta K}{K_{II}} = \frac{\delta\theta(\theta + \theta')}{\theta(\theta' + \delta\theta)}.$$

where K is the rigidity of the specimen, and δK its change due to magnetization. If $\frac{\delta\theta}{\theta}$ be neglected, we have

$$\frac{\delta K}{K_{II}} = \frac{\delta\theta}{\theta} \left(1 + \frac{\theta}{\theta'}\right).$$

If the change were greater than 1 per cent., the first equation was used, and if it were less, the second equation was employed.

70 Messrs. K. Honda and T. Terada on the Change of
The results of experiments are as follows:—

Nickel. Pl. III. fig. 35.

TABLE XXVIII.

$$\theta = 2^{\circ}5.$$

$$t = 21^{\circ}0 \text{ C.}$$

T=1022 gr./mm. ²		T=3096 gr./mm. ²		T=5465 gr./mm. ²	
H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$
8.3	+ 0.83	4.9	+ 0.84	9.3	+ 0.34
13.9	— 0.24	9.7	+ 1.41	15.0	+ 0.30
23.3	— 2.37	18.7	+ 0.00	21.5	— 0.30
43.7	— 3.11	43.0	— 8.81	26.4	— 1.26
86.8	— 0.41	85.4	— 10.04	42.4	— 7.00
125.6	+ 2.45	123.4	— 6.84	85.3	— 17.33
145.8	+ 3.58	143.2	— 5.37	123.8	— 15.21
207	+ 6.39	204.4	— 1.43	174.2	— 10.45
311	+ 9.17	304	+ 2.53	305	— 2.90
441	+ 11.00	432	+ 5.13	432	+ 0.43

$$\theta = 5^{\circ}0.$$

T=1022 gr./mm. ²		T=3096 gr./mm. ²		T=5465 gr./mm. ²	
H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$
6.2	+ 0.68	4.0	+ 0.67	5.6	+ 0.08
11.3	— 0.08	8.8	+ 1.39	10.1	+ 0.12
19.6	— 1.54	15.3	— 0.48	15.4	— 0.31
42.8	— 2.14	26.9	— 4.87	27.1	— 3.05
86.3	— 0.00	43.3	— 7.94	42.8	— 7.87
124.2	+ 2.18	86.2	— 7.33	85.7	— 13.51
145.8	+ 3.27	143.6	— 3.84	123.8	— 11.95
208	+ 5.68	207	— 0.70	174.2	— 8.53
310	+ 8.26	309	+ 2.76	306	— 2.25
439	+ 10.06	438	+ 5.38	432	+ 0.76

In nickel, the change of rigidity and the effect of tension are very remarkable. Under small tension, the rigidity first increases slightly, then decreases rapidly as the field is increased, attains a minimum and gradually increases, till in a certain field it recovers its initial value. As the field is

further increased, the rigidity increases by magnetization. If the tension is increased, the field corresponding to the maximum decrease moves towards higher fields, and the amount of decrease is remarkably increased. Tensions here used were so adjusted as to be nearly equal to those in the oscillation method. The effect of the amplitude of twist was to diminish the change of rigidity in its absolute amount.

Comparing the above results with those of the oscillation method, we notice that with weak tensions, the curve of the change of rigidity as given by Barus's method lies considerably above the curve of the change by the oscillation method, and that with greater tensions the contrary is the case.

Swedish Iron and Tungsten Steel. Pl. III. figs. 36 & 37.

TABLE XXIX.

Swedish Iron.

Tungsten-Steel.

$T=3271 \text{ gr./mm.}^2$ $t=20^{\circ}0 \text{ C.}$

$T=3405 \text{ gr./mm.}^2$ $t=20^{\circ}1 \text{ C.}$

$\theta=2^{\circ}5.$		$\theta=5^{\circ}0.$		$\theta=4^{\circ}0.$		$\theta=8^{\circ}0.$	
H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$
4.4	0.32	3.9	0.19	13.6	0.00	12.6	0.00
10.5	1.00	10.2	1.01	21.8	0.08	18.3	0.07
20.0	1.36	16.5	1.27	27.1	0.14	29.3	0.17
42.6	1.88	25.2	1.51	46.0	0.20	44.7	0.21
75.5	2.20	42.2	1.81	81.3	0.30	76.4	0.30
123.1	2.35	75.5	2.06	134.1	0.38	123.8	0.38
174.0	2.51	171.4	2.34	172.9	0.40	172.2	0.41
306	2.56	301	2.42	304	0.45	300	0.49
434	2.67	428	2.51	429	0.51	425	0.52

Thus the rigidity of these metals always increases with the field; the change for Swedish iron is tolerably large; but for tungsten-steel it is very small. As to the effect of amplitude of twist, it is slightly to diminish the change in Swedish iron, but is almost insensible in tungsten-steel. The change of rigidity obtained by Barus's method is several times greater than that by the oscillation method. Thus, in the case of ferromagnetic metals, relation (3) does not hold even approximately.

24.74 per cent. Nickel. Pl. IV. fig. 38.

TABLE XXX.

 $t = 21^{\circ}2$.

T=948 gr./mm. ²		T=1768 gr./mm. ²		T=4240 gr./mm. ²	
$\theta = 2^{\circ}5$.		$\theta = 3^{\circ}5$.		$\theta = 3^{\circ}5$.	
H. _I	$\frac{\delta K}{K_H} \times 10^2$.	H.	$\frac{\delta K}{K_H} \times 10^2$.	H.	$\frac{\delta K}{K_H} \times 10^2$.
3.4	0.08	3.8	0.12	3.7	0.11
10.5	0.31	9.5	0.25	10.3	0.23
23.0	0.42	20.5	0.41	19.7	0.30
86.1	0.56	42.7	0.45	42.2	0.34
145.0	0.61	83.5	0.45	86.1	0.40
...	...	142.1	0.45	145.4	0.41
307	0.56	307	0.48	311	0.41
434	0.56	437	0.48	441	0.41
$\theta = 7^{\circ}5$.		$\theta = 7^{\circ}5$.		$\theta = 7^{\circ}5$.	
3.2	0.09	3.8	0.09	3.8	0.12
9.9	0.24	9.3	0.22	10.8	0.21
21.4	0.34	19.0	0.31	20.6	0.28
43.2	0.41	42.9	0.40	23.8	0.31
124.9	0.45	83.5	0.42	84.6	0.35
206	0.45	143.0	0.42	143.9	0.36
310	0.46	307	0.44	309	0.36
436	0.46	436	0.44	436	0.36

50.72 per cent. Nickel-Steel. Pl. IV. fig. 39.

TABLE XXXI.

 $t = 19^{\circ}0\text{ C.}$

T=1136 gr./mm. ²		T=3445 gr./mm. ²		T=6084 gr./mm. ²	
$\theta = 2^{\circ}5.$		$\theta = 2^{\circ}5.$		$\theta = 2^{\circ}5.$	
H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$
3.2	0.30	3.4	0.24	3.0	0.03
7.3	1.32	8.5	1.27	7.7	0.53
14.1	2.93	14.6	2.31	14.1	1.26
22.6	4.35	22.9	3.33	22.9	2.08
35.3	5.87	35.6	4.45	35.3	2.89
49.7	6.95	49.3	5.24	49.9	3.59
69.6	7.90	69.1	6.00	69.6	4.23
94.1	8.66	93.8	6.62	94.1	4.79
132.7	9.42	133.4	7.26	132.7	5.30
202	10.06	203	7.84	203	5.90
302	10.52	303	8.23	303	6.30
430	10.83	429	8.51	429	6.57
$\theta = 5^{\circ}0.$		$\theta = 5^{\circ}0.$			
3.1	0.52	3.2	0.23		
7.0	1.35	7.4	1.05		
14.2	2.93	13.9	2.17		
22.3	4.27	22.7	3.28		
34.9	5.70	35.2	4.36		
49.0	6.77	49.5	5.17		
68.9	7.71	69.6	5.97		
93.8	8.51	94.1	6.60		
132.3	9.17	133.4	7.20		
202	9.87	203	7.83		
302	10.30	303	8.27		
428	10.60	430	8.54		

TABLE XXXII. $t=19^{\circ}2\text{ C.}$

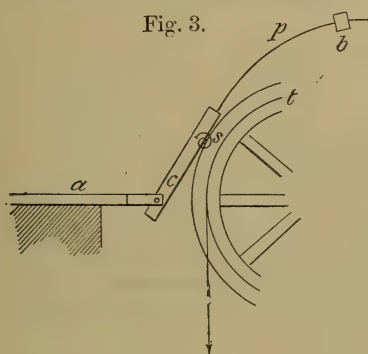
T=1109 gr./mm. ²		T=3361 gr./mm. ²		T=5936 gr./mm. ²	
$\theta=2^{\circ}75.$		$\theta=2^{\circ}65.$		$\theta=2^{\circ}55.$	
H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$
3·8	0·06	3·0	0·26	3·2	0·07
...	...	6·7	0·97	11·1	0·97
11·1	2·20	15·5	2·33	21·1	1·72
23·2	4·44	23·5	3·16	22·9	1·92
35·8	5·70	35·7	3·92	34·7	2·41
...	...	50·3	4·48	48·9	2·92
64·1	6·99	69·2	4·95	68·9	3·32
85·7	7·77	93·8	5·35	93·6	3·64
150·2	8·19	132·1	5·69	131·9	4·00
248	8·60	202	6·03	202	4·28
342	8·69	303	6·25	303	4·54
432	8·83	430	6·34	433	4·61

$\theta=5^{\circ}3.$		$\theta=5^{\circ}1.$		$\theta=5^{\circ}05.$	
4·5	0·36	3·3	0·37	2·7	0·17
...	...	7·8	1·18	9·2	0·82
12·9	2·18	15·8	2·26	18·0	1·50
23·1	3·69	22·8	2·90	23·5	1·83
39·8	5·02	34·2	3·64	34·8	2·32
...	...	48·5	4·23	49·1	2·80
57·9	5·77	68·9	4·70	69·6	3·17
93·8	6·55	93·8	5·06	94·1	3·48
132·0	6·94	131·6	5·39	132·7	3·80
203	7·28	202	5·71	202	4·11
304	7·53	302	5·94	298	4·30
432	7·67	431	6·05	429	4·43

Thus the change of rigidity in these nickel-steels is similar, in its general aspect, to that given by the oscillation method. In 50·72 per cent. Ni and 70·32 per cent. Ni, however, the small initial decrease of rigidity is not observed. Increased twist diminishes the change of rigidity; increased tension affects the change in a similar manner but in a greater degree. The change as given by Barus's method is generally greater than that by the oscillation method. The difference becomes less as the tension is increased. In 28·74 per cent. Ni, the results by the two different methods fairly agree, and the agreement becomes closer with increased tension. Thus, also in these alloys, relation (3) is not generally satisfied, except in 28·74 per cent. Ni. The difference, however, becomes less as the tension is increased.

(a) *Measurement of the Change of Rigidity by ordinary method.*

In a paper by Messrs. S. Shimizu, S. Kusakabe, and one of us *, the change of rigidity of ferromagnetic bars is given, having been determined by first applying the torsion and then the magnetic field. To investigate whether the results are the same or not, if the order of applying the torsion and the field is reversed, the same arrangement was used as that employed in the experiment above cited. The essential parts of the arrangement remained unchanged, except for the



mirror system. The rotating cylinder, to which the mirror was fixed, was supported horizontally by means of two very weak spiral springs (*s*) attached to the sides of a carriage (*e*) similar to that used in our measurement of the change of elasticity. To the carriage was rigidly fixed a horizontal axis, the conical ends of which fitted to agate cups on the arms of a Y-shaped bar (*a*). This

bar could be adjusted to any desired position, such that the axis of the mirror would be pressed by the plane side of the carriage perpendicularly on a point of the circumference of the torsion wheel (*t*) rigidly fixed to the specimen. To adjust the pressure properly, a sliding weight (*b*) was put on a pin (*p*) protruding from the carriage.

Two vertical scales 30 cm. long were erected at a horizontal distance of 6.45 m. in front of the mirror in the same vertical line; one at the same height as the mirror, and the other about one metre above it. The image of the scales was observed with a telescope at the same height as the mirror. An initial couple was applied and the mirror was so adjusted that the image of the lower scale was observable with the telescope. A suitable weight was then chosen for giving an additional couple, which twisted the rod to such an extent that by the consequent rotation of the mirror, the image of the upper scale just appeared in the field of the telescope. This weight was 500 gr. The readings of the two scales, corresponding to the removal and to the addition of the additional weight, were taken. When properly adjusted, their difference was fairly constant for repeated observations; the mean of several observations was always taken as the

* K. Honda, S. Shimizu, and S. Kusakabe, *loc. cit.*

deflexion due to the weight. Next, a field was applied and the procedure repeated; in this way, we obtained the values of deflexions corresponding to the different fields, the demagnetization being, of course, carefully made before each experiment. The difference of the deflexions in a certain field and in no field gives the change of rigidity.

The sensitiveness of the present arrangement was such that a deflexion of 1 mm. of the scale reading corresponded to a twist of $7''\cdot64 \times 10^{-3}$ per cm. of the specimen. The specimens tested in the present experiments were those used in the previous experiments above quoted, and had the following dimensions:—

Metals.	Diameter.	Length.	Rigidity.
Soft iron ...	0·986 cm.	20·43 cm.	$7\cdot92 \times 10^{11}$
Nickel.....	1·115 cm.	21·42 cm.	$7\cdot41 \times 10^{11}$

On the other hand, the determination of the change of rigidity by the ordinary method of first applying the couple and then the magnetizing field was always made, and the results obtained by the two different methods were compared. The magnetizing coil was the same as that used in the last experiment, so that the heating effect due to the magnetizing current was negligibly small.

In using the present arrangement, all necessary precautions were taken, which were referred to in the paper above quoted. The results are given in the following:—

Nickel. Pl. IV. figs. 41 & 42.

TABLE XXXIII. $t = 14^{\circ}\cdot7$ C.

N = $7\cdot00 \times 10^6$. $\theta = 12''\cdot83$.		N = $14\cdot09 \times 10^6$. $\theta = 25''\cdot83$.		N = $21\cdot27 \times 10$. $\theta = 38''\cdot98$.	
H.	$\frac{\delta K'}{K_H} \times 10^2$.	H.	$\frac{\delta K'}{K_H} \times 10^2$	H.	$\frac{\delta K'}{K_H} \times 10^2$.
6·3	+0·05	6·3	+0·07	5·6	+0·05
14·5	+0·12	13·5	+0·12	13·4	+0·11
20·5	-0·02	20·5	-0·01	22·0	-0·07
28·4	-0·34	26·7	-0·31	26·1	-0·24
34·9	-0·65	35·0	-0·68	33·8	-0·60
56·0	-1·37	55·7	-1·37	55·7	-1·24
82·8	-1·35	83·5	-1·33	81·0	-1·20
114·0	-0·60	115·8	-0·59	114·0	-0·50
164·8	+1·15	167·0	+1·16	164·8	+1·14
236	+3·40	238	+3·39	236	+3·22
288	+4·59	292	+4·61	289	+4·34
359	+5·81	365	+5·87	362	+5·57
407	+6·38	412	+6·48	410	+6·17
519	+7·39	523	+7·53	518	+7·21

Table XXXIII. (continued).

$$\Delta N = \pm 3.50 \times 10^6, \quad t = 16^{\circ}.7 \text{ C.}$$

$N = 7.00 \times 10^6.$ $\theta = 12''.83.$		$N = 14.09 \times 10^6.$ $\theta = 25''.83.$	
H.	$\frac{\delta K}{K_H} \times 10^2.$	H.	$\frac{\delta K}{K_H} \times 10^2.$
...	...	6.8	-0.08
15.1	-0.19	15.4	-0.31
22.2	-0.55	22.2	-0.57
48.2	-1.65	48.5	-1.60
82.4	-2.35	82.4	-2.29
127.1	-2.56	127.5	-2.38
192.5	-1.94	193.2	-1.64
288	-0.67	286	-0.34
398	+0.59	386	+0.59
508	+1.37	495	+1.54

In the above tables, N is the moment of force applied in C.G.S. units, and θ the corresponding twist of the specimen as calculated from the modulus of rigidity. The change of rigidity obtained by our direct method is denoted by $\frac{\delta K}{K_H}$, and that obtained by the indirect method, by $\frac{\delta K'}{K_H}$. The values of $\frac{\delta K'}{K_H}$ fairly coincide with those obtained by the previous experiment; but it is to be noticed that the field here given is not the effective one, but the external field applied. The difference between $\frac{\delta K}{K_H}$ and $\frac{\delta K'}{K_H}$ is remarkable; its amount is of the same order of magnitude as the change itself. The increase of the angle of twist slightly affects the change of rigidity, always diminishing it in absolute amount. The general feature of $\frac{\delta K}{K_H}$ is quite similar to that for nickel wire with weak tension, as obtained by the oscillation method.

It is curious to observe that the residual twist is always positive and uniformly increases with the field, as shown in Pl. IV. fig. 42.

It may be noticed that the course of the curve of $\frac{\delta K'}{K_H}$ closely resembles that of $\frac{\delta E}{E}$ as given by the flexure method. In both cases, the right order of applying the stress and the field is inverted.

Soft Iron. Pl. IV. figs. 43 & 44.

TABLE XXXIV.

 $t=19^{\circ}0\text{ C.}$

$N=7.00 \times 10^6.$ $\theta=19''.65.$		$N=14.09 \times 10^6.$ $\theta=39''.56.$		$N=21.27 \times 10^6.$ $\theta=59''.70.$	
H.	$\frac{\delta K'}{K_H} \times 10^2.$	H.	$\frac{\delta K'}{K_H} \times 10^2.$	H.	$\frac{\delta K'}{K_H} \times 10^2.$
7.3	0.01	8.4	0.02	9.3	0.01
22.0	0.07	22.0	0.07	22.0	0.06
40.0	0.11	39.6	0.14	39.6	0.12
73.3	0.23	73.3	0.27	72.6	0.25
111.3	0.63	111.0	0.68	110.6	0.65
160.5	0.95	160.5	1.03	160.5	1.02
252	1.34	252	1.42	252	1.43
322	1.51	315	1.58	315	1.60
396	1.66	395	1.73	396	1.75
511	1.79	508	1.87	509	1.87

Since the time of the previous experiment, the specimen had been once annealed at a temperature of about 1100°C. , so that the change of rigidity $\frac{\delta K'}{K_H}$ was somewhat increased in the present experiment. The effect of the amplitude of twist on the change of rigidity is very small; the increase of the twist slightly augments the change of rigidity up to the range of the twist used. The residual twist is always positive as in the case of nickel. $\frac{\delta K}{K_H}$ is always inappreciable in the present arrangement; if it exist at all, it cannot be greater than 0.05 per cent. for the maximum field used.

Thus, in nickel and iron rods, $\frac{\delta K'}{K_H}$ is always far greater than $\frac{\delta K}{K_H}$, so that relation (3) cannot be satisfied even approximately.

In concluding the descriptions of the results of our experiments, the following remarks may be added. Since we measured the changes of elasticity and of rigidity for the same wires of ferromagnetic metals and alloys, it will be now interesting to calculate the change of Poisson ratio by magnetization. As we have seen, the change of elastic constants is considerably affected by tension, it is necessary to

calculate the change of the ratio from the changes of elasticity and rigidity for the same tension.

If σ be the Poisson ratio, we have

$$\frac{\delta\sigma}{\sigma} = \frac{\frac{E}{K}}{\frac{E}{K} - 2} \left(\frac{\delta E}{E} - \frac{\delta K}{K} \right).$$

The value of $\frac{E}{K}$ used in the calculation of the following table is that for no field; and the values of $\frac{\delta E}{E}$ and $\frac{\delta K}{K}$ are those obtained by the tension effect of the magnetic elongation and by Barus's method respectively, for in these two sets of experiments, the stress was first applied and then the magnetizing field.

TABLE XXXV.

Specimen...	Nickel.	Swedish Iron.	Tungsten Steel.	28.74% Ni.	50.72% Ni.	70.32% Ni.
Tension...	3020 gr.	3270 gr.	3320 gr.	1770 gr.	3350 gr.	3280 gr.
H.	$\frac{\delta\sigma}{\sigma} \times 10^2$.	$\frac{\delta\sigma}{\sigma} \times 10^2$.	$\frac{\delta\sigma}{\sigma} \times 10^2$.	$\frac{\delta\sigma}{\sigma} \times 10^2$.	$\frac{\delta\sigma}{\sigma} \times 10^2$.	$\frac{\delta\sigma}{\sigma} \times 10^2$.
10		- 2.2			- 0.3	- 1.6
20		- 3.8		- 1.0		
30	- 6.1		0.0		- 1.9	
100	+ 37	- 7.4	0.0	- 1.3	- 8.3	- 14.5
150	+ 50					
200	+ 23	- 9.0	0.1	- 1.3	- 10.1	- 13.5
250	+ 7					
300			0.3	- 1.1		- 13.8
350		- 10.2			- 11.1	

From the above table, it will be seen that the Poisson ratio is generally diminished by magnetization. In nickel, it is very large, amounting to even 50 per cent.; but in tungsten-steel it is almost zero, indicating a tendency to increase. The change of the Poisson ratio also varies considerably with tension, and the above table shows only an example of the change.

§ 4. CONCLUDING REMARKS.

The results obtained in the present investigation may be summarized as follows:—

(i.) In Swedish iron and tungsten-steel (Pl. IV. figs. 45 and 46), the change of elastic constants is generally positive, but its amount is extremely small, less than 0·5 per cent. The change by the indirect method is several times greater than that by the direct method. It is interesting to notice that the change of elasticity and that of rigidity almost coincide with each other, not only in their general aspects, but also in their quantitative relations.

(ii.) In nickel (fig. 42), the change of elastic constants is remarkably large, amounting to about 15 per cent. in the change of elasticity and 7 per cent. in the case of rigidity. The elastic constants first decrease and then increase, as the field becomes greater. The elastic constants by the indirect method are numerically greater than those by the direct method.

(iii.) In 28·74 per cent. Ni (fig. 48), magnetization increases the elastic constants by a small amount. The changes given by the two different methods nearly coincide with each other for a moderate tension. In 50·72 per cent. Ni and 70·32 per cent. Ni (figs. 49 and 50), the increase of the elastic constants is remarkably large, approaching in amount the change of rigidity in nickel. Except with a very weak tension, the change by the indirect method is greater than that by the direct; but this difference becomes less as the tension is increased. With a tension of 3 or 4 kg. per square millimetre, it nearly vanishes.

From the results above given, it is evident that there are some cases in which relation (3), given in the earlier part of this paper, does not hold even approximately, so that equation (1) cannot be freely used in any quantitative discussion.

There are many analogous cases in the problem of magnetostriction. As will be seen from our subsequent paper to be published presently, the change of magnetization by stretching a magnetized wire does not agree with the change deduced from the results obtained by magnetizing the wire in the unstrained as well as the stretched state. Similar phenomena are also observable in the change of magnetism caused by the twist. Next, we may cite the case of the Wiedemann effect. It is well known that in iron and nickel, the twist produced

by magnetizing the wire traversed by an electric current is generally greater than the twist caused by passing the current through the magnetized wire. The difference is remarkable; in some cases* the former is several times greater than the latter. In nickel-steels* the difference is, however, very small. Again, take the case of magnetizing a wire traversed by an electric current†. The change of magnetization by the longitudinal current is considerably greater, in the case of magnetizing the wire traversed by the current, than in the case of passing the current through the magnetized wire. All these phenomena may perhaps arise from the hysteresis effect of magnetization; *i. e.* the final states attained by a magnetic substance differ according to the order of applying the field and the stress, or the field and the longitudinal current.

In conclusion, it may be remarked that, since the change of the elastic constants by magnetization is not so small as is generally believed, in any theory of magnetostriction aiming at the quantitative agreement between the theory and the experiment, these changes must necessarily be taken into account. In addition to this, an equality such as

$$\frac{\partial^2 Q}{\partial x \partial y} = \frac{\partial^2 Q}{\partial y \partial x},$$

in which Q is a quantity which depends upon two apparently independent variables x and y , cannot be used without experimental verification. These facts make the development of the theory very difficult.

In passing, the following remarks may be added. If we consider, in the above equation, Q as the length of a specimen, x the temperature and y the magnetic force, we have

$$\frac{\partial}{\partial H} \left(\frac{\partial l}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial l}{\partial H} \right),$$

provided l is independent of the order of applying the field and the temperature. Integrating, we get

$$\left(\frac{\partial l}{\partial t} \right)_{II} - \left(\frac{\partial l}{\partial t} \right)_0 = \frac{\partial (l_{II} - l_0)}{\partial t}.$$

If a be the coefficient of thermal expansion,

$$a_{II} - a_0 = \frac{\partial (l_{II} - l_0)}{l_0 \partial t}.$$

* K. Honda and S. Shimizu, Jour. Sc. Coll. xvi. Art. 14.

† K. Honda, *ibid.* xi. p. 284, 1899.

Thus the change of the coefficient of thermal expansion by magnetization is equal to the temperature coefficient of the magnetic elongation. As the latter coefficient is known from the experiment * by Mr. S. Shimizu and one of us, the values of δa are calculated and graphically drawn in Pl. V. figs. 51, 52, 53, 54, and 55.

By referring to the figures, we see that the change of the coefficients of thermal expansion by magnetization depends considerably upon temperature. Ordinates of the curves represent the change of the mean coefficient of expansion between two temperatures belonging to each curve.

The change of the mean coefficient of expansion in nickel (fig. 51) between the ordinary and liquid air temperatures first decreases, attains a minimum, and then gradually increases, as the field becomes greater, till it is greater than its initial value. At a temperature higher than the ordinary, the change of the coefficient of expansion steadily increases, soon approaching an asymptotic value. In a given field, its value increases with temperature, and after passing through a maximum, slightly decreases. The maximum amount of the change is of the order of 1 per cent. of the coefficient itself.

In soft iron and tungsten-steel (figs. 52 and 53), the change of expansion is very small. Up to a moderate temperature, the coefficient of expansion increases steadily with the field, except in weak fields, in which a small decrease is observed. At higher temperatures, the change becomes negative for all fields. In iron, a maximum decrease is observed.

In cast cobalt (fig. 54), the change of the coefficient of expansion at low temperature increases with the field, reaches a maximum, and then decreases. As the temperature becomes higher, the amount of the maximum gradually lessens; the position of the maximum shifts towards lower fields and the curve at last cuts the zero line. At a sufficiently high temperature, the change of expansion is always negative.

The change of the coefficient of expansion in annealed cobalt (fig. 55) is rather abnormal, and its maximum amount is relatively large, being about 2 per cent. The change of the mean coefficient of expansion between the ordinary and liquid air temperatures steadily decreases with the field, though its amount is small. As the temperature rises, the amount of the diminution increases rapidly and then decreases. Here the course of the curve shows a minimum decrease of

* K. Honda and S. Shimizu, Jour. Sc. Coll. xx. Art 10, 1903.

the change. As the temperature is further increased, a considerable amount of increase is observed, which steadily increases with field. It attains a maximum and again decreases, till it changes its sign.

The values of δa thus obtained may possibly differ from the values obtained by heating the specimens in a constant magnetic field. They are the changes of a by magnetization, when the temperature is first raised and then the field applied. However, any experiment for the determination of the change of a , in which the thermal expansion is directly observed in a magnetic field, must be welcomed, as affording a counterpart for the analogous comparison.

Again, if we put, in the last equation, the tension T per unit area for H , we get

$$\left(\frac{\partial l}{\partial t}\right)_T - \left(\frac{\partial l}{\partial t}\right)_0 = \frac{\partial(l_T - l_0)}{\partial t},$$

or

$$a_T - a_0 = \delta a = \frac{\partial(l_T - l_0)}{l_0 \partial t};$$

but since

$$\frac{l_T - l_0}{l_0} = \frac{T}{E},$$

we have finally

$$\frac{\delta a}{T} = \frac{\partial}{\partial t} \frac{1}{E} = - \frac{1}{E^2} \frac{\partial E}{\partial t}.$$

Thus the effect of tension upon the coefficient of thermal expansion can be found from the temperature coefficient of the modulus of elasticity. The relation was first obtained by Dahlander*, and verified by him to agree well with the experiment. In problems which do not relate to magnetism, the hysteresis effect is generally very small, so that the agreement might have been expected.

* Dahlander, *Pogg. Ann.* cxlv. p. 147, 1872.

III. *On the Measurement of Gas-Engine Temperatures.*
By BERTRAM HOPKINSON, Professor of Mechanism and
Applied Mechanics in the University of Cambridge.*

[Plate VI.]

AN important point in the experimental study of the gas-engine is the determination of the temperature of the gases at each point of the cycle. It is obvious that if the mean temperature at any one point is known, that at any other can be calculated from the indicator diagram, assuming the relation $\frac{pv}{\theta} = \text{constant}$ to hold throughout. This relation is probably true as a practical approximation for the gas-engine mixture since it consists, as to 80 per cent. of its volume, of nitrogen and oxygen, which pass unchanged through the combustion. For the starting point of the temperature determinations it is most convenient to choose the temperature at the end of the suction-stroke, when the charge of gas has just been drawn in and its compression is about to commence. The measurement of this temperature, or "suction temperature" as it is generally called, is best effected by determining the quantity of air and gas drawn into the engine at each stroke, from which, knowing the pressure and volume of the mixed gases when the inlet-valves have closed, their temperature can be calculated. The accurate measurement of the large volumes of air used by a gas-engine of considerable size is, however, a difficult operation; and it is desirable, if possible, to use some means of directly measuring the temperature of the charge at the end of the suction-stroke. Moreover, the subsequent calculations from the indicator diagram of the temperature at other points of the cycle, lead to mean values for the whole volume, and give us no information as to the difference of temperature between one part of the cylinder and another. Such differences may amount to several hundred degrees centigrade during the first half of the expansion or working stroke. They arise partly from the cooling effect of the walls, and partly from the manner in which the inflammable mixture is ignited. As I have shown elsewhere †, the temperature round about the point of ignition is necessarily some 500° higher than at a distance, because of the relative slowness of the propagation of flame as compared with the time of reaction of the gases when the flame has

* Communicated by the Author.

† Proc. Roy. Soc. A. vol. lxxvii. p. 387; Engineering, vol. lxxxi. p. 777.

once reached them. For the complete study of gas-engine temperature therefore some method of local thermometry is necessary.

The platinum resistance-thermometer is the only means available for the direct measurement of rapidly changing gas-temperatures, since it alone can be constructed with the requisite low thermal capacity. Moreover, it gives the local temperature as distinct from the mean value obtained from a study of the pressure. It has been employed by Callendar and Nicholson * with great success for the measurement of steam temperatures in the steam-engine cylinder; and Burstall has used it in the gas-engine, but experienced great difficulties †. I have also used it in studying the explosion of gaseous mixtures in a closed vessel ‡. The difficulty of the gas-engine experiments is that the temperature rises locally much above the melting-point of platinum, and the wire fuses unless it is so thick that it fails to reach the maximum temperature of the gas surrounding it. But if the latter condition be fulfilled, it is obvious that the thermometer does not give the temperature of the gas at other points of the cycle, unless corrections are applied for the time-lag. Professor Burstall found it necessary to use wires $\frac{1}{1000}$ of an inch diameter, in order to secure sufficient permanence in the thermometer to take a complete series of measurements of its resistance throughout the cycle by means of a rotating contact-maker. In my own closed-vessel experiments I was able to use the finest obtainable wire ($\frac{1}{1000}$ in. diameter), and it nearly always melted if placed near the point of ignition. But as a continuous record of the resistance was taken in each explosion prior to melting, this did not affect the measurements. Even with such fine wire the correction for time-lag was by no means negligible; and I was able to show that a wire double the diameter might be as much as two hundred degrees hotter or cooler than the gas surrounding it when the temperature changes so rapidly as in the working stroke of a gas-engine. The determination of the time-lag of the wire, or to speak more precisely of the relation between the temperature of the wire and that of the gas, is therefore a matter of much importance in these measurements. The difference of the two temperatures is due to the constant exchange of heat, which must go on between the wire and the gas to provide for the warming or cooling of the wire and for the loss of heat from the wire by radiation. If the temperature of the wire is known

* Min. Proc. Inst. C. E. vol. cxxxi.

† Phil. Mag. vol. xl. p. 282 (1895).

‡ Proc. Roy. Soc. A. vol. lxxvii. p. 387.

in terms of the time, it is easy to calculate the rate at which it is receiving heat from the gas. Let θ' be the temperature and t the time, then heat must be supplied to the wire at a rate $k \frac{d\theta'}{dt} + f(\theta')$, where k is the capacity for heat of the wire, and $f(\theta')$ the rate at which it loses heat by radiation *in vacuo* when at temperature θ' . The radiation term depends on the nature and extent of the surface of the wire and on the temperature; it is not important except at very high temperatures—for most purposes a sufficiently good approximation to its value can be obtained from the numerous published results for polished platinum, which are in fair agreement with one another, and give the radiation per sq. cm. of surface up to temperatures near the melting-point.

The difference of temperature between the wire and the gas at a distance from it will, under any given conditions, be proportional to the rate of exchange of heat between wire and gas. If θ' be the gas temperature, we have

$$k \frac{d\theta'}{dt} + f(\theta') = \lambda(\theta - \theta'),$$

whence θ can be obtained if λ is known. The constant λ is the rate at which heat passes between the wire and the gas, per degree difference of temperature between them, under the conditions of temperature, motion of the gas, &c., which actually obtain. These conditions vary from point to point of the cycle in the gas-engine, and λ also varies. It is necessary, in fact, in measuring gas-engine temperatures in this way to determine the relation between heat flow and temperature difference at each point of the stroke.

The experiments to be described in this paper consisted in a determination of the temperatures in the cylinder of a large gas-engine by means of a platinum thermometer, the gas-engine being motored round with the gas-supply cut off so that it simply compressed and expanded a charge of air. The object of the experiments was to test the platinum thermometer as a means of measuring rapidly varying gas-temperatures, rather than an enquiry into gas-engine phenomena. The absence of explosions secures a fairly uniform temperature throughout the cylinder contents. The mean temperature of the charge given by the indicator diagram therefore does not differ much from the local temperature in the neighbourhood of the wire; and the wire temperatures can be checked by reference to the indicator diagram. I have used a simple method of getting the correction constant λ which seems to give good results, and a description of this may be useful to others

who are desirous of measuring gas-engine temperatures. The conclusion at which I have arrived is that by the use of this method of correction and sufficiently accurate measurements of resistance, the temperature of the gas can be obtained within a few degrees with a wire $\frac{1}{1000}$ inch in diameter, if the wire does not melt. But in large engines using high compression, a wire of this size will inevitably melt; and if still thicker wire be used, the correction for time-lag becomes unmanageably large. It appears that the measurement of temperature in such engines by this means is a very difficult operation, involving measurements of accuracy far exceeding that of the ultimate result, unless wires of a more refractory metal than platinum be used.

The briefest description of the gas-engine on which the experiments were made is all that is necessary. It is one of Messrs. Crossley Bros'. well-known Otto cycle-engines, having a cylinder $11\frac{1}{2}$ inches diameter, and a stroke of 21 inches, and it gives about 40 H.P. when running at 185 revs. per minute. The compression space is 407 cub. ins. or $\frac{1}{5.36}$ of the stroke volume, and the gases are compressed to about 170 lbs. per square inch absolute before firing. The engine has been lent to the Engineering Department of Cambridge University through the kindness of Mr. F. W. Crossley. For the purpose of the experiments to be described it was belted to a dynamo (lent by Messrs. Mather & Platt, Limited) by which it was driven, the gas-supply being cut off. Thus the engine drew in a charge of air through the air inlet-valve, compressed it to 173 lbs. per sq. in., expanded it again down to near atmospheric pressure, and expelled it through the exhaust-valve.

The thermometer is a piece of bare platinum wire, $\frac{1}{1000}$ inch in diameter and 29 centims. long. Its ends were hard-soldered to thicker platinum wires which were sealed into glass tubes passing through the exhaust-valve cover and projected about $\frac{1}{20}$ of an inch into the compression-chamber of the engine. The wire was threaded through a hole in a thin mica plate, the plate being carried on the end of a steel rod 5 inches long and $\frac{1}{8}$ inch diameter, which was screwed into the exhaust-valve cover and projected into the compression-chamber. Thus the wire was supported in the form of a V, the ends being $2\frac{1}{2}$ inches apart and close to the walls of the compression-chamber and the apex in the centre of that chamber. The exhaust-valve is in the floor of the compression-chamber immediately below the wire. The inlet-valve is in the side of the compression-chamber. During the suction and exhaust strokes the motion of the air about the wire is violent and turbulent owing to its

rapid ingress and egress through the valves. During the compression and expansion strokes, the air is probably almost at rest except in so far as it is moved by the piston.

The wire formed one arm of a Wheatstone's bridge, the connexions being as in fig. 1 (Plate VI.). The rotating contact-maker A is a block of hard wood soaked in paraffin and carried on the valve-shaft of the engine. It had a slip of brass $\frac{1}{8}$ of an inch wide let into its periphery which made contact with the brushes B, and so completed the galvanometer circuit once in two revolutions of the engine, that is, once in the cycle described above. The moment of contact could be adjusted to any desired point in the cycle by altering the position of the brushes. The current in the wire with this arrangement of resistances was limited to about $\frac{1}{100}$ ampere, and it had no appreciable heating effect upon the wire.

The resistance of the wire at 0° C. was 4.26 ohms, the temperature coefficient .00343, and δ (the factor for correcting platinum to centigrade temperatures) was 1.57. The succession of temperatures determined in this way is shown in fig. 2, curve A (Plate VI.). Of the actual observations, those shown thus ● were all taken on the same day, and the conditions were approximately the same throughout. The temperature of the air in the air-pipe close to the engine was 17° C., the temperature in the exhaust-pipe $26\frac{1}{2}^{\circ}$ C., and the temperature of the jacket-water 46° C. The mean speed throughout was 180 revs. per minute. The observations shown thus ○ were taken on another day when the conditions would not be quite the same.

The connexions were now altered as shown in fig. 3 (Pl. VI.). The conditions were kept exactly the same as before; in fact, the engine was not stopped. The current now passing in the wire (measured by the ammeter M) is about 0.75 ampere and produces a considerable heating effect. The external resistance of 50 ohms sufficed to keep the current substantially constant in spite of the varying resistance of the wire. The succession of wire temperatures under these conditions is plotted in curve B, fig. 2 (Pl. VI.).

From curves A and B the curve of gas temperature can be at once deduced. Let θ be the gas temperature, and θ'' the wire temperature given by curve B. Then if C is the current in the wire and R its resistance at the moment, heat is being supplied to the wire electrically at the rate C^2R , and by conduction from the gas at the rate $\lambda(\theta - \theta'')$. Neglecting radiation losses, we have therefore

$$\lambda(\theta - \theta'') = k \frac{d\theta''}{dt} - C^2R, \quad . \quad . \quad . \quad . \quad (1)$$

5.3 ohms, and the current is 0.765 amp. Heat is therefore supplied to the wire by the current at the rate of 3.1 watts. The rate at which the wire is losing heat to the gas along PP' is accordingly $3.1 + 2.16 = 5.26$ watts. This is $\lambda(\theta'' - \theta)$. Along QQ' the rate of loss of heat is 1.61 watts, and this is the value of $\lambda(\theta' - \theta)$. The difference, viz. 3.65 watts, is equal to $\lambda(\theta'' - \theta')$. The mean value of $\theta'' - \theta'$ over the range is 40.5 , whence

$$\lambda = \frac{3.65}{40.5} = .09.$$

This is the rate (in watts) at which the wire loses heat to the gas per degree difference of temperature. The difference of temperature between wire and gas along QQ' is therefore $\frac{1.61}{.09} = 17.9$ C., and the gas temperature at the middle point of the range (615° crank-angle) is 12° C. Thus the point R on the gas curve is found.

The wire temperatures are probably correct within 1° . If all four temperatures were in error to that extent, and if the errors were disposed in the most unfavourable manner, λ would be wrong by .0085, or about 10 per cent. of its value, and the error in the correction $\theta' - \theta$ would be about 3° . It is unlikely that all the temperature errors will combine to produce the maximum error in the result, and probable that this particular gas temperature is correct to within one or two degrees. At other points the gas temperatures are not so accurate; the possible error increases with the slope of curve B, and where that curve departs greatly from a straight line a smaller time interval must be taken to get the slope. Thus at the end of the compression or the middle of the expansion strokes, when the correction amounts to nearly 100° , the error may amount to 10 or 15 degrees.

The pressure in the cylinder throughout the cycle is shown by the dotted curve in fig. 2. The pressure was obtained from indicator diagrams and (during the suction and exhaust strokes) by a water-pressure gauge which could be momentarily opened to the cylinder at any point in the cycle.

Several points of interest appear in a study of the curve of gas temperatures, and a comparison of it with the pressure curve. We may discuss these in order, beginning with the suction stroke [$540^\circ - 720^\circ$ crank angle]. During the suction stroke the pressure in the cylinder falls considerably below atmospheric; but the work done on the air in forcing it through the constricted opening of the inlet-valve is almost wholly

absorbed in heating it; for though the motion of the air in the cylinder during this part of the cycle must be very turbulent, the velocities cannot be such as to account for any considerable portion of the energy expended. The temperature of the stream of air just after it has passed the inlet-valve, and has spread out so that its velocity is greatly reduced, will therefore be very nearly equal to the external temperature; and it might be expected at first sight that a thermometer placed anywhere in the cylinder during the suction stroke would show that temperature. As a matter of fact, it will be seen from an inspection of the curve that the gas temperature shown by the thermometer falls in the middle of the stroke to about 10°C. , that is some 7° below the temperature of the air in the inlet-pipe. Of this fall there cannot be any doubt, as it was confirmed on a number of different occasions. I think that it is probably due to the fact that the pressure at the middle of the suction stroke is lower than at the beginning, so that the air first drawn in has its temperature lowered by rapid expansion after it has entered the cylinder and has been reduced nearly to rest. Towards the end of the suction stroke the reverse process occurs, since the air which entered at the middle of the stroke has had its pressure increased more or less adiabatically from about 11.3 lbs. to 14.7 lbs. per square inch. It will be seen that the temperature shown by the wire at this point is about 34°C.

The mean suction pressure exerted on the piston during the suction stroke is about $2\frac{1}{2}$ lbs. per sq. in., and the work done by the piston is 360 foot-lbs. per cubic foot of stroke volume. The pressure at the end of the suction stroke is very nearly atmospheric, but the air in the cylinder is then a little hotter than it was before it entered the cylinder, because some of the work done by the piston in suction has gone to increase its internal energy. The volume displaced by the piston exceeds the volume of air (reckoned at atmospheric temperature and pressure) which has entered the engine. by the increase of volume due to the rise of temperature, and work is done on the external atmosphere in the course of suction to an amount equal to such increase of volume multiplied by the atmospheric pressure. It is easy to see from this that the rise of temperature of the air in the cylinder (assuming that it receives no heat) must be equal to the work done upon it in drawing it in divided by its capacity for heat at constant pressure. Taking the temperature as 17°C. , the latter is about 24 ft.-lbs. per cubic foot, and the temperature at the end of the suction stroke should therefore be 14° above the inlet

temperature, or 31° *. This agrees pretty well with the temperature shown by the thermometer at this point, that is 34° . It is not to be expected that the agreement will be absolute because the air will have received some heat from the cylinder-walls, which have a temperature of 40° C.

The inlet-valve closes at crank-angle 25° (25° after the out-centre). The pressure is then almost exactly atmospheric, the temperature calculated from the wire thermometer is 37° C., and the volume is 1.45 cubic feet. From these data the mean temperatures can be calculated from the indicator diagrams at any point of the compression and expansion strokes until the exhaust-valve opens. It will be useful to compare the temperatures so calculated for one or two points with those given by the thermometer (curve C). At the in-centre (crank-angle 180°), the pressure reaches its maximum value of 173 lbs. absolute. The volume is then 0.236 cubic feet, and the mean temperature calculated from the pressure and volume is

$$\frac{173}{14.7} \times \frac{0.236}{1.45} \times 310 = 590^{\circ} \text{ Absolute, or } 317^{\circ} \text{ C.}$$

The temperature obtained from the thermometer is 320° C., but the correction here is about 100° , so that the possible error is considerable. The temperature near the wire is certainly above the mean, since the gas is losing heat to the cylinder-walls, and the layer in contact with the walls must be colder than the remainder. If we suppose that the gas in the neighbourhood of the wire has been compressed without loss of heat, the temperature there will be

$$\left(\frac{173}{14.7} \right)^{\frac{\gamma-1}{\gamma}} \times 310 = 628^{\circ} \text{ Absolute, or } 355^{\circ} \text{ C.}$$

The gas-temperature curve during expansion is, as it should be, nearly symmetrical with the compression portion. The pressures in expansion are slightly less than at the corresponding points in compression by reason of the loss of heat to the cylinder-walls; but the consequent differences of temperature are too small to be certainly indicated, especially in

* If θ be the rise of temperature the volume of air drawn in per stroke is $\frac{290}{\theta+290}$ of the stroke volume, since the absolute temperature in the inlet-pipe is 290° . The work done is 360 foot-pounds per cubic foot of stroke volume, or $\frac{360 \times 290}{\theta+290}$ per cubic foot of air drawn in. We have therefore $\frac{360 \times 290}{(\theta+290)24} = \theta$, whence $\theta = 14^{\circ}4$.

the latter parts of the stroke where the possible error is very great. The wire temperature (curve B) reaches a maximum at crank-angle 210° C. (30° after in-centre). The wire temperature must here be equal to that of the gas since there is no transfer of heat; it is 263° C. The pressure at this point was found to be 104.5 lbs. per sq. in. absolute, and the volume is 0.333 cubic feet. The mean temperature is therefore

$$\frac{104.5}{14.7} \times \frac{0.333}{1.45} \times 310 = 506^{\circ} \text{ Absolute, or } 233^{\circ} \text{ C.}$$

The temperature calculated on the assumption of adiabatic compression from 14.7 lbs. to 104.5 lbs. is 542° abs. or 269° C. The temperature given by the wire again falls, as it should do, between the two limits.

The exhaust-valve opens at crank-angle 310° (50° before out-centre). The pressure falls to near atmospheric at the out-centre, and then rises again during the exhaust stroke. At the out-centre the temperature shown by the wire is 40° , but is subject to a considerable possible error since the wire is still hot from the compression and is cooling rapidly. During the exhaust stroke the pressure rises, and there is a corresponding rise of temperature which continues till near the end of the stroke. The inlet valve opens at crank-angle 515° (25° before the in-centre) and provides an additional exit for the air. At the same time, the piston is slowing down and there results a rapid drop in the pressure, and a corresponding fall in temperature due to the expansion.

It appears therefore, that when corrected for lag in the manner described, the temperature of the wire gives that of the gas in its neighbourhood at least as accurately as the latter can be obtained from the indicator diagram. The corrections are, however, very large at certain points, and the temperature correspondingly uncertain; the possible error amounting sometimes to as much as 20° C. This large error is mainly due to uncertainty in the shape of the wire-temperature curve, which gives rise to errors in the final gas-temperature greatly exceeding the errors of measurement in the wire-temperature. It is doubtful whether the possible error can be reduced except by using a much smaller wire, with a correspondingly smaller correction for time-lag*. There was no

* Even with the somewhat rough methods of measurement that I used, accidental fluctuations in the wire-temperature were quite obvious—due, no doubt, to the eddying motion of the air in the cylinder. Such fluctuations will, of course, appear whatever the size of wire, but will introduce greater uncertainty, the larger the wire. More accurate resistance measurement would not eliminate them.

reason in these particular experiments why the wire should not have been $\frac{1}{1000}$ inch diameter instead of $\frac{4}{1000}$ inch. The larger size was chosen because it was intended ultimately to use it for measuring the suction temperatures when the engine worked in the ordinary way, taking in and firing a charge of gas. But it was found that even this large wire always fused before any observations could be taken. A still thicker wire might, of course, have been used for the purpose, but the correction would then be so great as to make the results valueless. It seems clear that in an engine of this size and compression ratio, it is a matter of much difficulty to get accurate measurements of temperature by the platinum thermometer, when the engine is firing. The temperature in this engine rises in places to 2200°C . after the explosion, and, probably, remains above 1710°C . (the melting-point of platinum) for $\frac{1}{30}$ second. It is therefore not surprising that even fairly thick wires are melted.

This paper may conclude with a short discussion of the value of λ , or the rate at which the wire loses heat per degree difference of temperature between it and the gas. This quantity is plotted in fig. 4 in terms of crank-angle. At the top of the compression λ is three times as great as at the latter end of the exhaust. This is, no doubt, mainly due to the fact that the thermal conductivity of the air increases with the temperature. The increased pressure also probably has some influence, since we have here to do with air in turbulent motion, and λ is not a function of thermal conductivity only. During suction λ is distinctly greater than during exhaust, probably because of the more violent motion of the air in the former operation.

It has sometimes been said in connexion with attempts to measure gas-temperatures by the platinum thermometer, that the rate at which a fine wire loses heat to a gas per unit length is independent of its diameter. This assertion is based on the assumption that convection currents play an unimportant part. In that case the rate of loss of heat will involve the logarithm of the radius of the wire and will not change greatly with that radius, provided it be small compared to the dimensions of the vessel in which the wire is situated. In experiments such as those here described, however, convection is exceedingly important; and the rate of loss of heat is, in consequence, considerably greater for a larger wire. I recently tried placing two platinum wires, one $\frac{2}{1000}$ inch and the other $\frac{4}{1000}$ inch diameter, in a long wooden box 6 inches square, along which a current of air with an estimated velocity of

10 feet per second was flowing. The wires were at right angles to the length of the box and were heated by an electric current, the resistance being measured at the same time. It was found that the 4 mil wire lost heat about 50 per cent. faster than the 2 mil wire at the same temperature. In the case of the larger wire, the rate of loss was .0028 watt per centimetre length for every degree centigrade by which the wire was hotter than the air current at a distance from it, and this relation held good up to a temperature of 600° or so, when the radiation became important. In the suction stroke of the gas-engine the loss is considerably greater, being about $\frac{0.11}{29} = .0038$ watt per centimetre per degree, a result easily explained by the more violent motion of the gas.

How great is the effect of convection is readily seen by comparing the figures just given for the rate of loss with that obtained by calculation from the thermal conductivity of air on the assumption that there is no convection. If a cylindrical wire of radius r , and at temperature θ , be surrounded by a concentric cylinder of radius r_0 at temperature θ_0 , the intervening space being filled with air at rest, the rate of flow of heat from the wire per centimetre length will be

$$2\pi k(\theta_1 - \theta_0) \frac{1}{\log_e \frac{r_0}{r_1}},$$

where k is the thermal conductivity of air. Taking k as .00005, r_1 as $\frac{2}{1000}$ inch, r_0 as 2 inches, we get for the rate of flow 4.15×10^{-5} calories per second, or 1.74×10^{-4} watts. This is only 1/16th of the rate found in a wire of the same size placed in a current of air flowing about 10 feet per second, the dimensions of the surrounding enclosure being of the same order as a cylinder of 2 inches radius.

I have great pleasure in acknowledging the help that I have received in this investigation from Messrs. A. L. Bird and A. R. Welsh, two students at the Engineering Laboratory, Cambridge. They made and reduced the whole of the observations on which the work is based.

Engineering Laboratory, Cambridge,
October 20, 1906.

IV. *On Surface Concentration, and the Formation of Liquid Films.* By S. R. MILNER, D.Sc. (Lond.)*.

THE nature of the mechanism by which solutions of soap and a few other substances are able to form durable films has long been a problem of great interest. It is easy to understand why in pure liquids, where the surface tension is rigorously uniform, lasting films should not exist; for the slightest local disturbance of equilibrium, such as that due to the weight of the films themselves, would make them collapse. A capability of local variation in the tension of its surface is thus seen to be essential for the stability of a film. Marangoni† in 1871 suggested that this capability is due to the presence on the surface of the film of a pellicle, composed of matter having a smaller capillary tension than that of water. By variations in the thickness of this pellicle which immediately ensue as it is pulled over the surface, the tensions will everywhere automatically take up the values necessary to balance all the other forces, and a stable equilibrium will result throughout the film. In 1890 Lord Rayleigh‡ considered the question, and gave a strong support to the theory by showing that the tension of a soap solution, measured less than $\frac{1}{100}$ second after the formation of the surface, approximates to that of pure water—for the formation of a pellicle may be reasonably expected to be a matter of time. The existence of a pellicle (or something equivalent to it) may be looked on as substantiated by this work, but little or nothing is known as regards its nature or method of formation. In the following paper some observations are recorded which, I think, throw a further light on these points and on the mechanism by which the stability of the film is maintained.

Valuable information on the conditions of the surface of solutions may be gained by studying their surface-tension curves in the light of the thermodynamic relations which exist between the surface tension and the concentration of the dissolved substance in the surface. Consider a solution of which the surface and the volume are capable of independent reversible alterations, the latter by means of a semi-permeable partition which separates the solution from pure water. Let s be the area of the surface, v the volume, τ the surface tension, p the osmotic pressure of the solute. The work done on the system by increasing the area by ds at constant volume is τds , that on increasing the volume by dv the

* Communicated by the Author.

† *Vide* Rayleigh, Proc. Royal Soc. xlvii. p. 281 (1890).

‡ *Loc. cit.*

surface being kept constant is $-p dv$. If these two reversible operations be performed successively, the final result is independent of the order in which they are performed, and by the second law of thermodynamics the same must be the case of the work done in reaching it. Consequently

$$\tau ds - \left(p + \frac{dp}{ds} ds \right) dv = -p dv + \left(\tau + \frac{d\tau}{dv} dv \right) ds,$$

and therefore

$$\frac{d\tau}{dv} = - \frac{dp}{ds}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(1) shows that the surface tension will vary with the volume of the solution (*i. e.*, with the concentration) only when the osmotic pressure depends on the surface. In order that this should be the case, we must suppose that in the thin surface-film which is the seat of the capillary forces, the concentration of the solute is different from what it is in the interior of the solution. In the case of an excess of concentration in the surface, any increase in the area will result in drawing away from the interior a certain amount of solute which was previously operative in producing osmotic pressure; with a defect the opposite will be the case. The actual concentration in the surface-film is indeterminate, in the absence of any knowledge as to how it varies with the distance away from the surface, but the whole excess of solute associated with the surface may readily be calculated from equation (1). Let σ be the "surface excess," *i. e.* the number of gram-molecules of the solute associated with each sq. cm. of the surface which are drawn out from the interior and made ineffective on the osmotic pressure. If N is the number of gram-molecules of the solute originally dissolved, the concentration in the interior, on which variable alone both the surface tension and the osmotic pressure depend, will be

$$c = \frac{N - \sigma s}{v}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Changing to the concentration as the variable by (2), we have

$$\frac{d\tau}{dv} = \frac{d\tau}{dc} \frac{dc}{dv} = - \frac{c}{v} \frac{d\tau}{dc}$$

and

$$\frac{dp}{ds} = \frac{dp}{dc} \frac{dc}{ds} = - \frac{\sigma}{v} \frac{dp}{dc}.$$

Hence the surface excess is given by the equation

$$\frac{d\tau}{dc} = -\frac{\sigma}{c} \frac{dp}{dc} \quad \dots \quad (3)$$

By this equation σ may always be calculated when the curve of the surface tension with the concentration is known.

If the osmotic pressure obeys the ordinary gas law, $\frac{dp}{dc} = RT$; but whether this is the case or not, it may always be obtained from freezing-point measurements. If i stands for the ratio of the molecular lowering of the freezing-point (lowering per gram-mol. per litre) to the normal lowering 1.86, $p = iRTc$, and consequently

$$\frac{dp}{dc} = iRT + RTc \frac{di}{dc}.$$

When, as is usually the case, the variation of i with the concentration is small, the second term of this equation may be neglected in comparison with the first, and we have approximately from (3)

$$\frac{d\tau}{dc} = -iRT \frac{\sigma}{c} \quad \dots \quad (4)$$

In all inorganic solutions so far measured the surface tension increases linearly with the concentration, that is $\frac{d\tau}{dc}$ is positive and constant; hence σ is negative, and $\frac{\sigma}{c}$ is constant, *i. e.*, there is a *defect* of salt in the surface-film of an amount proportional to the concentration.

As regards the actual magnitude of the defect, sodium chloride may be cited as an average case. The increase of surface tension* is 1.72 dynes per cm. for an increase in concentration of 1 gm.-mol. per litre. Also $i = 1.74$, taking an average value of the factor, $R = 8.32 \times 10^7$ ergs per degree C., $T = 290^\circ$. Substituting these numbers in (4) we obtain for a normal solution,

$$\sigma = -4.09 \times 10^{-11} \text{ gm.-mol. per sq. cm.}$$

This is equivalent to a defect of .024 mgm. per sq. metre.

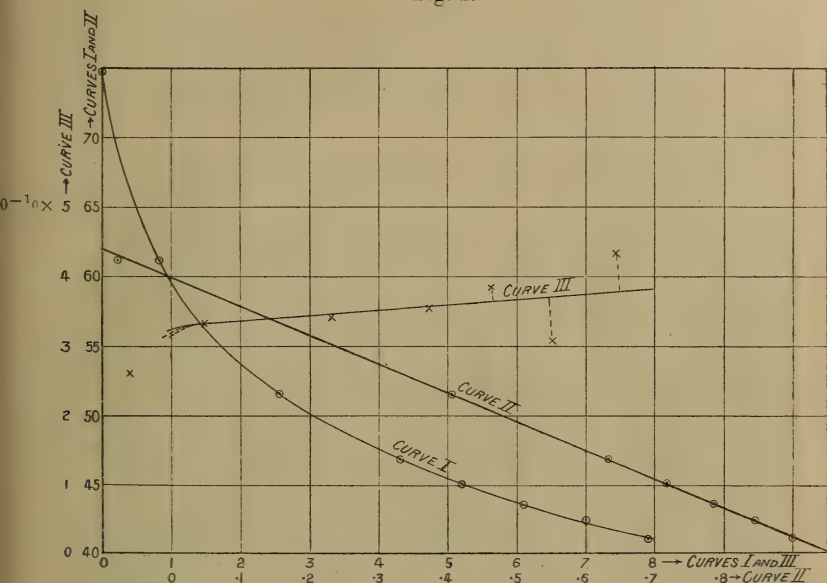
On the other hand, in several organic solutions the surface tension is less than that of water, and there is consequently an excess of solute in the surface. Acetic acid, the surface tension of which has been measured by Whatmough†, forms

* Forch, *Ann. der Physik*, xvii. p. 744 (1905).

† *Zeit. für Phys. Chem.* xxxix. p. 166 (1902).

an instructive example. Whatmough's results, plotted with the concentration, are shown in curve I. (fig. 1). The curve

Fig. 1.



Curve I. Surface Tension (dynes/cm.) with concentration (gm.-mol./litre).
 " II. " " " " " logarithm of concentration.
 " III. Surface Excess (gm. mol./sq. cm.) with concentration.

is very approximately logarithmic, as is better seen in curve II., where the same results are plotted with the logarithms of the concentrations. In acetic acid the molecular depression of the freezing-point, throughout the range of concentration included in the curves, is very approximately constant and has the value 2° per gm.-mol. per litre. The factor i is thus 1.08, and consequently

$$\sigma = -\frac{1}{1.08 RT} c \frac{d\tau}{dc} = -\frac{1}{1.08 RT} \frac{d\tau}{d \log_e c}.$$

The second form of this equation shows that if the surface tension vary linearly with the logarithm of the concentration, the surface excess will be constant. Curve II. shows that this must be very nearly the case in acetic acid. There is perhaps a slight curvature in the direction which implies that σ increases a little with the concentration, as shown in curve III., which represents its magnitudes at the different concentrations as determined from the differences of the

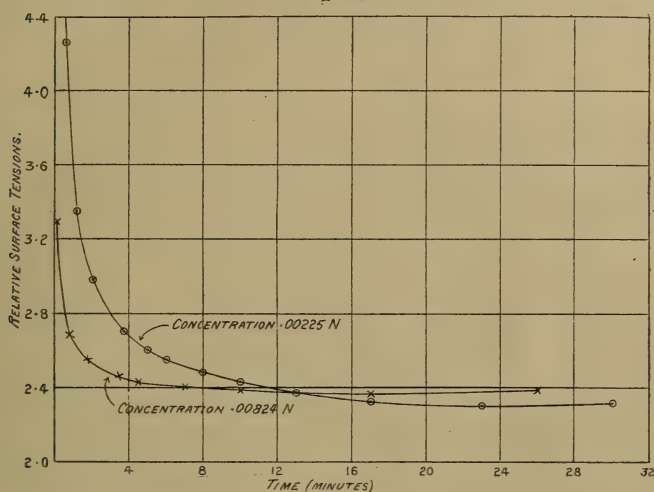
successive experimental numbers. The value is 3.3×10^{-10} gm.-mol. per sq. cm., or .20 mgm. per sq. metre in a normal solution, which is nearly ten times as much as the corresponding defect in salt solutions; and it alters by less than 15 per cent. even when the concentration is increased eight-fold. At concentrations lower than normal the curve becomes uncertain; but it is evident that, as the solution becomes more dilute, the ratio of the excess to the concentration must ultimately become very large.

It is interesting to observe that very dilute solutions of acetic acid will give a lasting foam, but that strong solutions will hardly foam at all. We may infer that the question is one of the relation of the surface excess to the concentration, and the consequent difference in its rate of formation by diffusion of the solute into the surface in strong and weak solutions. In strong solutions, on the surface excess being pulled away from any portion of a film by a local inequality of the forces, it may be formed again so quickly that the surface tension remains practically unchanged, and the film thins rapidly and collapses as in a pure liquid. In dilute solutions the excess must take an appreciable time to form, and the life of the film is correspondingly increased.

Sodium Oleate Solutions.—Here it is more difficult to determine the surface excess from the tension curve. Marangoni has shown that the surface tension, while very much less than that of pure water, is practically independent of the concentration, and indeed based on that fact an argument for the existence of the pellicle. For the calculation of the excess a knowledge is required of the rate of diminution of the tension with the concentration. I consequently made a series of measurements, and continued them to as low concentrations as possible, to see if there were any portion of the curve at which $\frac{d\tau}{dc}$ might be obtained. The solutions were contained in a U-tube, having one large and one capillary limb, the difference in the heights of the liquid in the two limbs being measured with a cathetometer. By forcing a little air into the wide limb, the liquid could be expelled over the top of the capillary and a perfectly fresh surface formed at any instant. It was found that the surface tension, after a fresh surface was formed in this way, always varied with the time, falling rapidly at first and afterwards more slowly until it reached a final value independent of the time. Fig. 2 shows two specimen curves of the variation in solutions of different concentrations. In moderately strong solutions the fall was initially very rapid, the tension reaching within

1 per cent. of its final value in one or two minutes, but in the dilutest solutions the fall extended over many hours. The

Fig. 2.



final value, when once obtained, was quite definite (so long as the surface was not disturbed); but on repeating the time-curves by making a fresh surface in the capillary without altering the other conditions, erratic differences of as much as 5 per cent. in the ultimate values obtained were often met with. Before reaching its final value the tension almost invariably passed through a slight minimum,—this is only just perceptible in the curves, but was very obvious in the measurements with the cathetometer.

The following numbers show the relative values of the surface tension ultimately reached in the different concentrations obtained in one series of measurements. They represent the differences of the heights of the liquid in the two tubes in cms. multiplied by the densities of the solutions.

Concentration.	Relative Surface Tension.
0.2 normal	2.330
.066	2.300
.00824	2.275
.00225	2.305
.00206	2.240
Water.	6.15

These numbers show no trace of any finite value for $\frac{d\tau}{dc}$ down to a concentration of .002 normal, and would in themselves seem to negative the idea of a surface excess. On the other hand, the very large negative, and probably infinite value which $\frac{d\tau}{dc}$ must possess in the neighbourhood of zero concentration, renders it probable that an excess of considerable magnitude exists even in the dilutest solutions. This conclusion is supported by the way in which in the different solutions the tension varies with the time. The time-curves suggest that the surface excess is so large, that in dilute solutions the diffusion of sufficient oleate into the surface to form it takes an appreciable time, and that the fall in the tension is proportional to the amount of oleate which has diffused into the surface in the interval*.

The theoretical method for the determination of the surface excess thus fails to give a result; but I have verified its existence experimentally, and made a rough estimate of its value, in the following way. A current of air, saturated by previously bubbling through water, was passed through a number of fine holes into a beaker about two-thirds full of a solution of sodium oleate. The foam rising to the top of the beaker was periodically removed, and the resistance of the solution was measured *in situ* at intervals during the experiment. In dilute solutions the resistance was always found to perceptibly increase after bubbling for about an hour's time—indicating that the oleate is removed from the interior by the continual re-formation of the surface. By measuring the rate of increase of the volume of the foam and the average size of the bubbles, and by assuming that each bubble as it forms removes a portion of the surface equal to its sectional area, the total area of the surface removed in a given time can be readily, if only roughly, estimated. Also, from the alterations in the concentration determined from the electrical measurements, the amount of oleate removed from the interior of the solution by the foam is known. Thus the surface excess, which is the amount removed per sq. cm. of fresh surface formed, is simply calculated. The following table shows the results obtained:—

* The curves roughly agree with those calculated on this supposition, but the effects are so much complicated by the alterations of the surface as the liquid moves in the tube, that I have not been able to make an accurate calculation.

Diameter of bubbles.	Volume of foam (litres).	Surface removed (square metres).	Resistance, corrected for temperature.	Concentration (gm.-mol. per litre).	Oleate removed (gram-mol.).	Surface Excess (gm.-mol. per sq. cm.).
·5 cm.			1374 ω	·002056		
	29·8	9·0	1456	·001940	$1·17 \times 10^{-5}$	$1·30 \times 10^{-10}$
	42·6	12·8	1467	·001925	1·34 "	1·05 "
	57·0	17·1	1509	·001872	1·86 "	1·09 "
·15 cm.			1171	·002220		
	4·45	4·5	1206	·002156	$0·64 \times 10^{-5}$	$1·42 \times 10^{-10}$
	11·16	11·2	1237	·002102	1·18 "	1·06 "
	17·24	17·2	1273	·002042	1·78 "	1·03 "

Mean volume of solution, 100 c.cs.

The mean result— $1·2 \times 10^{-10}$ gm.-mol. per sq. cm., or ·4 mgm. per sq. metre—is doubtless a moderately good estimate for the surface excess at the moment of formation of the bubbles, but it must necessarily be very much less than the ultimate value of the excess. The curves of fig. 2 show that at the concentration of these experiments the excess is not fully formed until after the lapse of $\frac{1}{4}$ hr., while in the bubbling the whole surface of the solution in the beaker was renewed in about 2 seconds,—so that only a very small fraction of the excess can have been formed.

The existence of a surface excess having been confirmed experimentally, a difficulty arises with regard to the theoretical interpretation of the surface-tension curve. According to the thermodynamical equation (4) an excess of the magnitude found should cause a very considerable diminution of the tension with the concentration; but, as a matter of fact, the ultimate values of the tension are practically independent of the concentration. Since the application of the thermodynamical reasoning is limited only by the reversibility of the processes concerned, we are driven to the conclusion that the formation of the surface excess in oleate solutions is not a reversible process. This idea is in conformity with the conclusion already arrived at, that the excess tends to be formed however dilute the solution; for this means that the reverse action (passage of the excess from the surface into the interior) is very small, if not actually zero. This leads us to expect that, since the surface cannot take up an indefinite amount of the oleate, it will ultimately in finite concentrations become saturated, when irreversibility will be produced by the continuous precipitation of the oleate into another phase. Several observations go to support this inference:—(1) It is well known that an insoluble scum is formed on the surface of

sodium-oleate solutions to an extent which increases with the time: this may well be the solid phase which causes the irreversibility. (2) If the surface of a soap-bubble, which has thinned so as to exhibit the reds and greens, is carefully watched as the bubble is allowed to decrease in size, it will be found to suddenly become covered with a large number of minute white flakes. These flakes are not formed in a film of constant size, except at places towards which the surface is being pulled by local want of balance of the tensions. On the above view, we should expect the precipitate to be formed in such cases, where, but for its occurrence, supersaturation* of the oleate in the surface would be produced. (3) In the experiments on the surface tension, in dilute solutions, on sucking the liquid down in the capillary it is brought back sharply to its previous position; but after pressing it up, it comes back only very slowly in accordance with the time-curve of fig. 2. This shows that while the tension increases when the surface is increased, it does not decrease, but even tends to increase, when the surface is diminished. Thus, whatever the cause of the want of reversibility may be, there can be little doubt as to the fact of its existence.

In the rest of this paper the above conceptions are applied to some results on the conductivity of soap-films obtained by Messrs. Reinold and Rücker, and to the explanation of the manner of thinning of films, and of the formation of the black spot.

Conductivity of Films.—Reinold and Rücker† measured the true, and what they called the “electrical” thickness (*i. e.*, the thickness calculated from the conductance on the assumption that the conductivity is the same as that of the liquid in mass) of a series of films formed from a solution containing 1 part of sodium oleate to 60 parts of water. They found that the electrical thickness was always greater than the true thickness measured optically; and from their results we may determine approximately the magnitude of the surface excess at different thicknesses of the film.

On the assumption that the conducting-power per molecule of the surface excess is the same as that of the rest of the solution, the conductance of a film of thickness t and concentration c and possessing an extra number σ of gram-molecules per sq. cm. of each surface, will be proportional to

* The minimum value of the surface tension and the variability of its ultimate value in the same solution described on p. 100 may be easily accounted for as due to a slight supersaturation of the surface.

† Phil. Trans. clxxxiv. p. 505 (1893).

$ct + 2\sigma$. The electrical thickness, t_e , was calculated by Reinold and Rücker on the assumption that the conductance was proportional to ct_e . Consequently,

$$ct + 2\sigma = ct_e;$$

and if $r = \frac{t_e}{t}$, the ratio of the electrical to the actual thickness,

$$\sigma = \frac{1}{2} ct(r-1).$$

In the experiments c was $\cdot 0164$ gm. per c.c.; thus, t being measured in micromillimetres,

$$\begin{aligned}\sigma &= \cdot 0082 \times 10^{-7} t(r-1) \text{ gm. per sq. cm.,} \\ &= \cdot 0082 t(r-1) \text{ milligrams per sq. metre.}\end{aligned}$$

The following table shows the values of the excess at different thicknesses, calculated from this equation. The experimental numbers are taken from tables xiv., xv., and xvi. of their paper, and arranged in order of decreasing thickness of film.

Optical thickness. $\mu\mu$.	Ratio of Electrical to Optical thickness.	Surface Excess (mgm. per sq. metre).	Ratio recalculated.
641	1.66	3.46	1.46
484	1.69	2.73	1.61
388	1.60	1.91	1.75
332	1.81	2.21	1.87
320	1.78	2.05	1.91
314	2.07	2.75	1.93
314	1.90	2.32	1.93
307	1.84	2.12	1.95
301	1.84	2.08	1.97
296	1.98	2.38	1.99
97	4.47	2.75	4.02
97	4.19	2.53	4.02
		Mean 2.4	
27.7	5.8	1.09	

These results show that the surface excess remains practically constant down to a thickness of $97 \mu\mu$. The first three numbers, it is true, show fairly large differences among themselves, but this is due to the experimental error being here a good deal magnified in the calculation; and an examination of the fourth column, which contains the theoretical ratios calculated using the mean value, 2.4 mgm. per sq. m., of the excess, shows that the differences between them and the experimental ratios nowhere exceed the possible errors of these

very difficult measurements. The mean value is several times larger than that obtained from the experiments on bubbling. This might have been anticipated, since, as was there pointed out, only a small fraction of the final excess can have been formed in those experiments.

The table shows further that in the black film of 27.7μ thickness, a reduction to less than half its value in thick films has occurred in the excess. This is, I think, a point of considerable importance in the explanation of the formation of the black film, and is considered more fully on p. 107.

Thinning of Films.—According to the theory of the formation of a pellicle, the surface of a vertical soap-film is in a state of statical equilibrium, and the thinning is due to the gradual draining down of the layer between the two surfaces impeded by the ordinary fluid viscosity of the liquid. In view of the continual re-formation of the surface excess when the surface is renewed, another cause will be seen to be operative in the thinning. Consider a horizontal film suddenly turned into a vertical position. In order for the film to be in equilibrium, the tensions at different levels must be different by the weight of the intervening film. This not being initially the case, the surface, and with it a portion of the excess which it contains, will be pulled down by the tension from higher to lower levels. The surface tensions of the higher levels will be thus increased until equilibrium results. This increase in the tension is, however, only temporary, since the excess removed from any portion of the surface is continually being replaced by diffusion of the oleate into the surface from the interior. But this will be similarly removed, and thus a continual circulation over the surface will be produced, the rate of which, other things being equal, will depend on the rate at which the surface excess is capable of being formed. Both circulation and draining will assist in thinning the film, and it is difficult to say *a priori* which of the two will be the more potent cause. We may, however, infer that, since the rate of formation of the surface excess is decreased by a decrease in the concentration of the solution, the same will be the case for the rate of thinning by circulation over the surface; so that the extent to which this cause is operative may be gauged by the extent to which the rate of thinning of films depends on the concentration. I made some experiments on this point by measuring the time required by films, formed in the same way from sodium oleate solutions of different strengths, to undergo a given degree of thinning. The films were formed on a rectangular frame of glass fibre, 2 cms. wide by 4 cms. high, having the handle attached at the bottom, and the whole

being so arranged that the framework could be raised out of the solution without removing a glass shade which covered the whole apparatus.

Concentration.	II.	III.
·20 normal,	15·1 secs.	12·7 secs.
·07	18·3	
·02	19·8	17·3
·007	20·9	
·002	21·0	19·8

In the above table the numbers in column II. are the times after the formation of the film at which the green of the 6th order (thickness $1100\ \mu\mu$) had reached a point 2 cms. from the top of the frame; those in column III. the times at which the blue of the 2nd order ($250\ \mu\mu$) had reached a point ·3 cm. from the top; they are each the mean of about ten measurements, which throughout nowhere differed among themselves by more than 1 second. The table shows that the time required for a given degree of thinning increases in a marked way with the dilution of the solution; and we may consequently infer that circulation over the surface is an appreciable factor in producing the thinning.

The circulation must, however, cease when the whole of the oleate has been removed from the interior of the film. The surface excess will then everywhere take up a *steady* value of such a magnitude as will enable the tensions to remain in statical equilibrium throughout the film; this point, however, does not appear to be reached—unless perhaps in very dilute solutions—before the black spot becomes formed.

Black Films.—Many observers have endeavoured to trace the effect on the surface tension of the approach to molecular magnitudes which films make as they thin; but the results have always shown that there is no appreciable difference in the tensions of thick and of the thinnest black films. Any other result would indeed be difficult to understand from the point of view of the equilibrium of the film introduced by Marangoni; but it is to be observed that it by no means indicates that a thickness of twice the radius of molecular action has not been reached and passed. If a film of pure water could be obtained with a thickness less than twice the radius of molecular action, its tension would doubtless be less than that of a thick water-film; but the tension of a soap-film

of the same thickness will depend on both the thickness and the magnitude of the surface excess. While a diminution of the thickness will tend to decrease the tension, a diminution of the excess will tend to increase it; and down to a certain limiting thickness there will always be a possible value of the excess which will enable the tension to keep uniform.

Now from the results on p. 105 for the value of the surface excess calculated from the conductivity of films, we must conclude that it is considerably less in black films than in thick ones. Another argument for the same thing may be adduced in the fact that when the black spot is formed its edge is (in my experience) always surrounded by a great many of the little white flakes, the formation of which as a precipitate formed from a supersaturated solution has already been adverted to. On this view they form a visible piece of evidence of oleate removed from the black spot. We must consequently conclude that the tension so far as it depends on the thickness has been diminished, and that in black films a thickness less than the range of molecular action has been reached. It would be interesting to know, even if only roughly, what the tension of a pure water-film of the same thickness would be. If we assume, as is suggested by the approximate agreement of the time-curves for the surface tension with diffusion curves, that the diminution of the surface tension of oleate solutions below that of pure water is proportional to the actual amount of the surface excess, an estimate of the value may be made from the results of Reinold and Rücker's conductivity experiments. On this assumption, if τ is the surface tension of a water-film, that of a soap-film will be $\tau(1-k\sigma)$, where σ is the magnitude of the surface excess; and since both in thick films and thin ones the tension must have the same value, 27.5 dynes per cm., we must have, for a thick film, where τ is 74 dynes per cm., and σ is 2.4,

$$27.5 = 74(1 - k \times 2.4),$$

and for a black film of $27.7 \mu\mu$ thickness, where $\sigma = 1.09$,

$$27.5 = \tau(1 - k \times 1.09).$$

This makes $\tau = 38.5$, only about half as great as the normal value. Although no great stress can be laid on the actual number, it is evident from the calculation that a thickness of $27.7 \mu\mu$ is well within the limits of the range of molecular action.

There remain outstanding the sudden formation, and the discontinuous thickness at the edge, of the black spot,—phenomena which have long been recognized as an indication

that the thickness is approaching molecular dimensions, but the actual cause of which has remained completely obscure. I have only been able to come across two suggested explanations of the discontinuity. Reinold and Rücker* suggest that it is due to the forces between the molecules becoming repulsive within certain limited regions, whereby the surface tension of the film alternately decreases and increases as the thickness is diminished. On the other hand, Bakker† connects it with the region of instability in the isothermal which represents the passage of liquid into vapour. An objection to both of these explanations is that they neglect to give a part to be played by the soap, although it has an obvious and essential role in the experimental production of the spectacle. It seems to me that an explanation on the following lines is more satisfactory.

Let us suppose (simply to make the argument more definite) that the tension of a thin soap-film of thickness t and surface excess σ is actually represented by the form $f(t)\{1 - k\sigma\}$. It is true that from a mathematical point of view an infinite number of corresponding values of t and σ may exist which will make this function have the same value as in a thick film. We have seen that if in any place the tension is accidentally less than this value, thinning occurs by the extreme surface layer, and with it the excess of oleate which it contains, being dragged away from the place by the surrounding tensions; but when we come to molecular magnitudes the alterations of the thickness and the excess produced in this way cannot be perfectly continuous, but must occur in finite, if very small, steps. Thus we may imagine that the thickness cannot alter by less than that of a layer 1 molecule thick, and similarly the surface excess by the amount of oleate contained in such a layer. But unless the increase in the tension produced by the second of these alterations is exactly of the right amount to neutralize the decrease produced by the first, the portion of the film thus thinned cannot remain in equilibrium with the thicker parts. Hence either the thinning will not take place, or the next layer will be removed simultaneously with the first, when the same considerations will again be applicable. It may not be until after a considerable number of layers of molecules are considered removed, that a point is reached at which the sums of the decreases of the thickness and of the surface excess which correspond to the removal of each layer exactly neutralize each other in their effects on the

* Phil. Trans. vol. clxxvii. p. 627 (1886).

† *Ann. der Physik*, Bd. xx. p. 35 (1906).

tension, and enable the thinned film to form in stable equilibrium with the thicker parts. If this be the case, the practical process of thinning will take place by the sudden removal of a finite thickness from a limited area, forming a spot with a sharply defined edge. From the point of view of this explanation there is no necessity to limit to a single case the number of films, having thicknesses finitely different from each other, which might be capable of being formed in this way. It is well known indeed that black films occur in two different thicknesses; and Stansfield* and Johonnott† have recently drawn attention to the existence of a series of at least three grey films of thicknesses intermediate between those of the white and the black.

The University, Sheffield.
August 1906,

V. *The Velocity and Energy of the α particles from Radio-active Substances.* By E. RUTHERFORD, F.R.S., Macdonald Professor of Physics, McGill University, Montreal ‡.

IN two previous papers §, an account has been given of measurements of the velocity and mass of the α particle expelled from selected products of radium, actinium, and thorium. It has been shown that, within the limit of experimental error, the α particles expelled from each of the substances examined have the same mass but differ in the initial velocity of their projection.

Under the normal conditions of experiment, the α particle carries a positive charge and behaves as if it were a charged atom of matter projected from the radioactive matter at very great velocity. The value of e/m —the ratio of the charge of the α particle to its mass—was found to be 5.07×10^3 . The probability that the α particle is an atom of helium is discussed in detail in one of the papers already mentioned.

The range of ionization in air of the α particles from a thin layer of active matter is a definite and easily measured quantity, which is a characteristic of each individual product. The ranges in air of the radium products have been accurately

* Proc. Royal Soc. March 1906, vol. lxxvii. p. 314.

† Phil. Mag. June 1906, p. 746.

‡ Communicated by the Author.

§ Rutherford, Phil. Mag. Oct. 1906. Rutherford and Hahn, Phil. Mag. Oct. 1906.

determined by Bragg and Kleeman*. Levin† has determined the range of the particles from polonium (radium F), while Hahn‡ has investigated in detail the ranges of the α particles expelled by the various products of thorium and actinium.

The velocities of the α particles in a homogeneous source of rays are all reduced by the same definite amount in passing through a material screen. As a result of the variation of velocity of the α particles from radium C in traversing matter, I have shown § that the velocity V of the α particle having a range r cms. in air after passing through a screen is given by

$$V/V_0 = \cdot 348 \sqrt{r + 1.25},$$

where V_0 is the initial velocity of the α particles emitted from radium C which have a range in air of 7.06 cms.

This relation has been confirmed by direct measurement of the velocities of the α particles from the various α ray products which have different ranges of ionization. The value of V_0 has been determined and shown to be 2.06×10^9 cms. per second. Consequently the velocity V of an α particle which is able to ionize air for a distance of r cms. is given by

$$V = \cdot 717 \sqrt{r + 1.25} \times 10^9 \text{ cms. per second.}$$

This is based on the assumption, which has been verified in a number of instances, that the α particles from the α ray products of the various radioelements have an identical mass.

The initial velocity of expulsion of the α particle from each α ray product is thus a definite constant which, like its period of transformation, serves to distinguish it from all other products.

The velocities of the α particles emitted from the various products are collected in the table given below (p. 112). The period of transformation of the product and the range of its α particles in air are added for comparison. The fourth column gives the kinetic energy of the α particle at the moment of its expulsion. Since the kinetic energy $\frac{1}{2}mu^2$ is equal to $\frac{1}{2} \frac{mu^2}{e}$, *e.*, and the value of $\frac{mu^2}{e}$ is determined directly from the known values of u and e/m , the kinetic energy is expressed in terms of e , the charge (in electromagnetic units) carried

* Phil. Mag. Sept. 1905.

† Amer. Journ. Sci. July 1906.

‡ Phil. Mag. July and Sept. 1906.

§ Phil. Mag. Aug. 1906.

TABLE I.

Product.	Period of transformation.	Range of α particles in air.	Velocity of α particles in cms. per second.	Kinetic energy of α particles.
Uranium	About 10^9 years. 22 days. 1300 years. 3·8 days. 3 minutes. 26 " 19 " 40 years. 6 days. 140 "	About 3·5 cms. (only β rays) .. 3·50 cms. 4·33 " 4·83 " (only β rays) 7·06 cms. (rayless) (only β rays) 3·86 cms. (rayless) (?) 3·9 cms. 5·7 " 5·5 " (rayless) 5·0 cms. 8·6 " (rayless) 4·8 cms. 6·55 " 5·8 " (rayless) 5·5 cms.	About $1·56 \times 10^9$ $1·56 \times 10^9$ 1·69 " 1·76 " $2·06 \times 10^9$ $1·62 \times 10^9$ $1·63 \times 10^9$ 1·89 " 1·86 " $1·79 \times 10^9$ 2·25 " $1·76 \times 10^9$ 2·00 " 1·90 " $1·86 \times 10^9$	About $2·41 \times 10^{14}$, e $2·41 \times 10^{14}$, e 2·82 " 3·06 " $4·21 \times 10^{14}$, e $2·59 \times 10^{14}$, e $2·61 \times 10^{14}$, e 3·52 " 3·42 " $3·17 \times 10^{14}$, e 5·00 " $3·07 \times 10^{14}$, e 3·95 " 3·57 " $3·42 \times 10^{14}$, e
Uranium X ?				
Radium				
Emanation				
Radium A				
" B				
" C				
" D				
" E				
" F				
Thorium	About 2×10^9 years. ? 3·6 days. 54 seconds. 11 hours. 1 hour. Very short.			
Radiothorium				
Thorium X				
Emanation				
Thorium A				
" B				
" C				
Actinium	? 19·5 days. 10·2 " 3·9 seconds. 36 minutes. 2 "			
Radioactinium				
Actinium X				
Emanation				
Actinium A				
" B				

by the α particle. The results, expressed in this form, do not involve any assumption as to the actual value of e .

The ranges of the α particles from all the various substances except uranium have been determined by the electrical method, first employed by Bragg and Kleeman. On account of the feeble activity of uranium, experiments made to determine the range of its α particles by the same method have not yet been successful. Bragg has, however, determined the stopping power of the α particle from uranium in terms of aluminium, and from this it can be deduced that the range of the α particle from uranium is in the neighbourhood of 3.5 cms.

It is of interest to compare the average velocity and energy of the α particle expelled in the successive transformations of the radioelements. The results are tabulated below. The third column gives the total energy of motion of the α particles liberated as a result of the successive disintegrations of a single parent atom. Since for any primary element in equilibrium with its succeeding products, the same number of atoms of each product break up per second, the average velocity or average energy of the α particles for each of the families of products is given by the sum of the velocities or energies divided by the number of products concerned.

TABLE II.

Element.	Average velocity of α particles in cms. per second.	Average energy of α particles in ergs.	Total energy liberated as the result of successive disintegrations of one atom of parent matter.
Uranium.....	About 1.56×10^9	$2.41 \times 10^{14} \cdot e$	$2.41 \times 10^{14} \cdot e$
Radium (including radium F)	} 1.74 "	3.02 "	15.1 "
Radium (excluding radium F)		3.12 "	12.5 "
Uranium (including radium and all its products)	} 1.71 "	2.92 "	17.5 "
Thorium		3.54 "	17.7 "
Actinium.....	1.88 "	3.51 "	14.0 "

We see from the above tables that

- (1) The initial velocities of expulsion of the α particles from the products of the radioelements all lie between 1.56×10^9 and 2.25×10^9 cms. per second, *i. e.* the maximum velocity of expulsion is only 1.44 times the minimum velocity. Of all products, the α particles emitted from thorium C have the greatest velocity, and those from uranium and radium the least velocity.
- (2) The average velocity and average energy of the α particles from thorium and actinium families are very nearly the same, and are greater than the corresponding values for the radium family. The average velocity of expulsion of the α particles from the radium family (excluding radium F) is about 6 per cent. less than that of the α particles from thorium and actinium.
- (3) The total energy liberated in consequence of the successive disintegrations of a radium atom is less than the corresponding value for the thorium atom, but is greater than that for actinium.

Notwithstanding the smaller number of α ray products in thorium, the total energy liberated for the disintegration of one atom of thorium is practically the same as the energy liberated for an atom of uranium, including in the latter radium as its product.

Connexion of the velocity of expulsion of the α particle from a product with the period of transformation.

In considering the possibility of some relation existing between the velocity of expulsion of the α particle and the character of the transformation which accompanies it, it is noteworthy that in a large proportion of the products, the velocity of expulsion of the α particle for each of the radioactive families increases progressively as the period of transformation decreases. Since the period of transformation of a product may be considered to be an inverse measure of the stability of the atoms composing it, this would show that the velocity of the expulsion of the particle is a function of the stability of the atoms and is least for the most stable atoms.

This is well exemplified in the case of the products uranium, radium, radium F, and radiothorium, which are transformed slowly compared with the other α ray products. The particles from uranium and radium have the minimum

range of about 3.5 cms., while those of radium F have the next lowest range, viz. 3.86 cms. A similar result holds in the case of thorium, where the α ray particles from the product of longest period have the least velocity. The period of transformation of radiothorium has not yet been measured, but it is apparently not less than one year, and may be much longer. In any case, the period is much longer than for any of the other α ray products of thorium.

This connexion is more clearly brought out in the table below by arranging the products of each of the radioactive families in the order of increasing velocity of the α particles expelled from them.

TABLE III.

Product.	Period of Transformation.
Uranium	about 10^9 years.
Radium	1300 years.
Radium F	140 days.
Emanation.	3.3 days.
Radium A	3 mins.
Radium C*	19 mins.

Product.	Period of Transformation.	Product.	Period of Transformation.
Radiothorium ...	More than 1 year.	Radioactinium ..	19.5 days.
Thorium B	1 hour.	Actinium B	2 mins.
Emanation	54 secs.	Emanation	3.9 secs.
Thorium X* ...	3.6 days.	Actinium X* ...	10.2 days.
Thorium C.....	Very short.		

The only exceptions to the rule that the velocity of expulsion of the α particle progressively increases with the decrease of period of transformation are the three products marked with an asterisk, viz. radium C, in the case of radium, and thorium X and actinium X, in the case of thorium and actinium respectively.

Assuming the above rule to be valid for all the products,

it is to be expected that the period of transformation of radium C should be less than one minute, while that of thorium X or actinium X should not be more than a few seconds. There is a possibility that further investigation may yet remove these products from the list of exceptions. It is possible that two successive changes may occur, for example, in radium C, the one change a rayless one of period 19 minutes, and other a ray change of very short period. A similar result may hold for thorium X and actinium X. Such possible double changes, where one of the changes is very rapid compared with the other, would be extremely difficult to detect. There is certainly no evidence so far obtained which in any way negatives such a possibility.

It will be seen that, in most cases, the period of transformation decreases very rapidly for a slight increase in velocity of the expelled α particles. I have not succeeded in finding any simple connexion between the velocity of expulsion of the α particle and the period of transformation of the product. Hahn has already drawn attention to the interesting fact that the corresponding products of thorium and actinium occupy the same relative position when the products of each element are arranged in order of increasing velocity of the α particles expelled from them (see previous table).

Rate of emission of energy.

We have seen that the energy of the α particles due to the successive disintegrations of one atom of the parent element is $12.5 \times 10^{14}.e$ ergs for radium (excluding radium F), $17.7 \times 10^{14}.e$ for thorium, and $14.0 \times 10^{14}.e$ for actinium. In a previous paper I have shown that the heating effect of radium is, in great part, a measure of the kinetic energy of the expelled α particles. We may consequently assume with confidence that thorium, uranium, and actinium also emit heat energy in amount proportional to the kinetic energy of the expelled α particles. Until the number of atoms which break up per second is determined accurately for each of these elements, the actual heat emission cannot be calculated with certainty. From measurements of the relative activity of thorium and uranium compared with radium, it can, however, be deduced that the heat emission per gram of radium is about two million times as great as the corresponding value for uranium and thorium.

Since the heat emission of radioactive substances is a measure of the energy of the expelled α particles, radioactive matter however sparsely distributed will emit heat energy

proportional to the quantity of matter present. It is consequently quite valid to assume, for example, that the heating effect of radium distributed in minute quantities through the earth's crust is proportional to the amount of radium present.

General considerations.

We have seen that the velocity of expulsion of the α particles from various radioactive substances all lie within a fairly narrow range, viz. between 1.56×10^9 and 2.25×10^9 cms. per second. In a previous paper I have shown that the α particle loses its power of producing ionization, of acting on a photographic plate and of producing phosphorescence when its velocity falls to about $\frac{1}{4} V_0$, where V_0 is the velocity of the α particles from radium C. The value of V_0 has been shown to be 2.06×10^9 cms. per sec., so that this critical velocity of the α particle is about $.82 \times 10^9$ cms. per sec. It is consequently seen that the velocity of expulsion of the α particles in general lies approximately between two and three times this critical velocity. An α particle escaping below this critical velocity would be difficult to detect and would produce little if any ionization. If the α particles from the active substances had been projected on an average with half their observed velocity, it would have been difficult to detect the presence of the α particles at all.

In the absence of any definite knowledge of the causes which lead to the successive disintegrations of the atom, it does not seem possible at the present time to give any adequate explanation of the modes of transformation observed in radioactive matter. At the same time, there can be little doubt that the data so far accumulated of the character and period of the changes and of the nature and velocity of the particles emitted will ultimately prove of very great value in obtaining a clearer idea of atomic constitution.

A study of radioactive phenomena has emphasized the importance of the α particle as one of the units of which the heavier atoms are built up, and it is not improbable that the α particle may play an equally important role in the constitution of other atoms besides those of uranium, thorium, radium, and actinium.

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November 1, 1906.

VI. *The Impulsive Motion of Electrified Systems.* By
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§ 1. **W**HEN an electrified system is suddenly set into motion, a pulse of electromagnetic disturbance travels outwards from the system in all directions with the velocity of light, and this pulse carries energy and electromagnetic momentum away to infinity. In the following paper, a method is given of calculating the radiated energy and momentum for any given system, and the results are applied to a number of simple systems. The principles of electromagnetism are then applied to deduce the electric and the magnetic energies and the momenta of the systems, when they are in steady motion. Some of the results, as will be indicated in footnotes, have already been published by Dr. Oliver Heaviside and by Dr. M. Abraham, but others are, I believe, now given for the first time.

The method employed in calculating the electric and magnetic forces in the pulse is simply an extension to finite speeds of Prof. J. J. Thomson's investigation † of the pulse given off when the infinitesimal velocity of a charged sphere is suddenly destroyed.

§ 2. When a charged particle has been at rest at a given point O for an infinite time, the lines of electric force are uniformly distributed round the particle. At the time $t=0$, let the particle be impulsively set into motion at a speed u , which is maintained constant in direction and magnitude. Then, at any subsequent time t , the field can be divided into two parts by a sphere of radius vt described about O as centre, v being the velocity of light. The field outside this sphere is the same as if the particle had remained at rest, while the field inside the sphere is the same as if the particle had been in steady motion for an infinite time.

On the other hand, suppose that a charged particle, which has been moving along a straight line, with uniform velocity u , for an infinite time, is suddenly stopped, at $t=0$, as it reaches a given point O. Then, at any subsequent time t , a sphere of radius vt will again divide the field into two parts. The field outside this sphere is the same as if the particle had continued in steady motion, while the field inside the sphere

* Communicated by the Author.

† *Elements of the Mathematical Theory of Electricity and Magnetism.* Third edition (1904), § 287.

is the same as if the particle had been at rest at O for an infinite time.

More generally, suppose that both the direction and the magnitude of the velocity are instantaneously changed so that the velocity of the particle is u_1 from $t = -\infty$ to $t = 0$, and u_2 from $t = 0$ onwards. Then, if a sphere of radius vt be described about O, the point where the impulse occurs, the field outside the sphere is the same as if the particle had continued to move with velocity u_1 , and the field inside the sphere is the same as if the particle had been in steady motion with velocity u_2 for an infinite time.

These results, which will be assumed in the present paper, can be established by the method of the "retarded potential" for points either inside or outside the sphere of radius vt^* . For *external* points they can also be established by Poisson's solution of the general equation $d^2\phi/dt^2 = v^2 \nabla^2 \phi$.

§ 3. A point charge, q , in steady rectilinear motion, produces an electromagnetic field which travels with the charge. In this field the lines of electric force are straight and pass through q , while the lines of magnetic force are circles in planes normal to the direction of motion. If R and ϕ , measured from q as an origin and from the line of motion respectively, be the coordinates of any point, then, as was first shown by Dr. Oliver Heaviside †, the electric force is given by

$$E = \frac{q(1-n^2)}{KR^2(1-n^2\sin^2\phi)^{\frac{3}{2}}}, \quad \dots \dots (1)$$

where u is the velocity of the charge and

$$n = u/v, \quad \dots \dots (2)$$

and K is the specific inductive capacity.

The magnetic force is given by

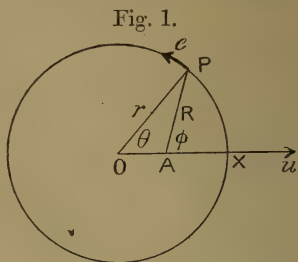
$$H = KuE \sin \phi. \quad \dots \dots (3)$$

The principles stated in § 2, together with these values of E and H , are sufficient for the solution of every problem connected with the motion of charged bodies.

* The most direct proof of the formulæ of the "retarded potential" is perhaps that given by Sommerfeld in his paper "Simplified deduction of the Field and the Forces of an Electron moving in any given way." Proceedings of the Koninklijke Academie van Wetenschappen te Amsterdam, Nov. 26, 1904.

† 'The Electrician,' Dec. 7, 1888, p. 148, or 'Electrical Papers,' vol. ii. p. 511.

Suppose that O (fig. 1) is the position of q at the time $t=0$ and A its position at any later time t , the velocity between O and A being constant and equal to u . Let P be any point on the sphere of radius $r=vt$, described about O as centre, and let the axis OA cut the sphere in the pole X. Then $AP=R$ and $PAX=\phi$, while $OA=ut=nr$.



Let $POX=\theta$. Then r and θ are the coordinates of P relative to O, while R and ϕ are its coordinates relative to A. If E_N be the electric force at P along OP, we have, by (1),

$$E_N = \frac{q(1-n^2)R \cos(\phi-\theta)}{KR^3(1-n^2 \sin^2 \phi)^{\frac{3}{2}}}.$$

But

$$R \cos(\phi-\theta) = OP - OA \cos \theta = r(1-n \cos \theta),$$

$$R \sin \phi = r \sin \theta,$$

and

$$R^2 = r^2(1+n^2-2n \cos \theta)$$

Hence

$$R^2(1-n^2 \sin^2 \phi) = r^2(1-n \cos \theta)^2.$$

Thus

$$E_N = \frac{q(1-n^2)r(1-n \cos \theta)}{Kr^3(1-n \cos \theta)^3} = \frac{q(1-n^2)}{Kr^2(1-n \cos \theta)^2}. \quad (4)$$

If the particle was at rest at O until $t=0$, the electric force just outside the sphere $r=vt$ is radial to O and normal to the sphere and has the value

$$E = q/Kr^2. \quad (5)$$

At the surface of the sphere there is therefore a discontinuity in the normal component of the electric force, and, since there is no charge on the surface, this discontinuity can arise only from a flux of electric displacement along the surface. By symmetry this flux must be along lines of longitude.

Let E be the tangential electric force along the lines of longitude of the sphere $r=vt$ and let

$$e = \int_0^p E dz, \quad (6)$$

where the integral is taken over the infinitesimal thickness, p , of the surface of the sphere, and dz is an element of length measured along p . In other words, $Ke/4\pi$ is the surface flux of

electric displacement across unit length of a line of latitude. When E and e are positive they are directed away from the pole X .

Now consider the spherical cap formed by the revolution of the arc PX (fig. 1) about OX . The flux of displacement out from this cap by its edge and by its convex surface is

$$(Ke/4\pi) \cdot 2\pi r \sin \theta + \frac{1}{2}q(1 - \cos \theta).$$

By (4) the flux of displacement into the cap by its concave surface is

$$(K/4\pi) \int_0^\theta E_N \cdot 2\pi r^2 \sin \theta \, d\theta,$$

or
$$\frac{1}{2}q(1 - n^2) \int_0^\theta \frac{\sin \theta \, d\theta}{(1 - n \cos \theta)^2},$$

or
$$\frac{q(1+n)(1 - \cos \theta)}{2(1 - n \cos \theta)}.$$

Equating the inward to the outward flux, we find

$$Ker \sin \theta = q \left\{ \frac{1+n}{1-n \cos \theta} - 1 \right\} (1 - \cos \theta),$$

and hence

$$e = \frac{q \sin \theta}{Kr(1 - n \cos \theta)} = \frac{qu \sin \theta}{Kr(v - u \cos \theta)}. \quad \dots (7)$$

This result is of great importance since all the formulæ for the radiation arising from the impulsive or gradual change of velocity of an electrified system can be deduced from it*.

Since e is at right angles to the plane of r and u we can express it in the vector form

$$e = \frac{q}{Kr} \frac{Vr_1 Vr_1 u}{v - ur_1} = \frac{q}{Kr} \frac{r_1(ur_1) - u}{v - ur_1}. \quad \dots (8)$$

Here **heavy type** denotes vectors, and r_1 is a unit vector in the direction of r . Also ur_1 is a scalar and $Vr_1 u$ a vector product.

§ 4. In the expanding spherical wave the disturbance is propagated at right angles to the electric force with velocity c , and hence the associated magnetic force, which is at right angles both to the electric force and to the direction of propagation, is given by the equation

$$H = cKE,$$

which holds in every case of propagation of electromagnetic waves.

* This result was first given by Dr. Heaviside, 'The Electrician,' Oct. 11, 1901.

In practical cases, however, the electrification is not concentrated in ideal point-charges, but is spread over surfaces or distributed through volumes. We therefore take a second plane GG' , parallel to FF' and at a distance dz from it, and a corresponding point g on OP . Then, at the time t , the only points on OP , which experience any wave disturbance from the electrification dQ contained between the planes FF' and GG' , are the points between f and g , and, further, the wave disturbance at points between f and g at time t is entirely due to the electrification dQ . It is clear that, in the limit, the electric force in the pulse at any point between f and g may be calculated by substituting dQ for q in (7) and then supposing that de , the corresponding total flux of electric displacement, no longer takes place along a mathematical surface, but is uniformly distributed in a layer of thickness dz . We thus obtain for the tangential electric force at any point of the pulse

$$E = \frac{u \sin \theta}{Kr(v - u \cos \theta)} \frac{dQ}{dz} \dots \dots (11)$$

In the same way, the corresponding total flux of magnetic induction, μdh , is uniformly distributed through a layer of thickness dz , and thus, by (10), the magnetic force in the pulse is given by

$$H = vKE = \frac{uv \sin \theta}{r(v - u \cos \theta)} \frac{dQ}{dz} \dots \dots (12)$$

Since $\mu K v^2 = 1$, it follows that, at any point of the pulse, $\mu H^2/8\pi = \mu v^2 K^2 E^2/8\pi = KE^2/8\pi$, and hence the magnetic energy per unit volume is equal to the electric energy per unit volume. Thus, when dQ/dz is known in terms of the angular coordinates of P , we can calculate the energy in the pulse due to the impulsive change in the velocity of the system.

§ 8. Within the region of wave disturbance there is a gradual transition of the field just outside the expanding pulse into the field just inside the pulse. In each of these fields the electric force ultimately varies as $1/r^2$, whereas the tangential electric force, arising from the sudden change in the velocity of the system, ultimately varies as $1/r$. Hence, when r becomes very great, the only electric force which need be considered is the tangential electric force E , given by (11).

If we wish to do so, we can easily calculate the actual electric and magnetic forces at any point f (fig. 2) within the pulse at any great time after the body has been set into

motion. For the pulse due to the part of the system which lies between f and the corresponding plane FF' has already passed over f , while the pulse due to the remainder of the system has not yet reached f . If Q_1 is the charge between f and the plane FF' and Q_2 is the remainder of the total charge, and if E_1 and E_2 are the electric forces due to Q_1 and Q_2 , then the electric force at f has the three components E_1 , E_2 , and E , the latter being given by (11). The electric force E_2 has the value Q_2/Kr^2 and is ultimately radial to O , but the force E_1 is ultimately radial to O' (fig. 2) and has the value

$$E_1 = \frac{Q_1(1-n^2)}{KR^2(1-n^2\sin^2\phi)^{\frac{3}{2}}},$$

where $R=O'P$ and ϕ is the angle between $O'P$ and OX .

The magnetic force at f has the three components H_1 , H_2 , and H , where H is given by (12) and

$$H_1 = KuE_1 \sin \phi, \quad H_2 = 0.$$

Both H_1 and H are in circles in planes normal to OX .

§ 9. The electric and magnetic forces in the pulse ultimately vary as $1/r$, while the thickness of the pulse, for given angular coordinates of P , ultimately becomes constant, and hence the electromagnetic energy in the pulse tends to a constant value, W , as vt , the radius of the pulse, becomes infinite. Since only the squares of E and H are concerned, it follows that the energy in the pulse is ultimately the same, whether the system is suddenly set into motion at speed u or is suddenly reduced to rest from speed u , the direction of u relative to the system being the same in both cases. The energy W , which is radiated away in the pulse, cannot be recovered and may therefore be spoken of as "lost."

§ 10. We now proceed to show that the *total* energy of the system, for steady motion at speed u , is simply equal to $W + U_0$, where U_0 is the electrostatic energy of the system at rest. The total energy at the speed u is $U + T$, the sum of the electric and magnetic energies, where

$$U = \frac{K}{8\pi} \iiint E^2 dx dy dz, \quad T = \frac{\mu}{8\pi} \iiint H^2 dx dy dz.$$

If the moving system be brought to rest suddenly, the charges *do not move* under the influence of the electric forces, which act on them after the system has been brought to rest, and therefore the electromagnetic field does no work upon the system after its motion is destroyed*. Hence the total

* See M. Abraham, *Theorie der Elektrizität*, vol. ii. p. 230.

energy in the field is unaltered by the sudden stopping of the system.

After a short time the pulse will get clear of the system and then it separates the "old field," which is external to the outer surface of the pulse, from the "new field," which is internal to the inner surface of the pulse. In the old field the electric and magnetic forces are those due to the undisturbed steady motion of the system, while in the new field the forces are those due to the system at rest. As the pulse advances, the energy in the old field continually diminishes to zero, the energy in the new field continually increases to U_0 , and the energy in the pulse tends to the value W . But the energy in the whole field has not been changed by stopping the system, and thus

$$U + T = W + U_0. \quad . \quad . \quad . \quad . \quad (13)$$

The force required to keep the system at rest does not necessarily vanish directly the velocity of the system has been destroyed, but only after the pulse has got clear of the system. Thus, in the case of a charged sphere of radius a the time $2a/v$ will elapse before the pulse gets clear of the sphere. After this time the sphere experiences no force.

§ 11. When the system is impulsively set into motion at speed u , a force may be required to maintain the speed at the constant value u until the pulse gets clear of the system. As long as the force is necessary, it will do work in causing the charges on the system to move while they are under the influence of the electric forces in the pulse*, and thus the action of the force changes the energy of the electromagnetic field. When a sphere of radius a is set into motion, the time $2a/(v-u)$ will elapse before the pulse gets clear of the sphere. After this time no force is required to keep the velocity of the sphere constant.

As soon as the pulse has got clear of the system, the energy of the electromagnetic field remains unchanged. In the old field, which is external to the outer surface of the pulse, the electric and magnetic forces are those due to the system at rest. As the pulse advances, the energy in the old field continually diminishes to zero, the energy in the new field continually increases to $U+T$, and the energy in the pulse tends to W . Hence, ultimately, the energy in the field has been increased from U_0 to $U+T$, while W has been lost by radiation. Hence, in setting the system in motion, the force

* Since H is perpendicular to u , the force on a unit moving charge due to H is $\mu u H$ at right angles to u . Hence no work is done in moving the charge while it is under the influence of this force.

does the work $U + T + W - U_0$, and this, by (13), is equal to $2W$. Thus, when a system is set into motion, half the work done by the applied force goes to increase the energy of the field, while the other half is lost by radiation.

§ 12. It can be shown that the force exerted on any system of charges by the electromagnetic field together with the force

$$\mathbf{G} = \frac{\mu K}{4\pi} \frac{d}{dt} \iiint \mathbf{V} \mathbf{E} \mathbf{H} \, dx \, dy \, dz,$$

where the integral is taken throughout the space within any surface S which encloses the system, is equivalent to the resultant of the Maxwell stress over S . Here, and throughout the paper, **heavy type** signifies vectors and $\mathbf{V} \mathbf{E} \mathbf{H}$ denotes the vector product of $\mathbf{E}$ and $\mathbf{H}$. The differentiation with regard to the time is to be performed on the values of $\mathbf{E}$ and $\mathbf{H}$ at points *fixed in space*.

When a system, which has been at rest from $t = -\infty$ to $t = 0$, is made to move in any given manner, from $t = 0$ onwards, by the application of a set of mechanically applied forces, we can take the surface S so great that, at any time t , the field at every point of S is the old statical field due to the system at rest, and then the resultant of the Maxwell stress over S vanishes*. In this case, the resultant force exerted by the electromagnetic field upon the system is $-\mathbf{G}$ and hence, if the system is to move in the given manner, there must be applied to the system by non-electromagnetic means, in addition to any force required to change the momentum of ordinary matter, a resultant force $\mathbf{F}$, where $\mathbf{F} = \mathbf{G}$. Hence we find that, to produce the given motion, we require a force $\mathbf{F}$, where

$$\mathbf{F} = \frac{\mu K}{4\pi} \frac{d}{dt} \iiint \mathbf{V} \mathbf{E} \mathbf{H} \, dx \, dy \, dz, \quad . \quad . \quad . \quad (14)$$

in addition to any force required for the motion of ordinary matter.

The force $\mathbf{F}$ is measured, on Newton's system, by the rate at which it could generate momentum in ordinary matter. Hence we may speak of $\mathbf{F}$ as generating momentum in the electromagnetic field and may say that the expression

$$\frac{\mu K}{4\pi} \iiint \mathbf{V} \mathbf{E} \mathbf{H} \, dx \, dy \, dz \quad . \quad . \quad . \quad (15)$$

is the momentum of the electromagnetic field.

* It is here supposed that there is no other system within a finite distance of the one under consideration.

We are thus led to regard the momentum as being distributed throughout the field. If $\mathbf{m}$ denote the amount per unit volume,

$$\mathbf{m} = (\mu K/4\pi) \nabla \mathbf{E} \mathbf{H} (16)$$

If $\mathbf{I} = \int \mathbf{F} dt$, so that $\mathbf{I}$ is the "impulse" of the applied force,

$$\mathbf{I} = \frac{\mu K}{4\pi} \iiint \nabla \mathbf{E} \mathbf{H} dx dy dz (17)$$

§ 13. When the system is set into steady motion from rest, the work done by the applied force $\mathbf{F}$ is $\int \mathbf{F} \mathbf{u} dt$, where $\mathbf{F} \mathbf{u}$ denotes the *scalar product* of $\mathbf{F}$ and $\mathbf{u}$. We have already found that this work is $2W$, and thus

$$\int \mathbf{F} \mathbf{u} dt = 2W (18)$$

Since $\mathbf{u}$ is constant, we have

$$\mathbf{u} \mathbf{I} = 2W = 2(T + U - U_0) (19)$$

This result is applicable whatever the form of the system; it is not restricted to systems of revolution.

As soon as the pulse has got clear of the system, the force $\mathbf{F}$ vanishes and the total momentum in the field preserves a steady value. By § 9 we see that the momentum in the pulse tends to a constant value, which we denote by $\mathbf{P}$, as rt , the radius of the pulse, becomes infinite. The momentum in the new field, which is enclosed by the inner surface of the pulse, also tends, therefore, to a constant value $\mathbf{M}$, which the system has when it is in steady motion. Hence

$$\mathbf{I} = \mathbf{M} + \mathbf{P} (20)$$

§ 14. In certain cases we can express $\mathbf{M}$, the momentum of the field, for the system in steady motion, in terms of T , the magnetic energy for steady motion. For steady motion we have, as in (3),

$$\mathbf{H} = K \nabla \mathbf{u} \mathbf{E} (21)$$

and hence, by (16),

$$\begin{aligned} \mathbf{m} &= (\mu K/4\pi) \nabla \mathbf{E} \mathbf{H} \\ &= (\mu K^2/4\pi) \nabla \mathbf{E} \mathbf{u} \mathbf{E} \\ &= (\mu K^2/4\pi) (\mathbf{u} \cdot \mathbf{E}^2 - \mathbf{E} \cdot \mathbf{E} \mathbf{u}) (22) \end{aligned}$$

Now let $\mathbf{E}_1$ be the component of $\mathbf{E}$ parallel to $\mathbf{u}$ and $\mathbf{E}_2$ and $\mathbf{E}_3$ the other rectangular components. Then $\mathbf{E} \mathbf{u} = \mathbf{E}_1 u$ and

$(\nabla u \mathbf{E})^2 = u^2 (E_2^2 + E_3^2)$, and hence

$$H^2 = K^2 u^2 (E_2^2 + E_3^2). \quad (23)$$

But, by (22), the component of $\mathbf{m}$ parallel to u is

$$m_1 = (\mu K^2 / 4\pi) (E^2 - E_1^2) u = \mu H^2 / 4\pi u, \quad (24)$$

and hence $M_1 = 2T/u. \quad (25)$

The other components of m have the values

$$m_2 = -(\mu K^2 / 4\pi) E_2 E_1 u, \quad m_3 = -(\mu K^2 / 4\pi) E_3 E_1 u. \quad (26)$$

In special cases, as when the system has an axis of symmetry parallel to u , the components M_2 and M_3 will vanish, and then we have simply

$$M = 2T/u. \quad (27)$$

§ 15. When $\mathbf{M}$ and $\mathbf{I}$ are both parallel to u , we can at once write down $\mathbf{P}$, the momentum in the pulse. For, by (20), $\mathbf{P} = \mathbf{I} - \mathbf{M}$, and hence, in this case, by (19) and (27),

$$P = 2W/u - 2T/u. \quad (28)$$

Thus, by (13), $P = 2(U - U_0)/u. \quad (29)$

If we can calculate P and W for the pulse, we can deduce from them not only the electric and the magnetic energies of the system in steady motion but its momentum as well. For, by (29), (28), and (27),

$$U = \frac{1}{2}uP + U_0. \quad (30)$$

$$T = W - \frac{1}{2}uP. \quad (31)$$

$$M = 2W/u - P. \quad (32)$$

§ 16. The impulse of the force required to suddenly stop a system, which is moving along an axis of symmetry, is easily found. For since both E and H in the pulse change signs, when the system is stopped instead of being started, we see that the momentum in the pulse is the same in both cases. Hence, before the stopping, the momentum is M , while at a great time after the stopping the momentum is P , since the Maxwell stress over an infinite surface enclosing the pulse vanishes, because E and H at points outside the pulse ultimately vary as $1/r^2$. Hence, if I' be the impulse, in the opposite direction to u , which is required to reduce the system to rest,

$$I' = M - P = 2(T - U + U_0)/u. \quad (33)$$

Hence the impulse required to start the system exceeds that required to stop it by twice the momentum carried away by the pulse in either case.

§ 17. We will now apply the principles explained in the earlier part of this paper to some simple systems, and will calculate the energy and momentum in the pulse formed when the system is set into motion at speed u . By § 7, the electric and magnetic energies in the pulse are equal, and thus the total energy in the pulse is ultimately equal to

$$W = \frac{\mu}{4\pi} \iint H^2 r^2 d\omega dz,$$

where $d\omega$ is an element of solid angle. Hence, by (11), since $u/v=n$, we have

$$W = \frac{\mu u^2}{4\pi} \iint \frac{\sin^2 \theta d\omega dz}{(1-n \cos \theta)^2} \left(\frac{dQ}{dz} \right)^2 \dots \dots (34)$$

In the pulse, $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to the radius r and to each other. Hence, attending to the directions of $\mathbf{E}$ and $\mathbf{H}$, we see that $\mathbf{VEH}$ is in the direction of the outward radius and that its magnitude is simply $\mathbf{EH}$. Hence, by (16), if $\mathbf{r}_1$ denote a unit vector in the direction of the radius, the momentum in the pulse is given, in vector notation, by

$$\mathbf{P} = \frac{\mu K}{4\pi} \iint \mathbf{r}_1 \mathbf{EH} r^2 d\omega dz = \frac{\mu u n}{4\pi} \iint \frac{\mathbf{r}_1 \sin^2 \theta d\omega dz}{(1-n \cos \theta)^2} \left(\frac{dQ}{dz} \right)^2 \dots (35)$$

Here and in (34) the range of z is from p_1 to p_2 , where $p_2 - p_1 = p$, the thickness of the pulse.

The component of $\mathbf{r}_1$ in the direction of $\mathbf{u}$ is $\cos \theta$, and hence when, for any reason, $\mathbf{P}$ is parallel to $\mathbf{u}$, it has the value

$$P = \frac{\mu u n}{4\pi} \iint \frac{\sin^2 \theta \cos \theta d\omega dz}{(1-n \cos \theta)^2} \left(\frac{dQ}{dz} \right)^2 \dots \dots (36)$$

When the system is symmetrical round an axis parallel to the direction of motion, we may put $d\omega = 2\pi \sin \theta d\theta$, and then

$$W = \frac{\mu u^2}{2} \iint \frac{\sin^3 \theta d\theta dz}{(1-n \cos \theta)^2} \left(\frac{dQ}{dz} \right)^2 \dots \dots (37)$$

$$P = \frac{\mu u n}{2} \iint \frac{\sin^3 \theta \cos \theta d\theta dz}{(1-n \cos \theta)^2} \left(\frac{dQ}{dz} \right)^2 \dots \dots (38)$$

where θ goes from 0 to π .

§ 18. *Small Velocities*.—Let the radius be defined by θ

and ϕ , where ϕ is measured round the axis of motion. Then

$$\int \left(\frac{dQ}{dz} \right)^2 dz$$

has the same value for the radius defined by $\pi - \theta$ and $\phi + \pi$ as for the radius defined by θ and ϕ . Hence, if we expand $(1 - n \cos \theta)^{-2}$ in the expression (35) for $\mathbf{P}$ and integrate term by term, we see that the integral of the *first term* is zero since, for a given value of $d\omega$, the parts of the pulse defined by θ and ϕ and by $\pi - \theta$ and $\phi + \pi$ make equal and opposite contributions to that term. Thus, going as far as the *second term*, we find that for small values of n

$$\mathbf{P} = \frac{\mu n n^2}{2\pi} \iint \left(\frac{dQ}{dz} \right)^2 r_1 \sin^2 \theta \cos^2 \theta d\omega dz. \quad (39)$$

Hence, for small speeds, $\mathbf{P}$ is proportional to n^3 , and therefore, by (29), $\mathbf{U} - \mathbf{U}_0$ is proportional to n^4 . By (13) it follows that $\mathbf{W} - \mathbf{T}$ is proportional to n^4 .

§ 19. *Sphere with a uniform surface-charge.*—This is, so far as I know, the only system to which the method explained in the present paper has been applied. The application was made by Dr. Heaviside*. If Q be the charge and a the radius of the sphere, $dQ/dz = Q/2a$, so that dQ/dz is independent of z , and thus the integrations with respect to z can be made at once. We thus obtain, by (37),

$$\mathbf{W} = \frac{\mu Q^2 a^2}{4a} \int_0^\pi \frac{\sin^3 \theta d\theta}{(1 - n \cos \theta)^2},$$

where $n = u/v$. Putting $1 - n \cos \theta = h$, we find

$$\begin{aligned} \mathbf{W} &= \frac{\mu Q^2 a^2}{4an} \int_{1-n}^{1+n} \left(\frac{n^2 - 1}{n^2 h^2} + \frac{2}{n^2 h} - \frac{1}{n^2} \right) dh, \\ &= \frac{\mu Q^2 a^2}{2a} \left(v \log \frac{v+u}{v-u} - 2 \right), \\ &= \frac{\mu Q^2 a^2}{a} \left(\frac{1}{3} + \frac{n^2}{5v^2} + \frac{n^4}{7v^4} + \dots \right). \end{aligned}$$

The electrostatic energy of the sphere at rest is given by

$$\mathbf{U}_0 = \frac{Q^2}{2Ka} = \frac{\mu v^2 Q^2}{2a}.$$

* 'The Electrician,' Nov. 29, 1901, p. 210.

Thus since, by (13), $U + T$, the total energy of the sphere in steady motion, is $W + U_0$, we have

$$U + T = \frac{\mu v^2 Q^2}{2a} \left(\frac{v}{u} \log \frac{v+u}{v-u} - 1 \right) = \frac{Q^2}{2Ka} \left(\frac{v}{u} \log \frac{v+u}{v-u} - 1 \right).$$

This value is identical with that which I found* by integration throughout the whole field of the sphere in steady motion.

Since the sphere moves along an axis of symmetry, the momentum carried off in the pulse is, by (38),

$$P = \frac{\mu n Q^2}{4a} \int_0^\pi \frac{\sin^3 \theta \cos \theta d\theta}{(1 - n \cos \theta)^2}.$$

Putting $1 - n \cos \theta = h$ and using $\mu K v^2 = 1$, we find

$$\begin{aligned} P &= \frac{Q^2}{4anuK} \int_{1-n}^{1+n} \left\{ \frac{n^2 - 1}{h^2} + \frac{3 - n^2}{h} - 3 + h \right\} dh, \\ &= \frac{Q^2 v}{4au^2 K} \left\{ \left(3 - \frac{u^2}{v^2} \right) \log \frac{v+u}{v-u} - 6 \frac{u}{v} \right\}, \\ &= \frac{Q^2 u^3}{aK v^4} \left(\frac{2}{3.5} + \frac{4u^2}{5.7v^2} + \frac{6u^4}{7.9v^4} \dots \right). \end{aligned}$$

Hence, as was proved in § 18, for small values of u/v the momentum carried off by the pulse varies as u^3 .

From the values of W and P those of U , T , and M can be deduced by the method explained in § 15.

§ 20. *Sphere with a uniform volume-charge.*—If z be measured from the centre of the sphere, we have, in this case,

$$\frac{dQ}{dz} = \frac{3Q(a^2 - z^2)}{4a^3}.$$

Hence

$$\int_{-a}^{+a} \left(\frac{dQ}{dz} \right)^2 dz = \frac{3}{5} \frac{Q^2}{a},$$

and thus, by (37),

$$W = \frac{\mu Q^2 u^2}{4a} \cdot \frac{6}{5} \int_0^\pi \frac{\sin^3 \theta d\theta}{(1 - n \cos \theta)^2},$$

and hence, by § 19,

$$W = \frac{6}{5} \cdot \frac{\mu v^2 Q^2}{2a} \left(\frac{v}{u} \log \frac{v+u}{v-u} - 2 \right).$$

* Proc. Royal Soc. vol. lix. p. 344 (1896), and Phil. Mag. Oct. 1897, p. 340.

Thus the energy carried off in the pulse in the case of the volume-charge is six fifths of the energy carried off in the case of a surface-charge.

In exactly the same way, it follows that P for a sphere with a volume-charge is six fifths of the value for a sphere with an equal surface-charge.

The electrostatic potential at a point within the sphere at a distance r from its centre, when the sphere is at rest, is easily found to be

$$Q(3a^2 - r^2)/2Ka^3,$$

and hence by the formula

$$U_0 = \frac{1}{2} \Sigma \text{ charge} \times \text{potential},$$

the electric energy is given by

$$U_0 = \frac{3}{5} \frac{Q^2}{Ka},$$

and this is six fifths of the energy when the charge is on the surface.

It therefore follows, by § 15, that the values of U , T , and M for a sphere with a volume-charge are six fifths as great as for the same sphere with an equal surface-charge*.

§ 21. *Pair of concentric spheres with uniformly distributed complementary charges.*—Let the outer sphere have the radius a and the charge Q , while the inner sphere has the radius b and the charge $-Q$. Then from $z = -a$ to $z = -b$ and also from $z = b$ to $z = a$,

$$dQ/dz = Q/2a,$$

but from $z = -b$ to $z = b$

$$dQ/dz = -Q(a-b)/2ab.$$

Thus
$$\int_{-a}^{+a} \left(\frac{dQ}{dz}\right)^2 dz = \frac{Q^2(a-b)}{2a^2} + \frac{Q^2(a-b)^2}{2a^2b} = \frac{Q^2}{2a} \cdot \frac{a-b}{b}.$$

For the outer sphere alone

$$\int_{-a}^{+a} \left(\frac{dQ}{dz}\right)^2 dz = \frac{Q^2}{2a},$$

and the value of U_0 for the pair of spheres is $Q^2(a-b)/2abK$. Thus the values of W , P , and U_0 for the system are $(a-b)/b$

* Dr. M. Abraham has proved similar results for an ellipsoid by a method not depending on the calculation of W and P for the pulse. See his *Theorie der Elektrizität*, vol. ii. p. 170.

times the values for the outer sphere alone, and hence, by § 15, the values of U , T , and M are $(a-b)/b$ times the values for the outer sphere alone.

In this calculation a uniform distribution of the charges has been assumed. But this is not the natural distribution, corresponding to the speed u , for a pair of conducting spheres. So far as I know, the distribution in the latter case has not been investigated.

§ 22. *Sphere with surface-density proportional to $\cos \theta$.*—Let σ , the surface-density, be given by $\sigma = \sigma_0 \cos \theta$. Then the same distribution can be obtained by taking two spheres of volume-densities ρ and $-\rho$ with their centres at a distance ξ apart, and proceeding to the limit $\xi=0$, while $\rho\xi$ has the constant value σ_0 . In terms of g , the “electric moment” of the sphere, we have

$$\rho\xi = \sigma_0 = \frac{3g}{4\pi a^3}.$$

Let O (fig. 3) be the centre of the sphere of density ρ and O' that of the sphere of density $-\rho$. Then, if A be the area of the section of the sphere O by a plane at a distance z from O , the normal making an angle θ with OX , we have $A = \pi(a^2 - z^2)$. The area of the section of the sphere O' by the same plane is $\pi\{a^2 - (z + \xi \cos \theta)^2\}$. Thus the charge dQ contained between two planes at a distance dz apart is

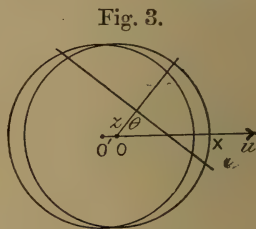


Fig. 3.

$$dQ = \pi\rho\{a^2 - z^2\}dz - \pi\rho\{a^2 - (z + \xi \cos \theta)^2\}dz,$$

whence, to the first power of ξ , we have

$$dQ/dz = 2\pi z \cos \theta \cdot \rho\xi.$$

Hence, since $\rho\xi = 3g/4\pi a^3$,

$$\frac{dQ}{dz} = \frac{3gz \cos \theta}{2a^3}.$$

Since

$$\int_{-a}^{+a} z^2 dz = \frac{2}{3} a^3,$$

we have, by (37),

$$W = \frac{3\mu u^2 g^2}{4a^3} \int_0^\pi \frac{\sin^3 \theta \cos^2 \theta d\theta}{(1 - n \cos \theta)^2}.$$

Putting $1 - n \cos \theta = h$, we find

$$\begin{aligned} W &= \frac{3\mu v^2 g^2}{4a^3 n^5} \int_{1-n}^{1+n} \left\{ \frac{n^2-1}{h^2} + \frac{4-2n^2}{h} - (6-n^2) + 4h - h^2 \right\} dh, \\ &= \frac{3\mu v^2 g^2}{2a^3} \left\{ \frac{v^5}{u^5} \left(2 - \frac{u^2}{v^2} \right) \log \frac{v+u}{v-u} - \frac{v^4}{u^4} + \frac{2}{3} \frac{v^2}{u^2} \right\}, \\ &= \frac{3\mu v^2 g^2}{a^3} \left\{ \frac{1}{3 \cdot 5} + \frac{3u^2}{5 \cdot 7 v^2} + \frac{5u^4}{7 \cdot 9 v^4} + \dots \right\}. \end{aligned}$$

Since the sphere moves along an axis of symmetry, the momentum carried off by the pulse is given by (38), and thus, after integrating with respect to z , we have

$$\begin{aligned} P &= \frac{3\mu u g^2 n}{4a^3} \int_0^\pi \frac{\sin^3 \theta \cos^3 \theta d\theta}{(1-n \cos \theta)^2} \\ &= \frac{3\mu u g^2}{4a^3 n^5} \int_{1-n}^{1+n} \left\{ \frac{n^2-1}{h^2} + \frac{5-3n^2}{h} - (10-3n^2) + (10-n^2)h - 5h^2 + h^3 \right\} dh, \\ &= \frac{3\mu u g^2}{4a^3 n^5} \left\{ (5-3n^2) \log \frac{1+n}{1-n} - 10n + \frac{8}{3} n^3 \right\} \\ &= \frac{6\mu n^3 g^2}{a^3 v^2} \left\{ \frac{1}{5 \cdot 7} + \frac{2u^2}{7 \cdot 9 v^2} + \frac{3u^4}{9 \cdot 11 v^4} \dots \right\}. \end{aligned}$$

When the sphere is at rest, the electrostatic potential at its surface is $V = g \cos \theta / K a^2$, and hence

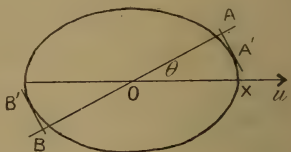
$$\begin{aligned} U_0 &= \frac{1}{2} \Sigma V q = \frac{1}{2} \int_0^\pi \frac{g \cos \theta}{K a^2} \cdot \sigma_0 \cos \theta \cdot 2\pi a^2 \sin \theta d\theta \\ &= \frac{g^2}{2a^3 K} = \frac{\mu g^2 v^2}{2a^3}. \end{aligned}$$

From these values of W , P , and U_0 we find, by § 15, that for the sphere in steady motion,

$$\begin{aligned} U &= \frac{\mu g^2 v^2}{2a^3} \left\{ 1 + \frac{6u^4}{5 \cdot 7 v^4} + \frac{12u^6}{7 \cdot 9 v^6} + \frac{18u^8}{9 \cdot 11 v^8} + \dots \right\}, \\ T &= \frac{3\mu v^2 g^2}{a^3} \left\{ \frac{1}{3 \cdot 5} + \frac{2u^2}{5 \cdot 7 v^2} + \frac{3u^4}{7 \cdot 9 v^4} + \frac{4u^6}{9 \cdot 11 v^6} + \dots \right\}, \\ M &= \frac{6\mu u g^2}{a^3} \left\{ \frac{1}{3 \cdot 5} + \frac{2u^2}{5 \cdot 7 v^2} + \frac{3u^4}{7 \cdot 9 v^4} + \frac{4u^6}{9 \cdot 11 v^6} + \dots \right\}. \end{aligned}$$

§ 23. *Ellipsoid of revolution with a surface-charge.*—When a charged ellipsoid is in uniform motion, the charge is distributed in the same way as when it is at rest*. The surface-density is everywhere proportional to the thickness of an infinitely thin ellipsoidal shell bounded by similar ellipsoids, the inner surface of the shell coinciding with the ellipsoid. Such a shell can be formed by uniformly straining a shell bounded by concentric spheres. If the spherical shell be cut by a series of parallel equidistant planes, they will divide the shell into portions of equal volume, and hence the series of parallel equidistant planes, into which the first series is transformed by the strain, divides the ellipsoidal shell into portions of equal volume. Thus, the charge contained between a pair of parallel planes which cut the ellipsoid is, for any given direction of the planes, proportional to the distance between them. Hence, if the distance between a pair of parallel tangent planes AA' , BB' (fig. 4) be p , the value of dQ/dz , where dz is an element of the normal to these planes, is constant for those planes and is equal to Q/p , where Q is the total charge on the ellipsoid.

Fig. 4.



§ 24. For simplicity, the ellipsoid of revolution will be supposed to move along OX , its axis of revolution. If a , b , be the axes of the ellipsoid, then, when the normal to the tangent planes makes an angle θ with OX

$$p^2 = 4\{b^2 + (a^2 - b^2) \cos^2 \theta\}.$$

Since dQ/dz is independent of z , we can integrate (37) at once with respect to z , and thus, writing $\cos \theta = x$, we find for the energy carried off in the pulse

$$\begin{aligned} W &= \frac{\mu v^2 Q^2}{4} \int_{-1}^{+1} \frac{(1-x^2) dx}{(1-nx)^2 \sqrt{b^2 + (a^2 - b^2)x^2}} \\ &= \frac{\mu v^2 Q^2}{4n^2} \{-J_1 + 2J_2 - (1-n^2)J_3\}, \end{aligned}$$

where

$$J_1 = \int_{-1}^{+1} \frac{dx}{\sqrt{b^2 + (a^2 - b^2)x^2}}, \quad J_2 = \int_{-1}^{+1} \frac{dx}{(1-nx) \sqrt{b^2 + (a^2 - b^2)x^2}},$$

and

$$J_3 = \int_{-1}^{+1} \frac{dx}{(1-nx)^2 \sqrt{b^2 + (a^2 - b^2)x^2}}.$$

* W. B. Morton, Proc. Physical Society, vol. xiv. p. 180 (1896), or Searle, Phil. Mag. Oct. 1897, p. 331.

When $a > b$, we put $a^2 - b^2 = c^2$, and then we have

$$J_1 = \frac{1}{c} \log \frac{a+c}{a-c}.$$

The case when $a < b$ is considered in § 26.

If in J_2 we put $b^2(1-n^2) = a^2 - l^2$, and change the variable from x to y , where

$$y + \frac{l^2 - n^2 a^2}{(1-n^2)l^2} = \frac{1}{1-nv},$$

we find, after some reductions,

$$J_2 = \frac{1}{\sqrt{l^2}} \int \frac{dy}{R},$$

where

$$R^2 = y^2 + \frac{n^2(a^2 - l^2)(l^2 - n^2 a^2)}{(1-n^2)^2 l^4}.$$

Thus, when l^2 is positive, or $a^2 > (1-n^2)b^2$, that is, when the ellipsoid is prolate or less oblate than the Heaviside ellipsoid,

$$J_2 = \frac{1}{l} \left[\log (y + R) \right]_{x=-1}^{x=1} = \frac{1}{l} \log \frac{a+l}{a-l}.$$

The case when l^2 is negative is considered in § 26.

The third integral J_3 can now be easily found, for on differentiating the integral J_2 with regard to n , we find that $ndJ_2/dn = J_3 - J_2$. Thus

$$J_3 = \frac{d}{dn} (nJ_2).$$

Since $dl/dn = nb^2/l$, we find

$$J_3 = \frac{l^2 - n^2 b^2}{l^3} \log \frac{a+l}{a-l} + \frac{2an^2 b^2}{l^2(a^2 - l^2)}.$$

Collecting these results, and using $\mu K v^2 = 1$, we have

$$W = \frac{Q^2}{4K} \left\{ \frac{l^2 + n^2 a^2}{l^3} \log \frac{a+l}{a-l} - \frac{2n^2 a}{l^2} - \frac{1}{c} \log \frac{a+c}{a-c} \right\}.$$

The electrostatic energy of the ellipsoid at rest is U_0 , where

$$U_0 = \frac{Q^2}{4Kc} \log \frac{a+c}{a-c}.$$

§ 25. With the notation of § 24, the momentum carried

off by the pulse is given by

$$P = \frac{\mu u Q^2}{4} \int_{-1}^{+1} \frac{(1-x^2)x dx}{(1-nx)^2 \sqrt{b^2 + (a^2 - b^2)x^2}}$$

$$= \frac{\mu u Q^2}{4n^2} \{-2J_1 + (3-n^2)J_2 - (1-n^2)J_3 - nJ_4\},$$

where

$$J_1 = \int_{-1}^{+1} \frac{x dx}{\sqrt{b^2 + (a^2 - b^2)x^2}} = 0.$$

Thus, using $\mu K v^2 = 1$, we find

$$P = \frac{Q^2}{4Ku} \left\{ \frac{n^2 a^2 + 2l^2 - n^2 l^2}{l^3} \log \frac{a+l}{a-l} - \frac{2an^2}{l^2} - \frac{2}{c} \log \frac{a+c}{a-c} \right\}.$$

By § 15 we obtain the results

$$U = \frac{Q^2}{4K} \left\{ \frac{n^2 a^2 + 2l^2 - n^2 l^2}{2l^3} \log \frac{a+l}{a-l} - \frac{an^2}{l^2} \right\},$$

$$T = \frac{u^2 Q^2}{4Kv^2} \left\{ \frac{a^2 + l^2}{2l^3} \log \frac{a+l}{a-l} - \frac{a}{l^2} \right\}.$$

By (27), $M = 2T/u$.

These values of U and T agree with those I obtained by integration throughout the whole field of the moving ellipsoid*.

§ 26. When $a < b$, the quantity c is imaginary, and when a becomes still smaller, so that $a < b \sqrt{1-n^2}$, the quantity l is imaginary also. The last case corresponds to an ellipsoid more oblate than Heaviside's. The expressions for U , W , and P can be adapted to the case where $b(1-n^2)^{\frac{1}{2}} < a < b$, and where c is imaginary but l is real, by putting $b^2 - a^2 = C^2$ and writing

$$\frac{2}{C} \tan^{-1} \frac{C}{a} \quad \text{for} \quad \frac{1}{c} \log \frac{a+c}{a-c}.$$

When $a < b(1-n^2)^{\frac{1}{2}}$, both c and l are imaginary and we must, in addition, put $b^2(1-n^2) - a^2 = L^2$ and write

$$\frac{2}{L} \tan^{-1} \frac{L}{a} \quad \text{for} \quad \frac{1}{l} \log \frac{a+l}{a-l}.$$

For a disk of radius b , we put $a=0$ and hence $C=b$ and $L=b(1-n^2)^{\frac{1}{2}}$. We thus obtain

$$U_0 = \frac{\pi Q^2}{4Kb}, \quad W = \frac{\pi Q^2}{4Kb} \left(\frac{1}{(1-n^2)^{\frac{1}{2}}} - 1 \right),$$

$$P = \frac{\pi Q^2}{4Kbu} \left(\frac{2-n^2}{(1-n^2)^{\frac{1}{2}}} - 2 \right).$$

* Phil. Mag. Oct. 1897, p. 340.

By the results of § 15 we obtain

$$U = \frac{1}{2}uP + U_0 = U_0 \frac{1 - \frac{1}{2}n^2}{(1 - n^2)^{\frac{1}{2}}} = U_0(1 + \frac{1}{8}n^4 + \dots),$$

$$T = W - \frac{1}{2}uP = U_0 \frac{n^2}{2(1 - n^2)^{\frac{1}{2}}} = n^2 U_0(\frac{1}{2} + \frac{1}{4}n^2 + \dots).$$

If we put $n = \sin \gamma$, we obtain

$$U = \frac{1}{2}(\sec \gamma + \cos \gamma) U_0,$$

$$m_t = 2T/u^2 = \sec \gamma \cdot U_0/v^2, \quad m_l = 2d(T/u)/du = \sec^3 \gamma \cdot U_0/v^2,$$

where m_t is the "transverse mass" and m_l the "longitudinal mass."

§ 27. *Circular disk moving in its own plane.*—In this instance the axis of symmetry of the system is not parallel to the direction of motion, as is the case in the other problems we have considered. We therefore require two angular coordinates, ϕ and ψ , where ϕ is measured from the axis of the disk and ψ is measured round the axis, starting from the plane which contains the axis, and is parallel to the direction of motion. Then, if the radius defined by ϕ and ψ makes the angle θ with the direction of motion, we have

$$\cos \theta = \sin \phi \cos \psi.$$

Taking the disk, of radius b , as the limiting form of an ellipsoid, we see that p , the thickness of the pulse, is equal to $2b \sin \phi$, and thus, by § 23, $dQ/dz = Q/2b \sin \phi$. Now $d\omega$, the element of solid angle, is equal to $\sin \phi d\phi d\psi$ and thus, by (34), since dQ/dz is independent of z ,

$$\begin{aligned} W &= \frac{\mu u^2 Q^2}{8\pi b} \iint \frac{\sin^2 \theta d\phi d\psi}{(1 - n \cos \theta)^2} \\ &= \frac{\mu u^2 Q^2}{8\pi b} \iint \frac{(1 - \sin^2 \phi \cos^2 \psi)}{(1 - n \sin \phi \cos \psi)^2} d\phi d\psi, \end{aligned}$$

where ϕ goes from 0 to π and ψ from 0 to 2π .

Expanding the denominator and integrating term by term, we find

$$W = \frac{\pi \mu u^2 Q^2}{4b} \left\{ \frac{3}{1} \left(\frac{1}{2} \right)^2 + n^2 \cdot \frac{7}{3} \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 + n^4 \cdot \frac{11}{5} \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 + \dots \right\}.$$

We can express this result in a finite form by means of the first and second complete elliptic integrals F and E . If $\Delta^2 = 1 - n^2 \sin^2 \alpha$, we have

$$F = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\Delta} = \frac{\pi}{2} \left\{ 1 + n^2 \left(\frac{1}{2} \right)^2 + n^4 \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 + n^6 \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 + \dots \right\}$$

$$E = \int_0^{\frac{\pi}{2}} \Delta d\alpha = \frac{\pi}{2} \left\{ 1 - \frac{n^2}{1} \left(\frac{1}{2} \right)^2 - \frac{n^4}{3} \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 - \frac{n^6}{5} \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 + \dots \right\}.$$

Comparing the series for W with those for F and E and using $\mu Kv^2=1$, we obtain

$$W = \frac{Q^2}{2bK} (2F - E - \frac{1}{2}\pi).$$

The momentum carried off by the pulse is, by symmetry, parallel to the direction of motion and hence, by (36), is given by

$$\begin{aligned} P &= \frac{\mu un Q^2}{8\pi b} \iint \frac{\sin^2 \theta \cos \theta d\phi d\psi}{(1 - n \cos \theta)^2} \\ &= \frac{\mu un Q^2}{8\pi b} \iint \frac{\sin \phi \cos \psi - \sin^3 \phi \cos^3 \psi}{(1 - n \sin \phi \cos \psi)^2} d\phi d\psi, \end{aligned}$$

where, as before, ϕ goes from 0 to π and ψ from 0 to 2π . Expanding the denominator and integrating term by term, we find

$$P = \frac{\pi \mu un^2 Q^2}{4b} \left\{ 2 \cdot \frac{7}{4^2} \binom{1}{2}^2 + 4n^2 \frac{11}{6^2} \binom{1.3}{2.4}^2 + 6n^4 \frac{15}{8^2} \binom{1.3.5}{2.4.6}^2 + \dots \right\}.$$

Thus, as was proved in § 18, the momentum carried off in the pulse is proportional to u^3 for small values of u/r . Comparing the series for P with those for F and E and using $\mu Kv^2=1$, we obtain

$$P = \frac{Q^2}{2bKu} \{ (3 - n^2)F - E - \pi \}.$$

Since the momentum of the field, when the disk is in steady motion, is parallel to the direction of u , we can employ the formulæ of § 15. Thus, since

$$U_0 = \pi Q^2 / 4bK,$$

we obtain

$$U = \frac{1}{2}uP + U_0 = U_0 \pi^{-1} \{ (3 - n^2)F - E \} = U_0 (1 + 7n^4/64 + \dots),$$

$$T = W - \frac{1}{2}uP = U_0 \pi^{-1} \{ (1 + n^2)F - E \} = n^2 U_0 (3/4 + 7n^2/32 + \dots).$$

Comparing these results with those of § 26, we see that the magnetic energy of a disk moving in its own plane is, for small speeds, greater than the magnetic energy of the disk moving parallel to its axis in the proportion of 3 to 2.

§ 28. To show how the increase of speed affects the electric and magnetic energies when the disk moves (1) along its axis, (2) in its plane, the following table is given. Since the "transverse mass," m_t , is equal to $2T/u^2$, it will be convenient to tabulate m_t/m_0 , where m_0 is the value of m_t for infinitesimal

speeds. To facilitate the use of Legendre's tables* for E and F, we put $n = \sin \gamma$.

		Motion along Axis.		Motion perpendicular to Axis.	
γ .	n .	U/U_0 .	m_t/m_0 .	U/U_0 .	m_t/m_0 .
10°	·1736	1·0001	1·0154	1·0001	1·0106
20	·3420	1·0019	1·0642	1·0016	1·0366
30	·5000	1·0104	1·1547	1·0085	1·0860
40	·6428	1·0357	1·3054	1·0278	1·1627
50	·7660	1·0992	1·5557	1·0712	1·2772
60	·8660	1·2500	2·0000	1·1590	1·4502
70	·9397	1·6329	2·9238	1·3317	1·7291
80	·9848	2·9662	5·7588	1·7068	2·2631
85	·9962	5·7804	11·4737	2·1263	2·8320

We notice that, as n increases, both U/U_0 and m_t/m_0 increase more rapidly when the motion is along the axis of the disk than when it is perpendicular to the axis. For some value of n between ·8660 and ·9397 the value of m_t/m_0 for motion along the axis becomes three halves of the value of m_t/m_0 for motion perpendicular to the axis, and then the magnetic energy for motion along the axis becomes equal to that for motion perpendicular to the axis.

The "longitudinal mass," m_l , is easily found. For we have †

$$\frac{dF}{dn} = \frac{E}{n(1-n^2)} - \frac{F}{n}, \quad \frac{dE}{dn} = \frac{E-F}{n}$$

and thus

$$m_l = 2 \frac{d}{du} \left(\frac{T}{u} \right) = \frac{2}{v^2} \frac{d}{dn} \left(\frac{T}{n} \right) = \frac{2U_0}{v^2} \cdot \frac{(1+n^2)E - (1-n^2)F}{\pi n^2(1-n^2)}.$$

§ 29. *Momentum and energy of a system of revolution moving slowly in any direction.*—When a system which is symmetrical round an axis, and is also symmetrical with respect to a plane normal to that axis, moves either along or at right angles to its axis, its electromagnetic momentum is in the direction of motion, but the amounts of momentum are different in the two cases. If, however, the axis makes any angle other than 0 or $\frac{1}{2}\pi$ with the direction of motion, the momentum has a component at right angles to that direction as well as one parallel to it.

A general investigation would be difficult, but we can easily

* Legendre, *Traité des Fonctions Elliptiques*, vol. ii.

† A. Caley, "Elementary Treatise on Elliptic Functions," chapter 3.

obtain some interesting results when n or u/v is so small that, as we see by (1), the electric force due to the system may be regarded as being the same as if the system were at rest. The axis of x will be taken in the direction of motion, the volume integral of E_1^2 through all space will be denoted by ΣE_1^2 , and η will be written for $K/4\pi v^2$, or $\mu K^2/4\pi$.

If M_1, M_2, M_3 be the components of the momentum when the axis of the system is parallel to the axis of x , we have, by (24), since $\mu K v^2 = 1$

$$M_1 = u\eta \Sigma (E_2^2 + E_3^2) = u\eta \Sigma (E^2 - E_1^2) = 2(u/v^2) U_0 - u\eta \Sigma E_1^2 \quad (40)$$

while, by (26), we have, for reasons of symmetry,

$$M_2 = -u\eta \Sigma E_1 E_2 = 0, \quad M_3 = -u\eta \Sigma E_1 E_3 = 0. \quad (41)$$

In the expression for M_1 we have put U_0 for U , since we know, by § 18, that, when n is small, $U - U_0$ is proportional to n^4 . Since there is symmetry round the axis, $\Sigma E_2^2 = \Sigma E_3^2$, and thus

$$u\eta \Sigma E_1^2 = 2uU_0/v^2 - M_1, \quad u\eta \Sigma E_2^2 = u\eta \Sigma E_3^2 = \frac{1}{2}M_1. \quad (42)$$

Now let the system be turned round the axis of y in the *positive* direction, so that the axis of the system makes an angle θ with the direction of motion and with the axis of x , and let G be a point rigidly connected with the system and turning with it. Since when u/v is very small the electric field turns round with the system without any other change, the new components of the electric force at G are given by the equations

$$\left. \begin{aligned} E_1' &= E_1 \cos \theta + E_3 \sin \theta, & E_2' &= E_2 \\ E_3' &= -E_1 \sin \theta + E_3 \cos \theta \end{aligned} \right\} \quad (43)$$

For M_1' , the component of the momentum in the direction of motion, we have, as in (40),

$$M_1' = 2uU_0/v^2 - u\eta \Sigma E_1'^2.$$

But $\Sigma E_1 E_3 = 0$, and hence, by (43),

$$\begin{aligned} M_1' &= 2uU_0/v^2 - u\eta \Sigma (E_1^2 \cos^2 \theta + E_3^2 \sin^2 \theta), \\ &= 2uU_0 \sin^2 \theta / v^2 + M_1 (\cos^2 \theta - \frac{1}{2} \sin^2 \theta). \end{aligned}$$

If N be the momentum *in the direction of motion* when the system moves at right angles to its axis, *i.e.*, when $\theta = \frac{1}{2}\pi$, we have

$$N = 2uU_0/v^2 - \frac{1}{2}M_1, \quad (44)$$

and thus for any value of θ ,

$$M_1' = M_1 \cos^2 \theta + N \sin^2 \theta.$$

By symmetry $\Sigma E_1 E_2 = 0$, and $\Sigma E_2 E_3 = 0$, and hence, by (43),

$$M_2' = -u\eta \Sigma E_1' E_2' = -u\eta \Sigma (E_1 \cos \theta + E_3 \sin \theta) E_2 = 0.$$

The momentum at right angles to the direction of motion is thus given by

$$\begin{aligned} M_3' &= -u\eta \Sigma E_1' E_3' = u\eta \Sigma (E_1^2 - E_3^2) \sin \theta \cos \theta \\ &= (N - M_1) \sin \theta \cos \theta. \quad . \quad . \quad . \quad . \quad . \quad (45) \end{aligned}$$

When $\theta = \frac{1}{2}\pi$, the component of the momentum at right angles to the direction of motion vanishes, and hence N is the resultant momentum in that case.

From (45) we see that the resultant momentum is inclined to the direction of motion at an angle ψ , where

$$\tan \psi = M_3' / M_1' = (N - M_1) \sin \theta \cos \theta / (M_1 \cos^2 \theta + N \sin^2 \theta).$$

If T be the magnetic energy for any value of θ , and T and S be the magnetic energies when the system moves along and at right angles to its axis, we have, by (25),

$$T' = \frac{1}{2} M_1' u, \quad T = \frac{1}{2} M_1 u, \quad S = \frac{1}{2} N u.$$

Thus

$$T' = T \cos^2 \theta + S \sin^2 \theta, \quad . \quad . \quad . \quad . \quad (46)$$

and

$$S = u^2 U_0 / v^2 - \frac{1}{2} T. \quad . \quad . \quad . \quad . \quad (47)$$

§ 30. We may illustrate the formulæ of § 29 by the results for a charged disk of radius b . Here we have, by § 26, for very small speeds

$$U_0 = \pi Q^2 / 4Kb, \quad T = \frac{1}{2} u^2 U_0 / v^2.$$

Hence

$$S = u^2 U_0 / v^2 - \frac{1}{2} T = \frac{3}{4} u^2 U_0 / v^2,$$

and thus the magnetic energy of the disk moving slowly in its own plane is half as great again as when it moves along its axis at the same speed, as we found in § 27.

Now $N - M_1 = 2(S - T) / u = \frac{1}{2} u U_0 / v^2$ and hence, when the axis of the disk makes an angle θ with the direction of motion,

$$M_1' = M_1 + (N - M_1) \sin^2 \theta = \frac{1}{2} (u / v^2) U_0 (2 + \sin^2 \theta),$$

$$M_3' = \frac{1}{2} (u / v^2) U_0 \sin \theta \cos \theta.$$

Thus the resultant momentum is inclined to the direction of motion at an angle ψ , where

$$\tan \psi = \frac{\sin \theta \cos \theta}{2 + \sin^2 \theta}.$$

It is easily found that $\tan \psi$ has a maximum value $(24)^{-\frac{1}{2}}$ or 0.20412, corresponding to $\psi = 11^\circ 32'$, when $\sin^2 \theta = 2/5$ or $\theta = 39^\circ 14'$.

For a sphere with a surface-charge where the surface-density is $\sigma_0 \cos \theta$, we have, by § 22, when u/v is very small,

$$T = 2u^2 U_0 / 5v^2.$$

Hence, by (47),

$$S = u^2 U_0 / v^2 - \frac{1}{2}T = 4u^2 U_0 / 5v^2 = 2T$$

and

$$N = 2M_1.$$

For a prolate ellipsoid, when n is very small, we may put $l=c$, and then we find, by § 25, that

$$T = \frac{u^2 Q^2}{4Kv^2} \left\{ \frac{a^2 + c^2}{2c^3} \log \frac{a+c}{a-c} - \frac{a}{c^2} \right\}.$$

Hence

$$\begin{aligned} S &= u^2 U_0 / v^2 - \frac{1}{2}T \\ &= \frac{u^2 Q^2}{4Kv^2} \left\{ \frac{3c^2 - a^2}{4c^3} \log \frac{a+c}{a-c} + \frac{a}{2c^2} \right\} \end{aligned}$$

where $c^2 = a^2 - b^2$.

For a very slender ellipsoid, where a/b is very great, we may put $(a+c)/(a-c) = 4a^2/b^2$, and need retain only the logarithmic terms. We then find

$$T = u^2 U_0 / v^2, \quad S = \frac{1}{2}T.$$

§ 31. Hitherto we have supposed that the speed is impulsively changed from 0 to u or from u to 0. But the principle of § 3 is easily extended to the case when the velocity of a point charge is impulsively changed from $\mathbf{u}$ to $\mathbf{u}'$, where $\mathbf{u}$ and $\mathbf{u}'$ may be inclined at any angle. We have, as Dr. Heaviside* has indicated, only to destroy the velocity $\mathbf{u}$ and then instantly restart the charge with the velocity $\mathbf{u}'$. By (8) we obtain a vector expression for $\mathbf{e}$ in the form

$$\mathbf{e} = \frac{q}{Kr} \left[\frac{\mathbf{r}_1(\mathbf{u}'\mathbf{r}_1) - \mathbf{u}'}{v - \mathbf{u}'\mathbf{r}_1} - \frac{\mathbf{r}_1(\mathbf{u}\mathbf{r}_1) - \mathbf{u}}{v - \mathbf{u}\mathbf{r}_1} \right]. \quad (48)$$

Writing $\mathbf{u}' - \mathbf{u} = \mathbf{w}$ and putting $v - \mathbf{u} \cos \theta = v - \mathbf{u}\mathbf{r}_1 = v\hbar$ we easily find

$$\mathbf{e} = \frac{q}{Kr} \left[-\frac{\mathbf{w}}{v\hbar - w \cos \gamma} + \frac{(\mathbf{v}\mathbf{r}_1 - \mathbf{u})w \cos \gamma}{v\hbar(v\hbar - w \cos \gamma)} \right]. \quad (49)$$

Here θ is the angle between $\mathbf{r}$ and $\mathbf{u}$ while γ is the angle between $\mathbf{r}$ and $\mathbf{w}$.

* 'Nature,' Nov. 6, 1902.

If w be infinitesimal compared with $v-u$, (49) becomes

$$e = \frac{q}{Kr} \left[-\frac{\mathbf{w}}{vh} + \frac{(v\mathbf{r}_1 - \mathbf{u})w \cos \gamma}{v^2 h^2} \right]. \quad (50)$$

Applying (50) to a sphere of radius a with a uniform surface-charge Q in the manner explained in § 7, we find that in the pulse, at a great distance from the sphere,

$$\mathbf{E} = \frac{Q}{2Kra} \left[-\frac{\mathbf{w}}{vh} + \frac{(v\mathbf{r}_1 - \mathbf{u})w \cos \gamma}{v^2 h^2} \right]. \quad (51)$$

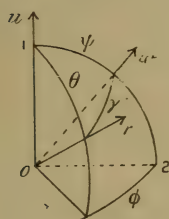
If ψ be the angle between $\mathbf{u}$ and $\mathbf{w}$, we find for the energy per unit volume in the pulse, since $\mu Kr^2 = 1$ and $n = u/v$,

$$\frac{KE^2}{4\pi} = \frac{\mu Q^2 w^2}{16\pi r^2 a^2} \left[\frac{1}{h^2} + \frac{2n \cos \gamma \cos \psi}{h^3} - \frac{(1-n^2) \cos^2 \gamma}{h^4} \right]. \quad (52)$$

One angular coordinate of r is θ ; if the other be ϕ and be measured round the direction of $\mathbf{u}$ from the plane of $\mathbf{u}$ and $\mathbf{w}$, we have

$$\cos \gamma = \cos \theta \cos \psi + \sin \theta \sin \psi \cos \phi. \quad (53)$$

Fig. 5.



The various angles are shown in fig. 5.

The thickness of the pulse is $2a$, and hence the energy radiated away in the pulse is given by

$$W = \iint \frac{KE^2}{4\pi} 2a r^2 \sin \theta d\theta d\phi.$$

From (53) we find

$$\int_0^{2\pi} \cos \gamma d\phi = \frac{2\pi \cos \psi (1-h)}{n}$$

$$\int_0^{2\pi} \cos^2 \gamma d\phi = \frac{2\pi}{2n^2} \{ (2-3 \sin^2 \psi)(1-h)^2 + n^2 \sin^2 \psi \}.$$

Integrating with respect to ϕ from 0 to 2π , and writing dh/n for $\sin \theta d\theta$, we find

$$W = A - B \sin^2 \psi,$$

$$A = \frac{\mu Q^2 w^2}{4an^3} \int_{1-n}^{1+n} \left(-\frac{1}{h^2} + \frac{2}{h^3} - \frac{1-n^2}{h^4} \right) dh,$$

$$B = \frac{\mu Q^2 w^2}{4an^3} \int_{1-n}^{1+n} \left\{ -\frac{3+n^2}{2h^2} + \frac{3-n^2}{h^3} - \frac{(1-n^2)(3-n^2)}{2h^4} \right\} dh.$$

Integrating with respect to h , we find

$$W = \frac{\mu Q^2 w^2}{3a(1-n^2)^2} (1-n^2 \sin^2 \psi)^*.$$

§ 32. In the pulse, $\mathbf{H} = vK\mathbf{V}\mathbf{r}_1\mathbf{E}$, and thus the momentum per unit volume is $(K/4\pi v)\mathbf{VEV}\mathbf{r}_1\mathbf{E}$ or $\mathbf{r}_1 K\mathbf{E}^2/4\pi v$, since $\mathbf{E}$ is perpendicular to $\mathbf{r}$. Hence, if $\mathbf{P}_1$ be the momentum in the pulse in the direction of $\mathbf{u}$,

$$\mathbf{P}_1 = \iint \frac{K\mathbf{E}^2}{4\pi v} 2ar^2 \cos \theta \sin \theta d\theta d\phi.$$

Using the values of $\int \cos \gamma d\phi$ and $\int \cos^2 \gamma d\phi$ given in § 31 and writing $(1-h)/n$ for $\cos \theta$, we find

$$\mathbf{P}_1 = \mathbf{C} - \mathbf{D} \sin^2 \psi$$

$$\mathbf{C} = \frac{\mu Q^2 w^2}{4av n^4} \int_{1-n}^{1+n} \left\{ \frac{1}{h} - \frac{3}{h^2} + \frac{3-n^2}{h^3} - \frac{1-n^2}{h^4} \right\} dh$$

$$\mathbf{D} = \frac{\mu Q^2 w^2}{4av n^4} \int_{1-n}^{1+n} \left\{ \frac{3+n^2}{2h} - \frac{9-n^2}{2h^2} + \frac{(3-n^2)^2}{2h^3} - \frac{(1-n^2)(3-n^2)}{2h^4} \right\} dh.$$

Integrating with respect to h , we find

$$\begin{aligned} \mathbf{P}_1 = \frac{\mu Q^2 w^2}{4av n^4} & \left[\log \frac{1+n}{1-n} - \frac{2n(3-5n^2)}{3(1-n^2)^2} \right. \\ & \left. - \sin^2 \psi \left\{ \frac{3+n^2}{2} \log \frac{1+n}{1-n} - \frac{n(9-12n^2-n^4)}{3(1-n^2)^2} \right\} \right]. \end{aligned}$$

If O2 (fig. 5) be in the plane of $\mathbf{u}$ and $\mathbf{w}$ and at right angles to $\mathbf{u}$, the cosine of the angle between $\mathbf{r}$ and O2 is $\sin \theta \cos \phi$. Hence, if $\mathbf{P}_2$ be the momentum in the direction of O2,

$$\mathbf{P}_2 = \iint \frac{K\mathbf{E}^2}{4\pi v} 2ar^2 \cos \phi \sin^2 \theta d\theta d\phi$$

From (53) we find

$$\int_0^{2\pi} \cos \gamma \cos \phi d\phi = 2\pi \cdot \frac{1}{2} \sin \theta \sin \psi.$$

$$\int_0^{2\pi} \cos^2 \gamma \cos \phi d\phi = 2\pi \cdot \sin \theta \cos \theta \sin \psi \cos \psi.$$

* Dr. O. Heaviside has calculated W for finite values of w , when $\mathbf{w}$ is in the same direction as $\mathbf{u}$. 'Nature,' Nov. 6, 1902.

Thus we obtain

$$P_2 = \frac{\mu Q^2 w^2 \sin \psi \cos \psi}{4av} \int_0^\pi \left\{ \frac{n}{h^3} - \frac{(1-n^2) \cos \theta}{h^4} \right\} \sin^3 \theta d\theta.$$

Since $\cos \theta = (1-h)/n$ and $\sin \theta d\theta = dh/n$, this becomes

$$P_2 = \frac{\mu Q^2 w^2 \sin \psi \cos \psi}{4avn^4} \int_{1-n}^{1+n} \left\{ \frac{(1-n^2)^2}{h^4} - \frac{3(1-n^2)}{h^3} + \frac{3-n^2}{h^2} - \frac{1}{h} \right\} dh.$$

Hence, finally,

$$P_2 = \frac{\mu Q^2 w^2 \sin \psi \cos \psi}{4avn^4} \left[\frac{2n(3-2n^2)}{3(1-n^2)} - \log \frac{1+n}{1-n} \right].$$

By symmetry the momentum in the pulse has no component perpendicular to the plane of $\mathbf{u}$ and $\mathbf{w}$.

§ 33. When n is small but, of course, not zero we find that

$$P_1 = \frac{w^2 u m_0}{10v^2} (4 - \sin^2 \psi), \quad P_2 = \frac{w^2 u m_0}{10v^2} \sin \psi \cos \psi,$$

where $m_0 = 2\mu Q^2/3a$, the electromagnetic mass of the sphere for infinitesimal speeds.

It is easily found that the whole momentum in the pulse can be expressed in the vector form

$$\mathbf{P} = (m_0/10v^2) \{ 3\mathbf{u} \cdot \mathbf{w}^2 + \mathbf{w}(\mathbf{u}\mathbf{w}) \}.$$

The angle β between $\mathbf{P}$ and $\mathbf{u}$ has its maximum value when $\tan \psi = 2/\sqrt{3}$, the maximum value being given by $\tan \beta = (48)^{-\frac{1}{2}}$. These values correspond to $\psi = 49^\circ 6'$ and $\beta = 8^\circ 13'$.

§ 34. When n is nearly equal to unity, so that $(1-n^2)^{-1}$ is large, we again obtain simple expressions for P_1 and P_2 . Since $\log x$ is negligible in comparison with x when x is large, we see, by § 32, that, when u is nearly equal to v ,

$$P_1 = \frac{m_0 w^2 (1 - n^2 \sin^2 \psi)}{2(1 - n^2)^2 v}, \quad P_2 = \frac{m_0 w^2 \sin \psi \cos \psi}{4(1 - n^2)^2 v}.$$

The resultant momentum in the pulse may be regarded as being in the direction of $\mathbf{u}$. For, if β be the angle between $\mathbf{P}$ and $\mathbf{u}$, we have

$$\tan \beta = \frac{P_2}{P_1} = \frac{(1 - n^2) \sin \psi \cos \psi}{2(1 - n^2 \sin^2 \psi)},$$

and $\tan \beta$ vanishes when $\psi = 0$ and when $\psi = \frac{1}{2}\pi$. The maximum value of $\tan \beta$ is the small quantity $\frac{1}{2}(1 - n^2)^{\frac{1}{2}}$, and this value occurs when $\tan \psi = (1 - n^2)^{-\frac{1}{2}}$.

§ 35. If the centre of the sphere be made to travel with velocity u round a polygon inscribed in a circle of radius r , and if 2α be the angle which each side subtends at the centre, the vector change of velocity at each angle is

$w = 2u \sin \alpha$, and the angle between $\mathbf{w}$ and $\mathbf{u}$ is $\psi = \frac{1}{2}\pi - \alpha$. Then, by § 31, when α is small, and therefore w is small, the mean rate, R , of radiation of energy is

$$R = \frac{2\mu Q^2 u^2 (1 - n^2 \cos^2 \alpha) \sin \alpha}{3ar(1 - n^2)^2}.$$

This result is applicable until α becomes so small that the pulse arising from one impulse has not got clear of the sphere before the next impulse occurs. The smallest permissible value of $\sin \alpha$ is $au/(v-u)r$, and in this case

$$R = \frac{2\mu Q^2 u^4}{3vr^2(1 - n^2)^2} \cdot \frac{1 - n^2 \cos^2 \alpha}{1 - n} = \frac{1 - n^2 \cos^2 \alpha}{1 - n} R_0.$$

Here the radius of the sphere appears only through $\cos^2 \alpha$. If we make a/r infinitesimal, we have $\cos \alpha = 1$ and

$$R = (1 + n)R_0.$$

Dr. Heaviside* has shown that R_0 is the rate of radiation of energy from a true *point-charge* moving with velocity u in a circle of radius r . Hence we see that, however small the sphere may be, the mean rate of radiation when the successive pulses just fail to overlap is greater than when they merge into a continuous disturbance.

§ 36. If a point-charge q has an acceleration $\mathbf{f}$, the change of velocity in time dt is $\mathbf{f} dt$. The disturbance arising during this time is confined to a shell contained between two spheres, one of radius vt described about the position of q at the beginning of the interval, and the other of radius $v(t - dt)$ described about the position of q at the end of the interval. The thickness of the shell is thus $(v - u \cos \theta) dt$ or $vh dt$ and the flux of displacement $\mathbf{K}e/4\pi$ is distributed through this thickness. Hence, putting $\mathbf{f} dt$ for $\mathbf{w}$ in (50) and allowing for the thickness of the shell, we find that the electric force at a great distance from an accelerated point-charge is

$$\mathbf{E} = \frac{q}{Kr} \left[-\frac{\mathbf{f}}{(v - u \cos \theta)^2} + \frac{(v\mathbf{r}_1 - \mathbf{u})f \cos \gamma}{(v - u \cos \theta)^3} \right]$$

where θ and γ are the angles made by $\mathbf{r}$ with $\mathbf{u}$ and $\mathbf{f}$. This result agrees with that given by Dr. Heaviside†, who has also calculated the energy and the momentum carried away in the expanding shell.

§ 37. In the preparation of this paper I have been greatly assisted by Mr. D. C. Jones of Pembroke College. He has not only written out the manuscript for the press but has also verified all the formulæ. My thanks are also due to Prof. J. J. Thomson for reading the paper in manuscript, and to Dr. A. H. Bucherer of Bonn for reading the proofs.

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* 'Nature,' Nov. 6, 1902, equation (10).

† 'Nature,' Nov. 6, 1902. See also M. Abraham, *Theorie der Elektrizität*, ii. p. 97.

VII. *Auroral and Sun-spot Frequencies Contrasted.*By C. CHREE, *Sc.D., LL.D., F.R.S.**

§ 1. **D**URING several recent investigations I have had occasion to contrast the annual variation in years of many and in years of few sun-spots of elements such as the diurnal range of the magnetic declination, or the frequency of occurrence of magnetic storms. The formula first advanced by Wolf

$$R = a + bS \quad . \quad . \quad . \quad . \quad . \quad (1)$$

as connecting R , the range in the mean diurnal inequality of declination throughout the year, with S the corresponding sun-spot frequency— a and b being constants—can be applied with considerable accuracy to the range in individual months of the year, and to magnetic inclination, horizontal force and vertical force, as well as declination. But taking any one element, a and b are different for the different months of the year, and b/a is in general decidedly larger for winter than for summer.

Suppose, now, that dashed letters refer to a winter, undashed to a summer month, and that suffixes 1 and 2 relate to two years in which sun-spots are respectively many and few. Then for the ratio of the ranges in the summer and the winter month concerned, we have in the year of many sun-spots,

$$R_1/R_1' = (a/a') \left(1 + \frac{b}{a} S_1 \right) \div \left(1 + \frac{b'}{a'} S_1 \right),$$

supposing for simplicity the sun-spot frequency the same for the two months. On the same hypothesis, we have for the corresponding ratio in the year when sun-spots are few,

$$R_2/R_2' = (a/a') \left(1 + \frac{b}{a} S_2 \right) \div \left(1 + \frac{b'}{a'} S_2 \right).$$

From these two equations we at once deduce

$$\frac{(R_2/R_2') - (R_1/R_1')}{R_1/R_1'} = \left(\frac{b'}{a'} - \frac{b}{a} \right) (S_1 - S_2) \div \left\{ \left(1 + \frac{b}{a} S_1 \right) \left(1 + \frac{b'}{a'} S_2 \right) \right\}.$$

As already mentioned, observation shows that b'/a' exceeds b/a , and by hypothesis $S_1 - S_2$ is positive, thus

$$R_2/R_2' > R_1/R_1' \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In temperate latitudes, whether sun-spots be many or few, the diurnal range of any magnetic element is larger in

* Communicated by the Physical Society: read November 23, 1906.

summer than in winter, *i.e.* R_1 exceeds R_1' and R_2 exceeds R_2' . Thus (2) shows that *relatively* considered the diurnal range is more variable throughout the year when sun-spots are few than when they are many. In other words, if the annual change of the diurnal range be illustrated by a curve whose ordinates represent the ratios borne by the ranges in individual months to their arithmetic mean for the twelve months, the maximum and minimum ordinates differ more when the year selected is one of few than when it is one of many sun-spots.

§ 2. Again, taking a list of the more considerable magnetic disturbances recorded at Greenwich from 1848 to 1903, as given by Mr. W. Maunder, I obtained the following figures * for the relative frequency at different seasons of the year, treating separately the fourteen years of largest (S max.) and the fifteen years of smallest (S min.) sun-spot frequency:

WINTER.		EQUINOX.		SUMMER.	
November to February.		March, April, Sept. Oct.		May to August.	
S max.	S min.	S max.	S min.	S max.	S min.
35	28	38	48	27	24

The figures denote percentages of the totals for the whole year. The average *absolute* numbers of storms per annum were

Whole period.	S max. years.	S min. years.
13.0	18.4	8.5

With increased sun-spot frequency, the absolute number of storms increased in all three seasons of the year; but *relatively* considered the increase was least in the equinoctial months—the season when magnetic storms are most numerous at Greenwich—and the tendency obviously was towards a more uniform distribution throughout the year. The phenomenon is thus analogous to that described above in the case of the regular diurnal range.

§ 3. In temperate latitudes, as is well known, magnetic storms of any considerable intensity are usually associated with auroras. It was thus of interest to determine whether auroral frequency showed phenomena corresponding to those just described in Terrestrial Magnetism. It has long been known that auroral frequency, as observed in temperate latitudes, varies in a general way with sun-spot frequency. The results obtained, however, have not shown a very exact correspondence between the years of maximum and minimum in the two classes of phenomena. In the case of

* Phil. Mag. Sept. 1905, p. 314.

sun-spot frequency—except as regards data for recent years—practically the only source available has been the data published by Wolf and Wolfer, which extend back to 1749. It is improbable that the unit in Wolfer's latest table* represents an absolutely unchanging value throughout the whole period, but great care has been taken to make the table as homogeneous as possible, and the epochs it gives for the occurrence of sun-spot maximum and minimum are presumably, in at least the great majority of cases, very approximately correct.

Auroral data are exposed to many more uncertainties. The observed frequency varies enormously at different parts of the Earth, and the number of auroras recorded in any specified area is largely dependent on the provision made for observing and recording them. Conspicuous auroras are unlikely to escape notice in populous countries where they are rare occurrences, but in Arctic latitudes where auroras are common many doubtless fail to be recorded. With the increase of population and the development of means of communication characteristic of the last 100 years, there has no doubt been a tendency to an increase in the proportion of auroras which come to be recorded.

Thus auroral frequency is a quantity which is certainly not expressed in terms of an invariable unit; and the various tables which have been published show irregularities due to temporary and local causes, whose disturbing influence it is practically impossible to assess. In the following investigations the methods adopted aim at reducing to a minimum the effect of the various uncertainties.

§ 4. The sun-spot frequencies made use of are derived exclusively from Wolfer's table, which gives data for each individual month during the 153 years 1749 to 1901. The auroral frequencies are from two sources, viz.: "Catalog der in Norwegen bis Juni 1878 beobachteten Nordlichter zusammengestellt von Sophus Tromholt, herausgegeben von J. Fr. Schroeter" (Kristiania, 1902), and Joseph Lovering's "On the Periodicity of the Aurora Borealis" (Mem. American Academy, New Series, vol. x. 1868).

Of the several tables in the former work, that employed is Table E, pp. 414-417, which gives auroral frequencies derived from the whole of Scandinavia from July 1761 to June 1878. In this table Schroeter has combined Tromholt's results for Norway with those of Rubenson for Sweden. In the original the yearly totals are for years commencing in July. In order, however, to obtain results more strictly

* *Met. Zeit.* 1902, vol. xix. p. 195.

comparable with Wolfer's mean annual sun-spot frequencies, I have calculated from the Scandinavian monthly totals data for years commencing in January. In addition to yearly totals—from July to June—for the whole of Scandinavia, Tromholt's Table E gives yearly totals for five subdivisions of the country, numbered I. to V. according to latitude. I. includes all districts north of $68^{\circ}5$, II. extends from $68^{\circ}5$ to 65° , III. from 65° to $61^{\circ}5$, IV. from $61^{\circ}5$ to $58^{\circ}5$, while V. includes the extreme south of Scandinavia from $58^{\circ}5$ to 55° .

For a considerable time subsequent to 1761, observations from district I. were very few, a fact due probably more to lack of observers than anything else. This possesses some importance for the following reason. The annual variation in auroral frequency is largely dependent on the fact that aurora is seldom vivid enough to be visible until the sun is several degrees below the horizon. In high latitudes there is daylight throughout the whole 24 hours near midsummer, and no daylight near midwinter, and the auroral frequency in these regions, as was pointed out many years ago by Lovering, has a single maximum near midwinter, and a single minimum answering to a total absence of aurora near midsummer*. This state of matters is at least approached in district I., and to a lesser extent in district II. (*cf. loc. cit.* Table G, p. 420). Further south in Scandinavia the annual variation is similar to that in England, showing two maxima near the equinoxes, a principal minimum at midsummer, and a secondary minimum at midwinter. The mean annual variation deduced for the whole of Scandinavia will clearly depend to some extent on how far the several districts contribute to the general result. Assuming an increasing relative contribution from district I., if annual variations be calculated from two different periods, one may not unreasonably expect the later period to show the equinoctial maxima less prominently and the midsummer minimum more prominently than the earlier period. This is one of my reasons for contrasting one period of a special type with two of an opposite type, the one earlier the other later.

Lovering gives annual variations for a number of separate stations. Most of these, however, are based on too few years' observations to suit the present enquiry.

Of the data for separate stations or districts, I propose to use only those for New York State (*l. c.* p. 181). These extend over 26 years, including 1205 separate observations,

* *Cf.* A. Angot's 'Les Aurores Polaires,' Paris, 1895, p. 125.

and represent a latitude much lower than that of Scandinavia. The other data employed are from Lovering's General Catalogue, pp. 195-200. This comes down to 1864, and extends to earlier than the 14th century. Though enumerating nearly 10,000 auroras it is probably, judging by Tromholt's figures, very far from complete. The data are doubtless of a very heterogeneous character, and the same precautions appeared necessary as in the case of the Scandinavian data.

Recently Prof. Schuster* has gone very fully into the existence of periodic variations in Wolfer's sun-spot frequencies, and has concluded that there is evidence of the existence not merely of the ordinarily recognized period of 11.125 years, but of others which like it are submultiples of a period whose most probable value is 33.375 years. This is one of the reasons why I have dealt with three successive 33-year periods, viz. 1761-1793, 1794-1826, and 1827-1859.

§ 5. Before comparing sun-spots and auroras, I would draw attention to some features of Table I., which gives mean sun-spot frequencies for the several months of the year, as calculated by me from Wolfer's table for a number of specified combinations of years. It has been remarked by Mr. Ellis†—who based his remarks on Wolfer's data *for groups of 3 months* (February to April, &c.)—that sun-spot frequency shows no real annual period. The results in the first line of Table I. cannot be said to be decisive against the existence of a small annual term, though inconsistent with the existence of a large term having this period. The extent of the difference between the mean results for the several months is more easily realized in Table II., which expresses the monthly values in Table I. as percentages of the arithmetic mean for the 12 months. One would, I think, hardly have anticipated, in means based on 153 years' data, the difference of $6\frac{1}{2}$ per cent. shown between the values for January and May. On the other hand, if there were a true annual period, one would expect the percentage figures in Table II. for the three successive 33-years periods to resemble one another more closely than they actually do.

The monthly means in these three periods show very considerable differences; in the earliest period, for instance, the difference between the March and November means amounts to 20 per cent. of the mean value for the epoch.

* Proc. Roy. Soc. A. vol. lxxvii. p. 145.

† Monthly Notices R. A. S. vol. ix. p. 142.

TABLE I.—Wolf and Wolfer's Sun-spot Frequencies.

	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Year.
153 years	44.41	46.67	45.40	46.13	47.41	46.02	45.46	45.83	45.91	46.52	46.36	46.23	46.04
33 " 1749 to 1801.....	63.94	64.58	57.93	63.80	70.92	65.70	63.79	62.56	68.21	66.97	71.11	67.46	65.59
" 1761 to 1793.....	20.32	21.67	20.76	19.57	18.67	20.21	21.19	19.68	17.73	20.99	19.43	23.13	20.28
" 1794 to 1826.....	55.53	56.95	56.95	56.28	54.76	53.09	50.94	55.67	57.38	60.96	55.43	58.86	56.07
" 1827 to 1859.....													
39 years S max. 1760 to 1894.....	81.76	85.78	84.13	89.24	89.66	89.21	88.35	89.03	88.04	90.93	90.23	90.08	88.05
" " S min. 1754 to 1890.....	13.63	14.36	14.44	12.09	13.06	11.34	9.87	10.26	9.63	12.45	12.95	12.12	12.19
33 years S max. 1760 to 1872.....	83.76	88.16	88.36	90.52	92.98	92.21	90.75	91.79	92.57	94.84	94.44	94.42	91.25
" " S min. 1754 to 1867.....	14.67	16.01	15.87	13.29	14.11	12.21	10.52	10.65	9.55	13.64	13.72	13.38	13.14

TABLE II.—Sun-spot Frequencies. Percentages of mean year.

	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Winter.	Summer.
153 years	96.5	101.4	98.6	100.2	103.0	100.0	98.8	99.6	99.7	101.1	100.7	100.4	99.8	100.2
33 " 1749 to 1801.....	97.5	98.5	88.3	97.3	108.1	100.2	97.3	95.4	104.0	102.1	108.4	102.9	99.6	100.4
" 1761 to 1793.....	100.2	106.8	102.4	96.5	92.1	99.7	104.5	97.0	87.4	103.7	95.8	114.0	103.8	96.2
" 1794 to 1826.....	99.0	101.6	101.6	100.4	97.7	94.7	90.9	99.3	102.3	108.7	98.9	105.0	102.5	97.5
" 1827 to 1859.....														
39 years S max. 1760 to 1894.....	92.9	97.4	95.6	101.4	101.8	101.3	100.3	101.1	100.0	103.3	102.5	102.3	99.0	101.0
" " S min. 1754 to 1890.....	111.9	117.9	118.5	99.2	107.2	93.1	81.0	84.2	79.0	102.2	106.3	99.5	109.4	90.6
33 years S max. 1760 to 1872.....	91.8	96.6	96.8	99.2	101.9	101.1	99.4	100.6	101.5	103.9	103.5	103.5	99.4	100.6
" " S min. 1754 to 1867.....	111.7	121.9	120.8	101.2	107.4	93.0	80.1	81.1	72.7	103.8	104.5	101.9	110.7	89.3

The central period 1794-1826 comprises that period of 33 consecutive years which gives a minimum mean sun-spot frequency. That mean is only a third of the mean appropriate to the combined periods 1761-93 and 1827-59.

As a rule, three successive years of conspicuously high frequency occur at each sun-spot maximum, and three of conspicuously low frequency at each minimum. Wolfer's table includes 13 groups of these 3 extreme years of sun-spot maximum, and 13 of sun-spot minimum. The respective means from these 39 years of maximum and 39 years of minimum form the fifth and sixth rows of Tables I. and II. The last two lines in these Tables refer to shorter groups of 33 years of sun-spot maximum and 33 of minimum which correspond more closely to the auroral data presently to be described.

§ 6. Probably the simplest way of investigating the relationship between sun-spot frequency and the magnitude of any element is to form two mean values for the element, the one corresponding to years of many, the other to years of few sun-spots, and then to assume that the difference between these means depends on the corresponding difference in sun-spot frequency. In applying this method to data from the whole period covered by Wolfer's table, one would employ the sun-spot frequencies given in the 5th and 6th rows of Table I. Doing so, we should get differences of sun-spot frequency varying from 81.76-13.63, or 68.13, in January, to 78.77 in August. The existence of so large a difference between the sun-spot data for January and August is immaterial, provided there is a direct connexion, which possesses no lag, between sun-spot frequency and the element concerned. If, however, the connexion is of a less simple character, for instance if the element depends on the sun-spot frequency for some months previously, the application of the above method would lead to an overestimate of the influence of sun-spot frequency in January and an underestimate in August. If we use more exact methods, *e. g.* the method of least squares,—still assuming purely synchronous variation—the above source of uncertainty is less easily recognized, but it exists all the same. Considered absolutely, the difference between the mean monthly values in Table I. is greater for the 39 or the 33 years of S max. than for the corresponding group of S min. years; but relatively considered—*cf.* Table II.—the variability from month to month is greater for the S min. group. Thus in the 39 years of S min. the means for March and September differ by 39 per cent. of the mean from the 12 months. Moreover in the S min. groups

TABLE III.—Auroral Frequency. Annual Variation.

	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Total Auroras.	Average per year.
Scandinavian, 1761-1877	12.40	11.78	13.18	7.82	0.93	0.04	0.26	4.17	11.87	13.52	12.23	11.80	9243	79.0
" 1761-1793	10.59	10.55	12.62	9.71	1.99	0.08	0.60	6.13	14.37	13.02	10.51	9.83	2512	76.1
" 1827-1859	13.29	11.42	12.07	7.54	0.43	0.00	0.18	3.66	11.67	14.26	12.68	12.78	2784	84.4
" Two periods } combined. }	11.94	10.99	12.34	8.63	1.21	0.04	0.39	4.90	13.02	13.64	11.60	11.30	5296	80.2
" 1794-1826	13.96	14.49	14.65	4.52	0.31	0.00	0.08	1.92	9.12	13.42	14.49	13.03	1304	39.5
Lovering's General Catalogue.														
" Up to 1864	9.54	9.73	11.96	9.37	5.09	3.22	3.72	6.45	10.39	11.37	9.87	9.29	9766	...
" 1761-1793	7.81	8.14	11.44	11.03	8.05	4.67	5.49	7.19	10.24	10.33	8.47	7.15	2421	73.4
" 1827-1859	10.61	9.31	10.44	8.46	4.95	3.72	4.62	6.13	9.79	10.54	10.18	11.26	3900	118.2
" Two periods } combined. }	9.21	8.73	10.92	9.75	6.50	4.20	5.05	6.66	10.01	10.44	9.33	9.21	6321	95.8
" 1794-1826	10.73	12.44	14.48	11.93	4.26	1.36	1.02	3.58	7.84	9.03	10.90	12.44	587	17.8

of years there is a decided difference—some 20 per cent.—between the means derived from the 6 winter and the 6 summer months. There is a smaller difference, but in the same direction, between the winter and summer means from the 33-year period 1794–1826 remarkable for its low average sun-spot frequency. This is unquestionably somewhat suggestive of an appreciable real annual period in sun-spot frequency in years when sun-spots are few; but whereas in the 39 or 33 years of *S min.* the mean frequency is large for May and November, and small for July, in the period 1794–1826 the exact opposite is seen. Again, an appreciable excess in the winter over the summer mean also appears in the period 1827–59, when the average sun-spot frequency considerably exceeded the average from the whole 153 years.

§ 7. Proceeding to Table III., we have in the first line the annual variation of auroral frequency in Scandinavia as derived from 117 years. The monthly values represent percentages of the value for the whole year. The largest values occur in October and March, and a secondary minimum is recognizable in December. The dip in the February value arises really from the smaller number of days in that month. When referred to an equal number of days, the February frequency exceeds that in January in the ratio of 104 : 100.

In the second and third lines we have similarly annual variations from the two 33-year periods of high average sun-spot frequency, their mean appearing in the fourth line. For the ratio, however, between the mean frequencies from the earlier and from the later of these two periods we have

From Sun-spots.	From Auroras.
100 : 85	100 : 111.

If we contrasted these two periods with one another we should thus associate an increase in auroras with a diminution in sun-spots.

Again, the annual variations from the two periods differ markedly. In the later period, as compared to the earlier, the summer frequencies fall and the winter frequencies rise.

The differences apparent between the two periods may represent a real change, but in all probability they are largely due to an increase in auroral observers, especially in the northern districts of Scandinavia. Taking the yearly data for the five districts mentioned above, we obtain for the total number of auroras observed the results given in Table IV. (p. 158).

TABLE IV.--Auroral Observations in Scandinavia.

	District.					All Scandinavia.
	I.	II.	III.	IV.	V.	
July to June.						
1761 to 1794.....	22	293	970	1453	555	2481
1794 to 1827.....	52	694	408	518	56	1316
1827 to 1860.....	827	873	352	1531	473	2811
1st & 3rd periods combined.	849	1166	1322	2984	1028	5292

An aurora is often seen in more than one district, but comparatively seldom in all. Obviously the increase in the auroral frequency in the latest as compared to the earliest of the three periods arises from the large increase of observations in the two most northern districts ; and this being so, the tendency naturally is to bring the annual frequency nearer to the Arctic type with a single maximum at midwinter.

Coming now to the intermediate period in Table IV., we see that the development of district I. was mainly subsequent to 1827. Thus what we should *a priori* expect to observe in the annual variation from this period in Table III. would be a form intermediate between those from the other 33-year periods, but approaching most closely that from the earlier period. What we actually do find is a variation differing from that of 1761-93 in the expected direction, but to an even *greater* extent than does the variation from the period 1827-59. Comparing the results from 1794-1826 in Table III. with the means from the preceding and succeeding 33-year periods we largely eliminate the influence of the change in observational conditions. The figures show that in the 33-year period characterized by few sun-spots, summer occurrences of aurora were relatively much fewer and winter occurrences more numerous than in the adjacent 33-year periods of high average sun-spot frequency. We thus have low sun-spot frequency associated with an exaggeration in the annual variation of auroras, the precise phenomenon already described in connexion with Terrestrial Magnetism.

§ 8. The mean sun-spot frequency for the two 33-year periods 1761-93 and 1827-59 combined was 60·83, whilst the corresponding auroral frequency for the year was 80·2.

Comparing these with the corresponding figures for 1794-1826, we have for a trebling of sun-spot frequency only a doubling of auroral frequency. This would suggest that if

auroral frequency is connected with that of sun-spots by a formula of type (1), then the constant a does not vanish; *i. e.*, a total absence of sun-spots would not be accompanied by a total absence of auroras. A difficulty, however, arises here, which the figures in Table IV. for the several districts of Scandinavia will serve to explain.

The ratio borne to the frequency of auroras in the average year of the period 1794–1827 by the corresponding frequency for the combined periods 1761–94 and 1827–60 is roughly 2 : 1 for the whole of Scandinavia; but is 9 : 1 for district V., 3 : 1 for district IV., 3 : 2 for district III., and less than 1 : 1 for district II. Any comparison for district I. would be misleading.

These figures suggest that with decrease in sun-spot frequency the diminution in auroral frequency is enormously greater in the south than in the north of Scandinavia. Several authorities have called attention to analogous phenomena, and the theory has even been advanced that the difference in auroral frequency in years of many and few sun-spots really arises from the alternate expansion and contraction of Fritz's isochasms* (curves of equal auroral frequency). The theory does not seem to be strongly supported, but there seems little if any doubt that a substantial difference really exists between the long period changes of auroral frequency in different regions. Results from several stations in Greenland—whose substantial accuracy seems accepted by Prof. A. Paulsen, one of the leading authorities on the subject—appear to indicate that auroral frequency is there very considerably *less* when sun-spots are many than when they are few.

§ 9. We now pass to the consideration of the figures from Lovering's general catalogue in the last five lines of Table III. As before, the monthly figures are expressed as percentages of the yearly total. The figures corresponding to the whole period covered by the table are given by Lovering † himself. Though derived from very heterogeneous data, they represent very fairly the type of annual variation which is characteristic of lower temperate latitudes. The midsummer minimum and the equinoctial maxima are less pronounced than they are even in the South of England.

Comparing the totals from the two periods 1761–93 and 1827–59, we see that the excess from the later period is even greater than it was in the case of Scandinavia. The annual

* Cf. Angot, 'Les Aurores Polaires,' p. 138.

† *L. c.* pp. 200 & 216. The total for December on p. 200 should apparently be 907, and not 1007 as printed

variations from the two periods also differ, and in the same direction as before. From May to September the relative frequency is decidedly less, and at midwinter considerably greater, for the later period than for the earlier, and the equinoctial maxima are but indistinctly shown in the later period. As in the case of Scandinavia, northern stations may have contributed more to the means for 1827-59 than to those for 1761-93. It is clear, however, from the substantial frequencies in the summer months, that even in the period 1827-59 data from temperate latitudes must largely have prevailed. Thus the closer approach to the Arctic or single maximum type is difficult to wholly explain, unless we admit that it is partly a real phenomenon, representing a real difference in the distribution of auroras throughout the average year of the two periods 1761-93 and 1827-59. As regards the intermediate period 1794-1826, we see that its mean annual variation differed from that of the neighbouring periods in the same direction as it did in the case of Scandinavia. The fall in the summer frequencies is very pronounced, even as compared to the period 1827-59 alone. Of the equinoctial maxima, that in March is much enhanced, but that in October seems to have vanished, unless it is represented by the maximum now shown in December.

The average year from the two periods 1761-93 and 1827-59 combined shows fully five times the auroral frequency of the average year of the period 1794-1826. This is intermediate between the ratios deduced for the Scandinavian districts IV. and V.

§ 10. Table V. shows the annual variation of auroral frequency in several pairs of groups of years characterized respectively by many and by few sun-spots. The first line gives results for the whole of Scandinavia from 30 years, made up of the 3 years of largest sun-spot frequency from each of the ten 11-year cycles covered by Tromholt's table: the second line gives the results for the corresponding 30 years of few sun-spots. Any *gradual* change in the nature of the observations should affect the two sets of results nearly equally. The third and fourth lines give corresponding data calculated from Lovering's general catalogue, also for 30 years of many and 30 years of few sun-spots.

Of the ten 11-year cycles employed in the two cases nine are identical; the Lovering data cover one cycle prior to the common nine, the Scandinavian one later. This was unfortunately necessary owing to the dates when the Lovering table ended and the complete Scandinavian table commenced. The two sets of data combined represent eleven consecutive

TABLE V.—Auroral Annual Variation in years of many and few Sun-spots.

	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Total Auroras.	Average per year.
<i>Scandinavian.</i>														
30 years S max. 1769-1872 ...	11.81	11.08	12.58	7.98	0.67	0.10	0.27	4.57	12.01	13.44	12.91	12.58	2997	99.9
" S min. 1764-1867 ...	12.47	12.26	14.05	7.59	1.03	0.00	0.16	3.85	12.04	13.72	12.69	10.14	1844	61.5
<i>Lowering's General Catalogue.</i>														
30 years S max. 1760-1861 ..	8.44	8.56	9.80	9.84	5.68	4.48	4.84	7.48	10.08	11.12	9.92	9.76	2500	83.3
" S min. 1754-1856 ...	10.86	10.13	12.92	10.92	5.33	3.34	3.82	4.97	9.76	8.79	9.95	9.22	1649	55.0
<i>New York State.</i>														
9 years S max. 1828-1849 ...	6.60	7.45	7.66	11.28	5.53	7.87	8.51	10.21	11.49	9.79	7.66	5.96	470	52.2
6 " S min. 1832-1844 ...	5.99	6.45	10.14	12.45	7.37	7.37	8.76	6.45	12.45	11.98	5.53	5.07	217	36.2

TABLE VI.—Auroral Annual Variation in years of many and few Auroras.

	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Total Auroras.	Average per year.
<i>Scandinavian.</i>														
30 years Auroral max. 1768-1872	12.98	10.72	12.78	8.07	0.71	0.09	0.30	3.99	11.52	13.97	12.66	12.21	3358	111.9
" " min. 1765-1866	11.00	10.17	12.67	8.19	1.09	0.00	0.19	4.80	12.99	14.97	13.63	10.30	1363	52.1
<i>New York State.</i>														
9 years Auroral max. 1829-1850	6.35	9.20	10.87	9.87	7.69	7.02	8.36	11.04	9.03	10.20	5.02	5.35	598	66.4
" " min. 1827-1851	5.26	3.86	7.02	11.23	10.18	3.16	6.67	10.88	16.49	12.98	4.91	7.37	285	31.7

11-year cycles, and the sun-spot data for the 33 years of many and the 33 years of few sun-spots from these eleven cycles are those given in the last two lines of Tables I. and II. For the purposes of the present enquiry, these sun-spot data answer sufficiently closely to either the Scandinavian or the Lovering auroral data.

The last two lines of Table V. give auroral data for New York State, derived from 3×3 years of largest sun-spot frequency and 2×3 years of least sun-spot frequency included between 1828 and 1849.

Each of the three pairs of comparative figures in Table V. shows a decided rise in the spring maximum in the years of few as compared to the years of many sun-spots.

The average monthly frequency from May to August is also less in the years of few sun-spots, very decidedly so in the case of Lovering's data; in the New York data the depression of the midwinter minimum is the more decided. On the whole, the phenomena resemble those already described, *i. e.* the annual variation is accentuated in the years of few sun-spots. The differences, however, between the selected years of many and the selected years of few sun-spots are less conspicuous than those between the 33-year periods; and such differences as exist may not unreasonably be partly ascribed to the differences in the annual variations of the sun-spot figures in the last two lines of Tables I. & II. This explanation cannot, at the same time, go very far, in view of the fact that there is no marked depression in the auroral frequency in September in the groups of 30 years of few sun-spots.

The differences between the average number of auroras in the year from the first two groups of contrasted years in Table V. are much less than the corresponding differences in Table III., notwithstanding that the sun-spot differences are larger in the case of Table V. The ratio of the mean frequency from the period or group of years of many sun-spots to the corresponding frequency from the period or group of years of few sun-spots takes the following approximate values in the several cases (p. 163).

Obviously the conclusions one would draw as to the extent of the influence of sun-spot on auroral frequency would vary immensely according to one's method of attacking the problem.

§ 11. Table VI. deals with the same Scandinavian and New York data as the previous tables, but instead of selecting groups of 3 years of S max. and S min., it selected groups of

	From the 33-year periods. Tables I. and III.		From 10×3 years of S max. & S min. Tables I. and V.	
	Sun-spot ratio.	Auroral ratio.	Sun-spot ratio.	Auroral ratio.
Scandinavian data ..	3 : 1	2 : 1	7 : 1	1·6 : 1
Lovering's data.....	3 : 1	5·5 : 1	7 : 1	1·5 : 1

3 successive years of largest and 3 successive years of least auroral frequency. One of the groups for New York State was made up of the last and the two earliest years of the period of observation. The differences between the mean sun-spot frequencies for the contrasted groups of years in Table VI. are really much less than in the corresponding cases in Table V., but the opposite is true of the mean auroral frequencies. For the ratio between the auroral frequencies in the contrasted groups of years we have 21 : 10 in Table VI., both for Scandinavia and New York State, whereas in Table V. the corresponding ratios are respectively only 16 : 10 and 15 : 10.

As to annual variation in Table VI., the group of years of few auroras shows, as compared to that of many auroras, an enhanced maximum at one or both equinoxes, and a lower minimum both in winter and summer.

Lovering's general catalogue is not considered in Table VI. owing to the erratic way in which the auroral frequencies in it vary from year to year. In some of the 11-year cycles years of many and few auroras seem to occur almost promiscuously, and the selection of 3 successive years as representative of either high or low frequency presented difficulties. Even in the comparatively homogeneous Scandinavian data, the same phenomenon occurred to a certain extent.

§ 12. The difference between the results in Tables V. and VI., and the great irregularity in the variations from year to year of auroral as compared to sun-spot frequency, point to one of two conclusions,—either

- (1) Auroral data are so heterogeneous, or so intrinsically defective, that consecutive years' results are affected by large differential errors when treated as measures of the same quantity; or
- (2) Auroral frequency depends, and to no small extent, on something more than the contemporaneous value of sun-spot frequency.

As to the question of a possible lag: In the case of Scandinavia the groups of 3 years selected from consideration of sun-spot frequency for use in Table V., represented an earlier epoch than those selected from consideration of auroral frequency for use in Table VI. in 14 cases, the same epoch in 2 cases, and a later epoch in only 4 cases. This is, to say the least, not unfavourable to the view that auroral frequency tends to lag behind sun-spot frequency.

In every case we have found the annual variation in auroral frequency, monthly values denoting percentages of the total number for the year, to be more uniform when sun-spots are numerous than when they are few. The differences, however, between the frequencies in the contrasted years are in some instances not very conspicuous, and may be partly due to chance. Also, supposing it to be a fact that the annual variation in temperate latitudes becomes more accentuated as sun-spots diminish, this may mean one of two things. There may be, as sun-spots decrease, a greater relative diminution in summer than in winter of the physical phenomena which appeal to our eyes as aurora, or there may only be a general diminution in the brightness of auroras throughout the whole year. From what happens during magnetic storms, it can hardly be questioned that the cause—presumably electric discharges in the upper atmosphere—to which auroral phenomena are due is often active when aurora is invisible. It may even conceivably be in continuous operation, though incapable of appealing to the eye, however favourable the visual conditions, until a certain minimum intensity is reached.

Only exceptionally brilliant auroras have much chance of being seen until the sun is far below the horizon, so that a general reduction of intensity might well be more prejudicial to visibility at midsummer than at other seasons.

In conclusion, I should like to draw attention to the utility for investigations such as the present of trustworthy auroral observations taken on a uniform plan, desirably throughout more than one sun-spot cycle, at a considerable number of stations suitably distributed over the earth.

VIII. On Radioactinium. By OTTO HAHN, Ph.D.*

IN a previous paper† I gave a brief account of the methods of separation and the radioactive properties of a new product of actinium, which was called "radioactinium," as its position in the disintegration series of actinium is the same as that of radiothorium in the thorium series.

The object of the present paper is to consider in more detail the properties of radioactinium, and also the experimental and theoretical curves, which throw light on its properties as the intermediate product between actinium and actinium X.

I. PREPARATION OF RADIOACTINIUM.

The experiments on radioactinium were made both with actinium of Debiere (kindly lent to me by Sir William Ramsay) and actinium (emanium) of Giesel. Again, as in previous cases, no differences in the radioactive behaviour of the two substances could be observed—a result which proves beyond question the identity of the radioactive constituents of the two preparations.

On dissolving in hydrochloric acid an actinium preparation in the form of a fine powder which has previously been heated, the resulting solution is generally not clear. A small portion, depending on the strength and nature of the acid and on the active preparation employed, remains undissolved.

If a small quantity of the *filtrate* is evaporated to dryness and tested from time to time in an electroscope, it is soon found that the activity continues to increase for a long time. The activity curves first obtained in this way were not very regular, but they showed a different shape from that to be expected; if the rise in activity were only due to the production of actinium X and its successive products.

In addition, the change in activity of the insoluble part was very different from the well-known behaviour of actinium X or of a mixture of actinium and a surplus of actinium X.

In some cases, the activity of the filter, instead of *decreasing* remained nearly constant or actually *increased*. This result was very puzzling, as the activity of the filtrate was also increasing at the same time.

All these results were quite unexpected, and at the time no reasonable explanation suggested itself.

In order to find out whether physical conditions, such as diffusion, moisture, &c., were in any way responsible for this

* Communicated by the Author. *Cyfr*

† 'Nature,' ~~May~~ 1906; Ber. xxxix. p. 1605.

anomalous behaviour, a large number of experiments were undertaken to test the variation of activity under different conditions and with different methods of preparation.

As the general result it was seen, that the undissolved part of the preparation contained a new radioactive product. If, in addition, actinium X also remained in the insoluble portion, the activity of the filter was nearly constant for some time, and then decayed slowly, finally diminishing according to an exponential law with a period of about 20 days.

If there was no actinium X in the insoluble part, the activity of the latter at first increased, reached a maximum after about two or three weeks, and later decayed with a period of about 20 days.

This product with a decay period of about 20 days was later called "radioactinium," when its position in the actinium series had been settled.

The method of preparation of this new product, previously described, was not very satisfactory, and generally gave a mixture of the whole series of actinium products together with a surplus of radioactinium.

The following two methods proved much more satisfactory and rendered it possible to obtain the product radioactinium free, or at least nearly free, from actinium and actinium X.

Actinium, which had been standing for a few months in solution in order to reach the equilibrium state of activity, was precipitated several times with ammonia. By this method Godlewski and, independently, Giesel have shown that actinium X remains in solution, while actinium itself remains in the precipitate. The precipitation had to be repeated several times to make the separation of actinium X as complete as possible.

The actinium precipitate is dissolved in hydrochloric acid, and to the acid solution is added a very little sodium thio-sulphate solution. The latter is decomposed by the acid, and amorphous sulphur soon appears. The sulphur in the cold solution is allowed to settle down for some time and is then filtered and washed. The surplus is then burnt in a porcelain crucible. An appreciable residue is always observed because some of the rare earths, like thorium, are precipitated with the sulphur.

This residue shows considerable activity, depending of course on the amount of actinium taken for the experiment. The activity increases to about twice the initial strength and then decays slowly, finally diminishing according to an exponential law with a period of about 20 days.

The other method, which proved satisfactory for the separation of radioactinium, is described already in my paper on "The Ionization Ranges of Actinium"*. A small quantity of ammonia is added to an actinium solution in radioactive equilibrium in amount sufficient to produce only a partial precipitation. The ammonia filtrate is redissolved and part again precipitated. After repeating this process several times, all the actinium X and the greatest part of the actinium remain in solution, while the radioactinium remains in the precipitate. The latter is then treated in the same manner as the sulphur precipitate described above.

The methods of separation of radioactinium, which we have outlined, are not altogether as satisfactory or reliable as could be wished. In some cases a very complete separation is obtained, but in others some actinium is carried down or other difficulties are encountered. The right conditions are not always easily found, but, in general, the separation can be effected by the methods described above.

Some recent experiments of Levin seem to show that radioactinium can also be separated to some extent by electrolysis.

II. RADIOACTIVE PROPERTIES OF RADIOACTINIUM.

a. *Nature of the Radiations.*

Radioactinium emits α rays. In the previous paper† I described the methods of preparing a thin film of this radioactive product, sufficiently active for the determination of the range of its α particles by the electric method. It was shown that freshly prepared radioactinium emits one homogeneous type of α rays, which are able to ionize the air for a distance of 4.8 cms. at ordinary pressure and temperature.

While newly prepared radioactinium always shows a very considerable α ray activity, the β ray effect at first is very weak, and is probably due to the presence of a slight amount of actinium B. This is in accordance with the results of Levin obtained in Rutherford's laboratory, for he has shown that actinium B is the only product of actinium which emits β rays.

A small amount of actinium emanation is also present. Since actinium X is the direct parent of the emanation, the amount of emanation in the preparation is a measure of the amount of actinium present.

As we shall prove later in this paper, actinium X is formed by the radioactinium: it is therefore very difficult to obtain a

* Phil. Mag. Sept. 1906.

† *Loc. cit.*

radioactinium preparation completely free from emanation, but a preparation can easily be obtained which initially shows only a very small emanating power compared with the value observed two or three weeks later.

The amount of emanation and of β ray activity observed rise proportionally, since actinium B which emits the β rays is a product of the emanation.

The α and especially the β ray activity of the radioactinium preparation increase rapidly for a week or more, and then more slowly as the maximum is reached (see figs. 1 & 2, pp. 173 & 175). The activity measured by the α rays reaches a maximum slightly earlier than the activity measured by the β rays or by the amount of emanation present.

b. *Decay Period of Radioactinium.*

It was very soon recognized that the period of the decay was not the 10.2 day period of actinium X but considerably slower. The activity, whether measured by the α or β rays or the amount of emanation present, finally decays according to an exponential law. It generally does not decay quite to zero, but a small residual activity of a few percent. was observed, indicating that the preparation contained small traces of actinium. By using actinium preparations carefully freed from radium, the possibility of this residual activity being due to radium was excluded.

After subtracting the small residual activity, the exponential nature of the decay curve was easily to be observed. The following values of the decay constant λ (sec.^{-1}) were found for several decay curves of radioactinium, which had been prepared by different methods and using both the actinium of Debierne and the actinium of Giesel:—

$$0.425 \times 10^{-6}$$

$$0.413 \times 10^{-6}$$

$$0.399 \times 10^{-6}$$

$$0.407 \times 10^{-6}$$

$$0.412 \times 10^{-6}$$

$$\text{Average} \quad . \quad 0.411 \times 10^{-6}$$

From this it follows that the time for radioactinium to be half transformed is 1.68×10^6 secs. or 19.5 days.

This value is a little lower than the value of 20 days, which I stated in my first paper on the subject. The period 19.5 days may still be slightly in error. Further measurements are in progress to determine the period with more accuracy.

For an example the following table gives the experimental (I.) and theoretical (II.) values of the activity ; II. supposing the period is 19.5 days. This only refers to the part of the decay curves obtained about one month and a half after the preparation of the radioactinium, when the decay is exponential. The numbers at I. are corrected by subtracting a small constant residual activity. In both cases, the value after 19.5 days is taken as 50.0.

Time in days.	I. Observed.	II. Calculated.
0	98.2	100
5	84.8	83.8
8	77.2	75.1
19.5	50.0	50.0
45	21.2	20.0
62	11.4	11.0
67	9.30	9.25
80	5.85	5.80
89	4.25	4.25
109	2.45	2.1

Thus over an interval of nearly four months the numbers are in a satisfactory agreement, showing that the decay is exponential with the period stated above.

c. Position of Radioactinium in the Radioactive Series.

The following experiments were made to throw light on the position of radioactinium in the actinium series. The radioactinium was prepared after one of the methods previously described and was tested on the filter. It exhibited the characteristic variation of α and β ray activity, viz. a rapid rise measured by the β rays and a more gradual rise measured by the α rays. The maximum value was reached after two and a half weeks. Shortly after reaching the maximum, the filter containing the sodium thiosulphate precipitate was burnt in a porcelain crucible and a solid white residue remained, which probably consisted in part of thorium oxide and other rare earths.

This residue, which weighed only a few milligrams, was then treated with strong nitric acid. A part of the residue remained undissolved. After dilution and filtration, the filtrate was precipitated with ammonia and left standing for a short time. A very small quantity of a colourless jelly-like precipitate appeared and was filtered through the first filter, which contained the trace of undissolved residue. The

ammonia filtrate was evaporated, heated and tested in the electroscope, the filter was burnt, and the small residue also tested in a similar way.

Both preparations showed considerable α ray activity, while the filter also showed considerable β ray activity.

The activity of these two preparations varied, however, in a very different way. The activity of the filter increased in exactly the same way as that of the initial radioactinium, when first prepared; the activity of the filtrate, however, steadily decreased according to an exponential law with a period of about 11 days.

There is no doubt that this decaying product was actinium X, for which Godlewski found a period of 10.2 days.

On repeating the above-described experiment, the activity of the ammonia filtrate was found to decay to half value in about 12 days and then slightly slower, indicating that the product consisted mainly of actinium X but also contained a little radioactinium.

For comparison, some actinium X was now prepared after the method of Godlewski and of Giesel from actinium solution itself and not from radioactinium. On account of the great number of other preparations under investigation, care could not be taken to avoid slight irregularities of the activity arising from the effect of moisture and other influences on the actinium X. The period found for actinium X was therefore not very accurate; 11 days, or even a little longer, was found instead of 10.2 days, as found by Godlewski. It is intended at an early date to repeat these experiments in order to determine with greater accuracy the decay and rise curves of the products actinium, radioactinium, and actinium X.

The separation of actinium X from decaying radioactinium, which originally was almost entirely free from actinium X, shows clearly that actinium X is continuously produced by radioactinium. This is confirmed by the fact that old radioactinium, from which the actinium X has been removed, again increases in activity in exactly the same way as a newly prepared specimen. This increase of activity is due to the formation of actinium X and its successive products; after reaching a maximum, the actinium X and radioactinium are in approximate radioactive equilibrium, and the activity finally decays according to the period of radioactinium.

In order to prove this connexion beyond question, the ionization curves of the α particles of radioactinium were determined at different intervals after preparation.

In my paper on "The Ionization Ranges of the α Particles of Actinium," I described the methods of preparing a thin

film of the active substance and gave the ionization curves obtained with it.

The success was complete. Immediately after preparation of the radioactinium the α rays were found to be homogeneous and had a range of 4.8 cms. This result showed that only one α ray product, viz. radioactinium, was present.

Progressive measurements showed the gradual entrance of new kinds of α rays, the most penetrating of which were shown to be due to actinium X.

If the actinium X was separated the ionization curves showed exactly the same variation with time as before.

d. *Comparison of the Experimental with the Theoretical Curves.*
(1) *α Ray Curves.*

Knowing the period of radioactinium and its connexion with the other actinium products, it is not difficult to deduce theoretically the rise and decay curves, measured by the α and β rays, that should be exhibited by radioactinium.

We must remember that Ra Act. is the parent of Act. X, which in turn is transformed through the stages emanation, actinium A, and actinium B.

Let λ_1 , λ_2 be the constants of transformation of radioactinium and actinium X respectively.

Let P and Q be the number of *atoms* of radioactinium and actinium X respectively existing at any time t after the preparation of the radioactinium. Then it is known from Rutherford's theory of the successive changes * that

$$P = n \cdot e^{-\lambda_1 t},$$

$$Q = \frac{n \cdot \lambda_1}{\lambda_1 - \lambda_2} (e^{-\lambda_2 t} - e^{-\lambda_1 t}),$$

where n is the number of atoms of radioactinium initially present.

Now the α ray activity of the radioactinium some time after preparation is due to radioactinium, actinium X, the emanation, and actinium B. The period of actinium X is 10.2 days, of the emanation 3.9 secs., of actinium A 36 minutes, and of actinium B 2 minutes. The changes following actinium X are thus very rapid compared with actinium X itself, and we may, for the purpose of calculation, assume without sensible error that the changes succeeding actinium X are instantaneous. We may therefore include the activity due to these products with that of actinium X.

Now the activity I_t (measured by the electrical method) of

* Rutherford, 'Radioactivity,' 2nd edition, p. 232.

the preparation of radioactinium at any time t is proportional to

$$\lambda_1 P + k \cdot \lambda_2 Q,$$

where k is a constant, which may be defined as the ratio of the ionization resulting from the disintegration of an atom of actinium X through the stages emanation and actinium B to that resulting from the disintegration of an atom of radioactinium.

If I_0 is the activity of the radioactinium immediately after preparation,

$$\begin{aligned} \frac{I_t}{I_0} &= \frac{\lambda_1 P + k \cdot \lambda_2 Q}{\lambda_1 \cdot n} \\ &= e^{-\lambda_1 t} + \frac{k \lambda_2}{\lambda_1 - \lambda_2} (e^{-\lambda_2 t} - e^{-\lambda_1 t}). \end{aligned}$$

This equation expresses the variation of activity of radioactinium, measured by the α rays.

It now remains to form an idea of the magnitude of the constant k . If each of the α ray products of actinium emitted α particles which produced exactly the same ionization in the surrounding gas, the value of k should be 3. This is obviously the case, since the ionization due to the emanation and actinium B is included with actinium X. The ionization, however, is *not* the same for all the actinium products, for I have shown that the ranges of ionization of the products radioactinium, actinium X, emanation, and actinium B are 4.8, 6.55, 5.8, and 5.50 cms. respectively. With a thin film of radioactive matter, the ionization due to each α particle escaping into the air should be approximately proportional to the range in air.

This relation has been shown by Boltwood* to hold closely for radium preparations. Consequently, if the measurements are made with a very thin film of radioactinium and under such conditions that the α particles are completely absorbed between the plates of the testing vessel, the value of k should

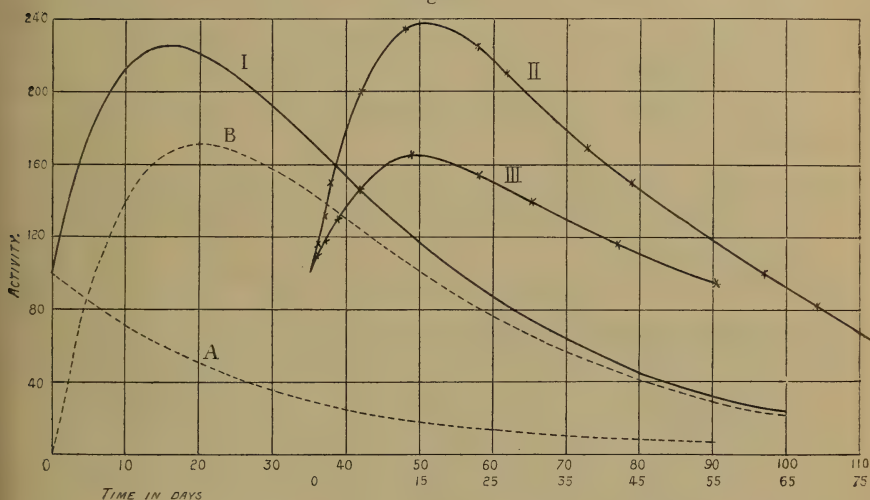
$$\text{be } \frac{6.55 + 5.8 + 5.50}{4.8} = 3.72, \text{ instead of } 3.$$

If, however, a comparatively thick layer of radioactinium is employed, the value of k should be still greater. This follows from the fact that the α particles of greatest range in air not only produce more ionization in the gas but, in addition, more escape into the gas, since they can reach the surface from a greater depth of radioactive matter than the α particles of smaller range.

* Boltwood, Amer. Journ. Sci. June 1906.

The above theory depends on the assumption that all of the radioactive products are completely retained in the radioactive matter. In practice, however, the preparations always lose a portion of their emanation by diffusing into the air. In consequence of this, the activity due to the succeeding product actinium B is less than the theoretical value if no emanation escaped. For a thin film of radioactinium we should consequently expect a value of k less than 3.72, and this value will be variable within limits depending upon the emanating power of the compound.

Fig. 1.



The theoretical curve is drawn in fig. 1, Curve I., the initial activity due to radioactinium being taken as 100. The shape of the curve agrees fairly closely with the experimental curve (Curve II.) plotted in the same way. The maximum ordinate of the theoretical curve is slightly less than the experimental value, showing that the value of k is slightly greater than 3.5.

The theoretical curve can be simply analysed into its components. The exponential curve A shows the activity at any time due to radioactinium itself. The difference curve B shows the activity at any time due to actinium X and its succeeding products.

Curve III. is another example of an experimental curve, which was obtained under conditions where the radioactinium probably contained some actinium and actinium X.

The maximum of the theoretical curve (Curve I.) is reached

after about 17 days, and the maximum of the experimental Curve II. is reached after the same interval. In Curve III. the maximum is reached in 14 days. The earlier occurrence of the maximum is to be expected if any sensible amount of actinium X is present initially with the radioactinium. In this case not only is the maximum reached earlier, but the increase of activity is also smaller. This is seen to be the case in the experiments, for the activity of Curve II. rises to more than twice its initial value, while in Curve III. it rises to only about 1.5 times.

The α ray curves are thus in very satisfactory agreement with the theory. The agreement of Curve II. with the theoretical curve is especially good over the whole range of observations. The theoretical activity ultimately decays exponentially with the period of radioactinium—a result in entire harmony with the experiments discussed earlier in the paper.

The results obtained are thus in complete accordance with the view that radioactinium is an α ray product of period 19.5 days, which is transformed into actinium X of period 10.2 days.

(2) β Ray Curves.

Radioactinium emits only α rays, and Levin has shown that the β ray activity of any preparation of actinium arises only from the product actinium B. Disregarding a small time-lag, the amount of actinium B present is proportional to the amount of the parent product actinium X. But we have seen that the number of atoms Q of actinium X present at any time is given by the formula

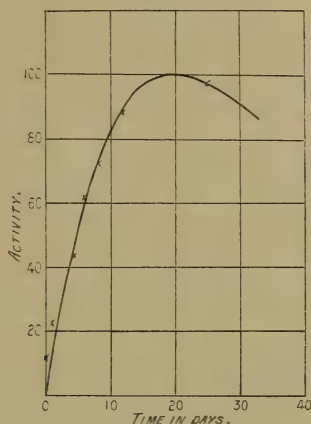
$$Q = \frac{n\lambda_1}{\lambda_1 - \lambda_2} (e^{-\lambda_2 t} - e^{-\lambda_1 t}).$$

The β ray activity of the radioactinium preparation is proportional at any time to the value of Q .

The theoretical curve for the variation of the β ray activity of radioactinium is shown in fig. 2. The activity starts from zero, rises to a maximum after 20 days, and ultimately decays exponentially with the period of radioactinium, viz., 19.5 days. The experimental observations are shown by crosses on the curve and are seen to lie closely on the theoretical curve. The observations of this particular preparation were not continued after 30 days, as the preparation was utilized for a further chemical examination. Other observations have shown that the β ray activity ultimately decays exponentially with

the same period as the α ray activity, viz., with the period of 19.5 days. This is in agreement with the theory.

Fig. 2.



Connexion of Radioactinium with Actinium.

It is now necessary to consider the connexion of the new product radioactinium with actinium itself.

Until recently, it was generally believed that actinium was the direct parent of actinium X. The intermediate product radioactinium had been overlooked. It was believed that actinium itself did not emit any rays, and that therefore by separation of actinium X it could be nearly completely freed from activity. This, however, is not correct.

M. Levin*, working in the same laboratory as the writer, determined with great accuracy the residual α ray and β ray activity of actinium after actinium X had been completely separated. By the well-known method of precipitation with ammonia, by continued precipitations he was able to reduce the emanating power of an actinium solution to one thousandth of its initial value. The separation of actinium X was very complete, but his actinium was still quite active. It always showed a strong α ray activity not less than about 22–25 per cent. of its initial value. Levin states that this activity is due to the presence of radioactinium which is not separated from actinium by means of ammonia. This conclusion is in full harmony with the observations of the writer.

In order to obtain actinium inactive, the radioactinium as well as actinium has to be separated. It is difficult to effect a complete separation of the radioactinium, when employing

* * * *Loc. cit.*

very small quantities of working material. But the writer finally succeeded in reducing the α ray activity to 2.5 per cent. of the initial value. The method employed to separate the radioactinium and actinium X from actinium was as follows :—

Actinium solution in equilibrium was precipitated several times with ammonia. Care has to be taken not to filter too early, as it takes a considerable time for the actinium to come out*. By this method, actinium X remains in the filtrates, actinium and radioactinium remain on the filter. The filter was again dissolved in dilute hydrochloric acid, and to the solution a little ammonia was added in order to weaken the free acid. To the weak acid solution a small quantity of dilute sodium-thiosulphate solution was added at ordinary temperature. The small sulphur precipitate carries the radioactinium down; actinium remains in solution. If the filtrate was still decidedly active, the process of precipitating traces of amorphous sulphur was repeated. This method of separation works very satisfactorily. In order to avoid the presence of sodium sulphite in the actinium solution, the separation may be carried out in a different way by fractional precipitation with ammonia. The first part always contains an excess of radioactinium, while the actinium is concentrated with the later parts. But this method does not effect a very rapid separation, and one sulphur precipitate is in many cases more effective than many careful ammonia precipitations.

The activity of the actinium obtained in this way is very small. The activity only increases slightly in the course of the first few days, but then rises more rapidly for over a month and reaches nearly a constant value after four months (compare Curve II. fig. 3, p. 177). It is of interest to compare the experimental curve of recovery of activity with that to be expected from theory. As before, the changes following actinium X may be supposed without any sensible error to be simultaneous. Actinium changes so slowly compared with its products that we may suppose that it breaks up at a constant rate, giving rise to radioactinium which in turn is transformed into actinium X. Since actinium itself is rayless, the curve of recovery of activity is complementary to the curve showing the decay of activity of radioactinium and actinium X initially in radioactive equilibrium (see Rutherford, 'Radioactivity,' vol. ii. p. 336).

Let P_0 , Q_0 represent the number of atoms of radioactinium and actinium X respectively present in a stage of equilibrium.

* See also Levin, *loc. cit.*

If λ_1 and λ_2 respectively are the constants of change of these substances, then

$$\lambda_1 P_0 = \lambda_2 Q_0.$$

Let $P + Q$ represent the number of atoms of radioactinium and actinium X present at any time t after the actinium is removed. Then

$$P = P_0 \cdot e^{-\lambda_1 t},$$

$$Q = \frac{Q_0}{\lambda_2 - \lambda_1} (\lambda_2 \cdot e^{-\lambda_1 t} - \lambda_1 \cdot e^{-\lambda_2 t})^*.$$

The activity I_t at any time t is proportional to

$$\lambda_1 P + k \cdot \lambda_2 Q,$$

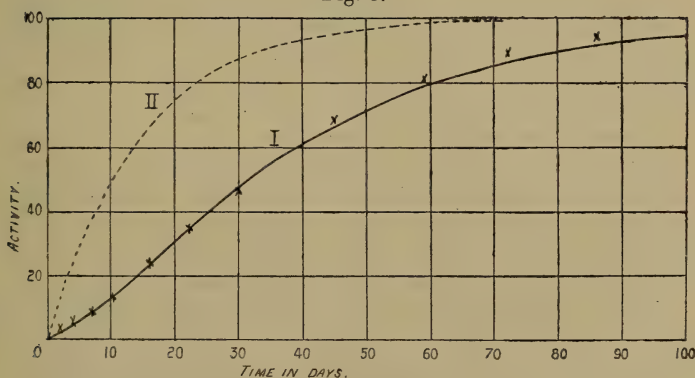
where k is a constant. If I_0 is the initial activity,

$$\frac{I}{I_0} = \frac{\lambda_1 P + k \cdot \lambda_2 Q}{\lambda_1 P_0 + k \cdot \lambda_2 Q_0} = \frac{P}{P_0} + \frac{kQ}{Q_0}.$$

The rise-curve is complementary to this, and consequently

$$\begin{aligned} \frac{\text{Activity at any time } t}{\text{Final activity}} &= 1 - \frac{I_t}{I_0}, \\ &= 1 - \frac{\left[e^{-\lambda_1 t} + \frac{k}{\lambda_2 - \lambda_1} (\lambda_2 \cdot e^{-\lambda_1 t} - \lambda_1 \cdot e^{-\lambda_2 t}) \right]}{1 + k} \end{aligned}$$

Fig. 3.



Taking the periods of radioactinium and actinium X as 19.5 days and 10.2 days respectively, the theoretical curve is given in fig. 3, Curve I., the maximum activity being taken as 100. The value of k is assumed to be 3.5—the same value

* These equations are given in 'Radioactivity,' vol. ii. p. 334.

as previously employed. The experimental values are shown by crosses on Curve I.; the agreement between the theory and experiment is clearly seen. The initial activity, experimentally found, is 2.4 per cent.; after about 4 days the experimental values lie closely on the curve for about one month. After that time, the experimental values lie somewhat above the theoretical curve.

We previously pointed out that the value of k is uncertain between limits, and varies for different experiments. Taking this factor into consideration, there is an excellent agreement between theory and experiment. For comparison, a curve is drawn in fig. 3 (Curve II.), which represents the rise-curve for actinium, if the latter produced actinium X directly, as was generally believed before the separation of radioactinium. The difference between the two curves is striking. While in Curve II. the activity has reached half of its final value after 10.2 days, the experimental Curve I. shows that more than 30 days are required for the activity of actinium to reach half of its maximum value.

We therefore see conclusively that the rise-curve for actinium is not complementary to the decay-curve of actinium X, but behaves just as it should do theoretically under the assumption that radioactinium is a product between actinium and actinium X.

III. OTHER PROPERTIES OF RADIOACTINIUM.

It is not possible to make any definite statement in regard to the chemical properties of radioactinium. The various methods employed for its separation are very similar in nature, and cannot be considered to be definite chemical reactions in the ordinary sense.

It is well known that barium sulphate possesses the property of dragging down many radioactive products from a solution, and the methods used for the separation of radioactinium appear to depend on a similar principle. A satisfactory explanation of these facts has not so far been given. It must not be forgotten that, with the exception of such slowly-changing substances as radiolead, polonium, and radiothorium, most of the products of the various radioelements exist in almost infinitesimal proportion mixed with the parent matter.

Methods of separation, similar in principle to those employed in the case of radioactinium, ought never be taken as a means of description of these products, and any system of classification based on such facts is liable to be very erroneous*.

* A discussion of these points for the case of actinium has been given too by Godlewski, *Jahrb. d. Rad. u. Elektr.* vol. ii. 1906.

temperatures may possibly effect a separation of radioactinium from actinium, but this point has not yet been examined.

For convenience, the disintegration series of actinium is given, together with the period of the products and the radiations they emit :—

Product.	Time to be half transformed.	Radiation.
Actinium.....	?	Rayless.
Radioactinium	about 19·5 days.	α rays.
Actinium X	10·2 days.	α „
Emanation	3·9 secs.	α „
Actinium A } active	36 mins.	Rayless.
Actinium B } deposit.	2 mins.	α , β , & γ rays.

Summary of Results.

The results brought out in this paper are chiefly summarized below :—

- (1) A new product called “radioactinium” exists in the disintegration series of actinium. This new product is produced directly from actinium and is the direct parent of actinium X.
- (2) The transformation of actinium into radioactinium is rayless.
- (3) Radioactinium emits only α rays and is half transformed in about 19·5 days.
- (4) The methods of separation employed for the isolation of radioactinium throw no definite light on its properties.

The present research was done at the McGill Physics Building at Montreal (Canada). Now that my visit to Montreal has come to an end, I record with much pleasure my gratitude to Prof. Rutherford for his kindness, interest, and ready help during the whole of my stay at the McGill University.

Berlin: Chemical Institute of the University.

IX. *The Carbon Cell.*
*By W. H. PATTERSON, B.Sc.**

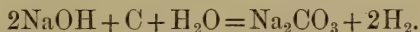
IN two publications, Haber and Bruner † have given reasons for considering that the so-called Jacques Carbon Cell is in reality an oxy-hydrogen battery.

This cell consists of an iron and a carbon pole in molten caustic soda.

Iron, as Liebenow and Strasser ‡ have already shown, is in molten caustic at first active and of base potential. After some time, however, such iron becomes coated with an oxide skin and then functions as an air electrode; the small quantities of manganate which are always present acting as oxygen carriers.

On the other hand, the carbon develops hydrogen by a slow reaction with the molten alkali, which forms the source of potential of the carbon electrode.

The reaction is according to the equation:—



Water, according to Le Blanc and Brode, is retained by molten sodium hydroxide with great pertinacity.

The slower the reaction the less the hydrogen is able to charge up the carbon dipping into the molten alkali, because the manganate and ferrite contained in it, as well as the air surrounding its upper surface, are striving to remove the hydrogen, thereby making the potential approach the value for air.

If there results, however, carbon monoxide from the action of the oxidizing substances contained, the formation of hydrogen is increased, because carbon monoxide reacts rapidly and completely with molten caustic soda, forming carbonate and hydrogen.

The author has made some experiments in order to establish this interpretation of the Carbon Cell. First of all, it was shown that there is a slow production of hydrogen if carbon and sodium hydroxide are maintained at 350° C. in an atmosphere of nitrogen. On the other hand, the measurements of Haber and Bruner on the potential of hydrogen in molten sodium hydroxide were repeated. In the various experiments, the alkali was contained in a large silver crucible

* Communicated by the Author.

† Haber & Bruner, *Zeit. f. Electroch.* vol. x. p. 697 (1904); also vol. v. p. 78 (1906).

‡ Liebenow & Strasser, *Zeit. f. Electroch.* p. 353 (1897, Feb.).

heated by a small Bunsen flame; temperatures were determined thermoelectrically. After the method of Haber and Bruner, hydrogen was led into the molten alkali through a platinum tube, and the E.M.F. of this tube measured by the compensation method against a N/10 electrode. In other experiments, the hydrogen was produced by throwing into the molten alkali a little formate and using a platinum wire as electrode. In both cases a value near to that of Haber and Bruner, *i. e.* -1.5 volt, was attained without perceptible temperature coefficient.

Experiments were then carried out on the oxy-hydrogen combination, with sodium hydroxide as electrolyte, in order better to establish its dependence on the temperature.

Haber and Bruner, in the latter communication, have calculated this potential thermodynamically from water-vapour dissociation at high temperature, with the assumption that the molten alkali is in equilibrium with the water-vapour of the air, the latter having an average tension of 0.031 atmos. The calculation rests on the integration of the Helmholtz formula for the free energy:—

$$A = Q_v + T \frac{\delta A}{\delta T},$$

which is integrated in the form

$$\frac{d \frac{A}{T}}{dT} = - \frac{Q_v}{T^2}.$$

By help of the well-known value of the reaction-heat of the formation of water-vapour from its elements and from the data found by Langen for the mean specific heat of the gases taking part in the reaction, the integral becomes

$$* A = 57084 - 2.976 \times \log_e T + 0.00125 T^2$$

$$- 4.56 \times T \times \log \frac{p_{H_2O}}{p_{H_2} \times p_{O_2}^{\frac{1}{2}}} + \text{const.} \times T.$$

The constant of this expression follows from the measurements carried out by Nernst and Wartenberg on the dissociation of water-vapour. It amounts to 7.6 . In the equation, T denotes the absolute temperature, and p the partial pressure of the gases. The free energy A is calculated in small calories.

* For further information relative to the above calculation *cf.* English edition of Prof. Haber's 'Thermodynamics of Technical Gas Reactions' (Longmans, Green & Co., 1907).

The values of the E.M.F. of the cell are derived from those of A by dividing with $2F$, *i. e.* 46220.

The following numbers are those obtained :—

° C.	E.M.F. (in volts).	
327	1.096—0.059	$\log \frac{p_{\text{H}_2\text{O}}}{p_{\text{H}_2} \cdot p_{\text{O}_2}^{\frac{1}{2}}}$
427	1.068—0.069	„ „
527	1.039—0.079	„ „
627	1.010—0.089	„ „

Taking with Haber and Bruner,

$$p_{\text{H}_2\text{O}} = 0.031,$$

$$p_{\text{H}_2} = 0.969,$$

$$p_{\text{O}_2} = 0.2,$$

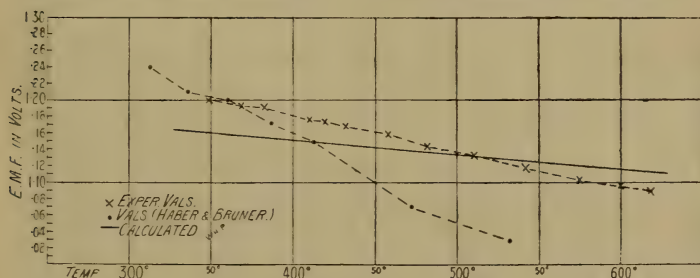
it follows—

$$\log \frac{p_{\text{H}_2\text{O}}}{p_{\text{H}_2\text{O}} p_{\text{O}_2}^{\frac{1}{2}}} = -1.1455 ;$$

so there is obtained for the E.M.F.

° C.	E.M.F. (in volts).
327	1.164 *
427	1.147
527	1.129
627	1.112

These figures are plotted in the accompanying diagram.



* No stress is laid on the absolute value of the third decimal place; as it has a relative value, however, the last figure is given here and subsequently in smaller type.

The values of Haber and Bruner are marked by points; they are above the curve at low temperatures and beneath it at high temperatures.

The reason for the high values at low temperatures can be understood, the molten alkali, according to Le Blanc and Brode, being at this range of temperature hydroscopic. The values at higher temperatures Haber and Bruner themselves regard as uncertain, their method seeming unfit to yield more exact results.

In order to obtain better values the author tried to find electrodes which would, by insertion in molten alkali, behave as hydrogen and as air-electrodes respectively, so that the difference of potential between them would give at once the E.M.F. of the oxy-hydrogen cell looked for.

It seemed probable that palladium saturated with hydrogen might be used as a hydrogen electrode. Palladium wires were charged, either in dilute sulphuric acid or dilute alkali, dried and inserted. Such wires develop hydrogen rapidly, but the amount of gas cathodically incorporated with the wire is so far above the amount which is in equilibrium with hydrogen under atmospheric pressure, at the high temperature of the molten alkali, that an excess of E.M.F. amounting to some centivolts is brought about. Of course the supercharge only lasts for a very short time, the hydrogen being driven off rapidly and the E.M.F. falling slowly; finally it reaches the value of the air electrode. There was no marked break in this fall of potential at the value belonging to that in equilibrium with hydrogen at atmospheric pressure. Hydrogenized palladium was therefore not suitable as a hydrogen electrode. Iron was tried, a metal which, as previously shown by Haber and Bruner, gives a value very near to that of hydrogen gas. The difficulty which ensues from the iron becoming too rapidly passive was overcome by coating the iron wires used with electrolytic iron, after the method of Classen, from a solution containing ferrous oxalate or from a solution containing ammonium chloride. Such wires show the hydrogen potential and for a longer time.

The possibilities with regard to the air electrode were more numerous. Silver, platinum, palladium, nickel, and iron were tried. Wires of these metals, after insertion into molten alkali, all showed the air-potential—the iron and nickel of course after some time—necessary to let them become passive. There is possibly a slight deviation in the case of palladium which, being probably slightly attacked by

the alkali, shows some inclination to deviate towards the hydrogen value.

Combinations consisting of such electrically coated iron wires, acting as a hydrogen electrode, and platinum, functioning as the air electrode, were finally found to be the best arrangement for measuring the temperature coefficient of the cell.

The following figures were thus arrived at (values plotted on diagram as curve with crosses).

Temp. . .	348°	369°	382°	410°	420°	432°	458°
E.M.F. . .	1.201	1.193	1.191	1.176	1.174	1.168	1.159
Temp. . .	482°	510°	542°	575°	600°	618°	
E.M.F. . .	1.144	1.132	1.118	1.103	1.096	1.090	

The results are in agreement with those which Haber and Bruner obtained by their different methods of working up to about 400°; above this the values are higher than those of Haber and Bruner, and are in closer proximity with theory.

In connexion with this, it was thought that perhaps addition of sodium oxide (Na_2O) could render the molten alkali free from water, and therefore change the E.M.F. of the cell, which, as a consequence of theory, would thereby be increased. This plan, however, did not bring about the desired result. Adding a little oxide did not cause a marked change; on the other hand, if all the associated water was eliminated, the iron, even coated electrolytically, quickly assumed the passive state.

The influence of associated water is more marked with potassium hydroxide. Caustic potash freshly melted, thereby containing relatively much water, gives rise to values which are much smaller than those obtained from caustic potash which has been kept molten for a long time and is therefore poorer in water.

This influence of water is easily misleading. The presence of water always causes iron to remain longer active and the hydrogenized palladium to retain its charge. Thus it is easier to work with molten alkali containing water, but the results may be several centivolts too low, the equilibrium pressure of the water-vapour above the molten alkali being in excess of the atmospheric vapour-pressure to some extent.

Caustic potash, moreover, from which water has been eliminated by prolonged heating, causes iron to become much more rapidly passive than does caustic soda.

The following values, in the case of caustic potash, were, *Phil. Mag. S. 6. Vol. 13. No. 73. Jan. 1907.* O

however, obtained with the same arrangement, *i. e.* electrolytic iron against platinum.

Temp.	405°	410°	425°	480°
E.M.F.	1.18 ₂	1.18 ₀	1.17 ₀	1.13 ₅

On comparing these numbers with those obtained from sodium hydroxide, it is seen that the difference is not very great.

The above results are in close agreement with those found by Haber and his co-workers Fleischmann and Foster * concerning the oxy-hydrogen cell with glass and porcelain as electrolyte. In especial between 340° and 575° C. the same thermodynamical expressions have been confirmed with these gas-cells (described by Haber in *Zeit. f. Electrochem.* 1906, No. 24).

It seems rather interesting that such distinct cases as the Carbon Cell and the dissociation of water-vapour at high temperatures are governed by the same thermodynamic relation; that is to say, they are both consequences of the free energy of the formation of water-vapour from its elements, the amount of which is found to be almost identical with the results of experiments described and quoted above.

In conclusion the author wishes to acknowledge his obligations to Professor Haber for kind assistance and advice.

Karlsruhe, Chem. Techn. Institut
der Hochschule, July 1906.

X. Notices respecting New Books.

The Royal Society, Or, Science in the State and in the Schools.

By Sir WILLIAM HUGGINS, K.C.B., O.M., D.C.L., LL.D., Sc.D.,
F.R.S., &c. With twenty-five illustrations. London: Methuen
& Co. 1906. Pp. xvi + 132.

THIS sumptuously printed and beautifully illustrated volume deserves to be widely circulated, not so much, perhaps, on account of its intrinsic artistic value, as by reason of the weighty matters of national importance with which it deals. It is practically a plea, by its distinguished author, for the wider recognition of the claims of Science by the State, the responsible public, and the schools. The bulk of the subject-matter has been taken from Sir William Huggins's Presidential Addresses. The introductory part contains an interesting sketch of the early history of the Royal

* *Zeit. f. anorgan. Chem.* vol. li, pp. 245 & 289 (1906).

Society. The fine photographic reproductions of portraits in oil of "Fellows of immortal fame in science" will be greatly appreciated by all readers of the book. We may add that the price (4s. 6d.) at which the book is published appears to us ridiculously low; and this fact should materially aid in securing for the work the wide circulation which it is highly desirable that it should have.

Abhandlungen über Theoretische Physik. Von H. A. LORENTZ, Professor an der Universität Leiden. Erster Band. Erste Lieferung. Mit 8 Figuren im Text. Leipsic and Berlin: B. G. Teubner. 1906. Pp. 298.

ALL physicists will welcome the appearance of the Collected Papers of Professor H. A. Lorentz, of which the first volume has now been issued. This contains 10 papers, in three different languages, for—as the author says in his brief prefatory note—"we Dutchmen are forced to write in foreign languages if we would take part in the common scientific work of the nations." The first paper, in English, is entitled "Some Considerations on the Principles of Dynamics, in connexion with Hertz's 'Prinzipien der Mechanik,'" and is a critical examination of Hertz's dynamical method. Paper II., in German, deals with the motion of a viscous liquid, and Paper III., in the same language, with the origin of turbulent motion in liquids, and the effect of turbulent motion on the flow through pipes. Paper IV. (French) is on the equations of motion of a gas and the propagation of sound according to the kinetic theory of gases. Papers V., VI., VII., VIII., and X., which are all in German, deal with the application of the virial theorem to the kinetic theory of gases, with the equilibrium of the kinetic energy of gaseous molecules, with the magnitude of regions in n -dimensional space, with the entropy of a gas, and with the virial theorem respectively. Paper IX. (French) is on the molecular theory of dilute solutions. Of special interest to advanced students is Paper XI., which occupies nearly one-third of the entire volume, and which is based on the author's lectures; this deals with the Second Law of thermodynamics, and with its relation to molecular theories. All the papers have been carefully revised by the author. The appearance of the next instalment of papers is promised shortly.

Leitfaden der Wetterkunde. Gemeinverständlich bearbeitet von Dr. R. BÖRNSTEIN, Professor an der Königl. Landwirtschaftlichen Hochschule zu Berlin. Mit 61 in den Text eingedruckten Abbildungen und 22 Tafeln. Zweite umgearbeitete und vermehrte Auflage. Braunschweig: F. Vieweg und Sohn. 1906. Pp. xii + 232.

THE first edition of this excellent manual was noticed in these columns some five years ago. The fact that a second edition has now been called for is ample proof of the fact that there is a widespread interest in meteorology, and that the author of the present

work has been successful in making his subject interesting and intelligible to a large class of readers. The new edition is modelled on precisely the same plan as the old one, there being no changes in the arrangement of the subject-matter; but the work has been considerably enlarged by the inclusion of much interesting information derived from investigations carried out since the appearance of the first edition. Among the special subjects deserving attention in the new edition may be mentioned the effects of forests on temperature and rainfall, the temperature distribution and currents in the higher strata of the atmosphere, the size and shape of rain-drops, and electrical disturbances in the atmosphere. The number of illustrations has also been increased, and in its new form the work is sure to prove even more popular than when it first made its appearance. We could certainly recommend no more attractive or simply and clearly written introduction to meteorology.

Some Founders of the Chemical Industry. By J. FENWICK ALLEN. Sherratt & Hughes, London & Manchester. 1906. Pp. xxiv + 289.

THIS volume contains biographical sketches of William Gossage, Josias Christopher Gamble, James Muspratt, Andreas Kurtz, Henry Deacon, James Shanks, Christian Allhusen, and Peter Spence. The biographies originally appeared in the 'Chemical Trade Journal,' from which they are republished without alterations. The method of treatment varies a great deal, for while in some cases the author deals mainly with a record of inventions, without any vivid presentation of the personality of the inventor himself, in others the human interest is given a generous share of attention. On the whole, these biographies will form interesting reading for the general public, and will be greatly appreciated by all who are directly or indirectly connected with the great alkali trade of Lancashire, of which the present volume is in some sense a history.

Die Geschwindigkeit der Röntgenstrahlen. Experimentaluntersuchung von ERICH MARX. Mit 6 Figuren im Text. Leipzig: B. G. Teubner. 1906. Pp. 52.

IN this interesting memoir, which is reprinted from the transactions of the *Königl. Sächsische Gesellschaft der Wissenschaften*, the author gives an account of an ingenious experimental method devised by him for determining the velocity of the Röntgen rays. Full details of the apparatus are given. The result of the investigation has been to establish the identity of the velocities of the Röntgen rays and of light, with a probable error in the determination of about 1 per cent. The author is to be congratulated on the skill, patience, and ingenuity which he has displayed in carrying out an extremely difficult experimental investigation.

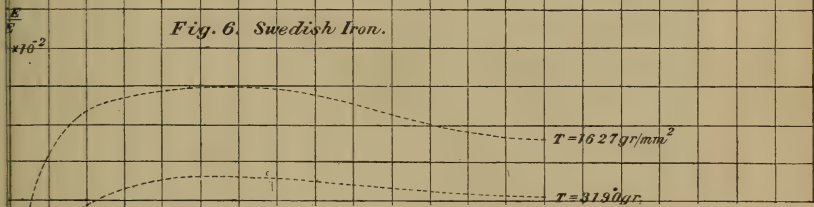
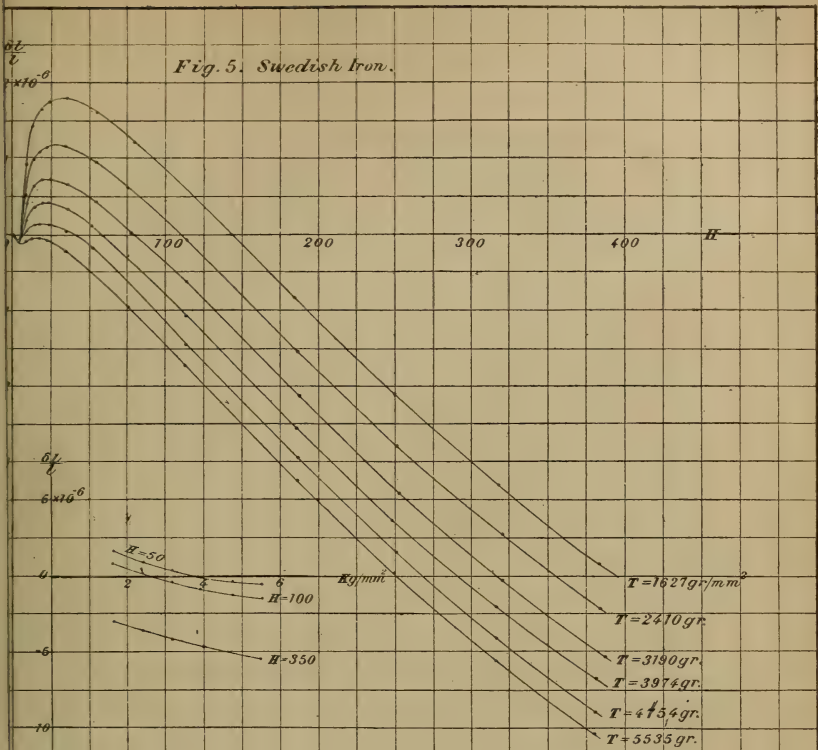


Fig. 1. Pure Nickel.

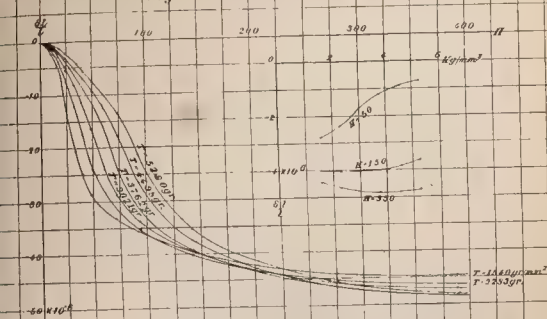


Fig. 2. Pure Nickel.

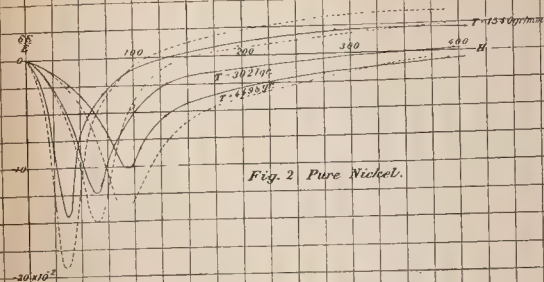


Fig. 4. Commercial Nickel.

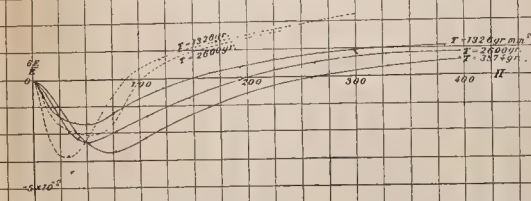


Fig. 8. Tungsten Steel.

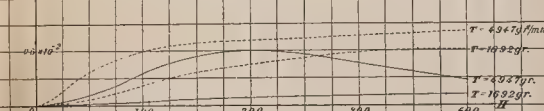


Fig. 9. 28.7% Ni.

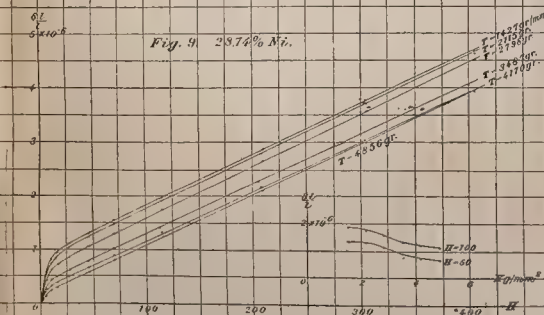


Fig. 5. Swedish Iron.

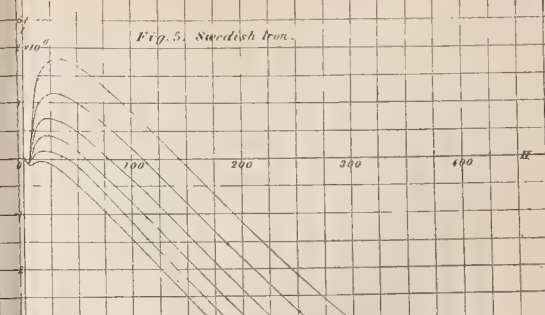


Fig. 6. Swedish Iron.

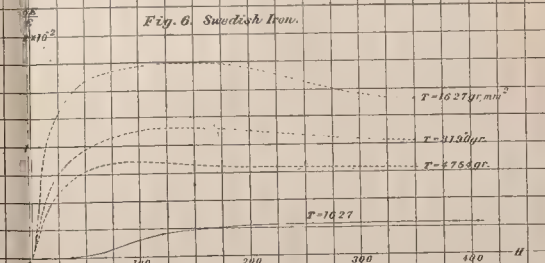


Fig. 10. 28.7% Ni.

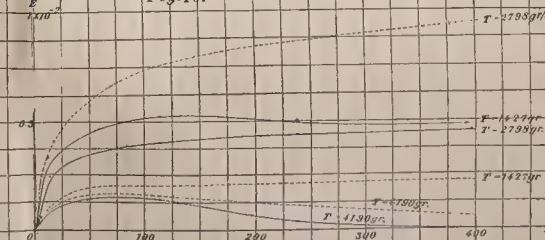


Fig. 3. Commercial Nickel.

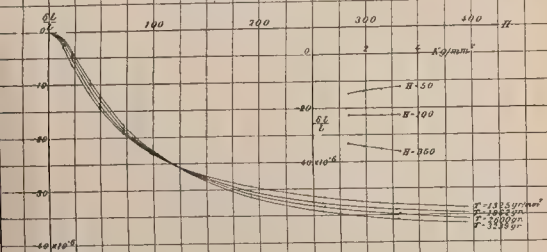


Fig. 7. Tungsten Steel.

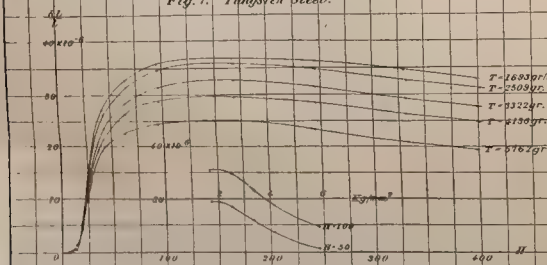


Fig. 17. 70.32% Ni.

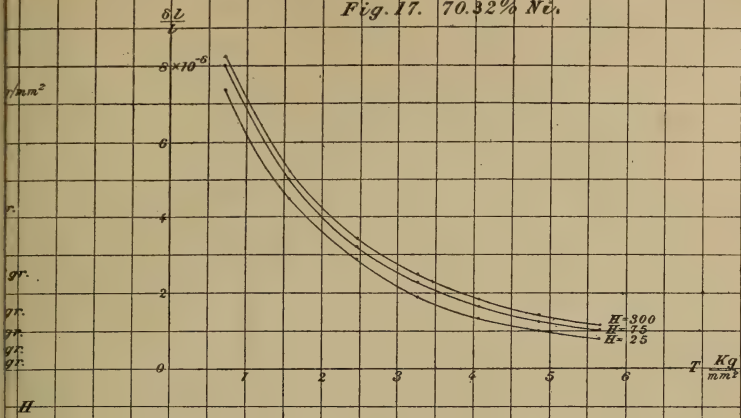


Fig. 13. 50.72% Ni.

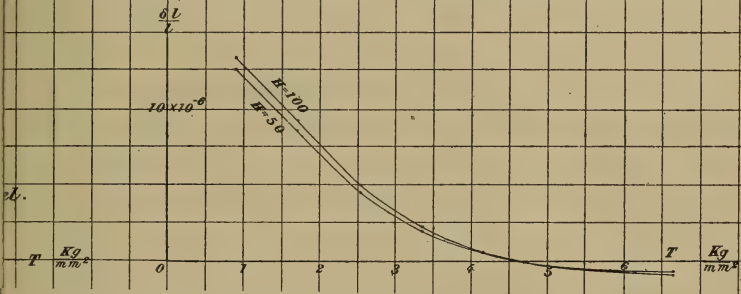


Fig. 11. 50.72% Ni.

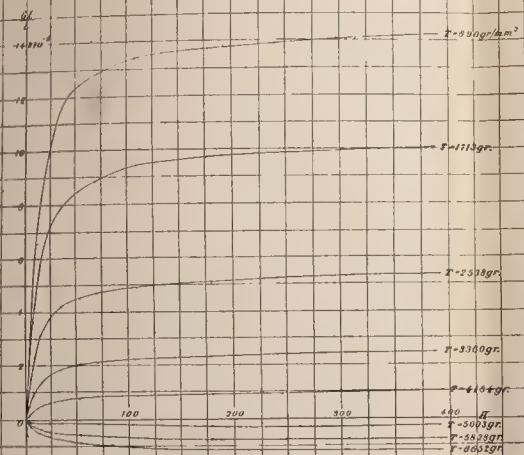


Fig. 12. 50.72% Ni.

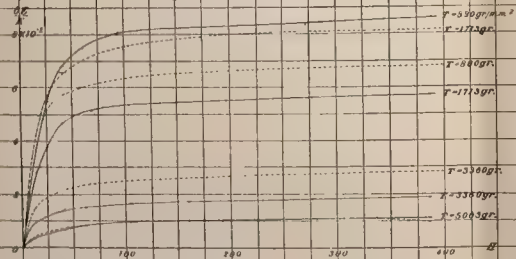


Fig. 13. 70.32% Ni.

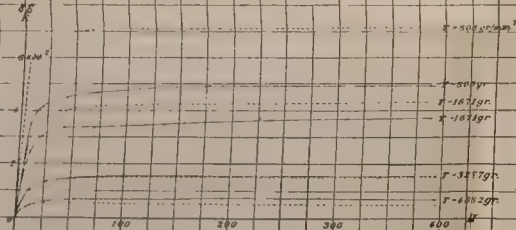


Fig. 14. 70.32% Ni.

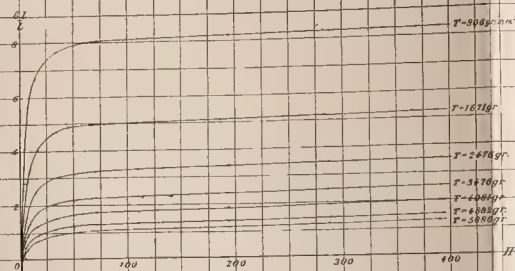


Fig. 15.

50.72% Ni.

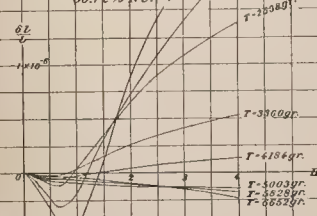


Fig. 16.

70.32% Ni.

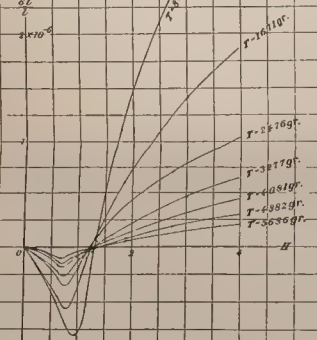


Fig. 17. 70.32% Ni.

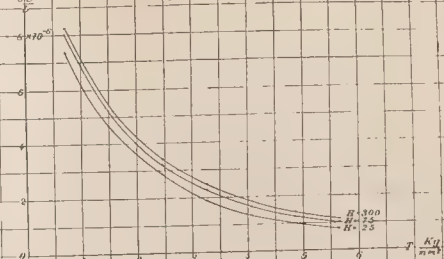


Fig. 18. 50.72% Ni.

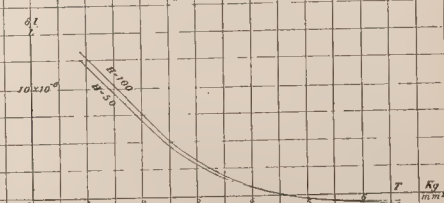


Fig. 19.

Pure Nickel.

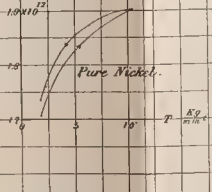


Fig. 20.

Commercial Nickel.

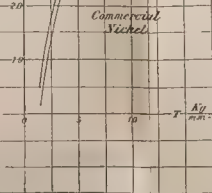


Fig. 21.

Swedish Iron.

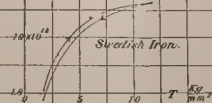


Fig. 22.

Pure Iron.

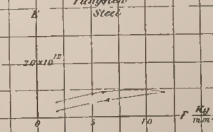


Fig. 23.

28.74% Ni.

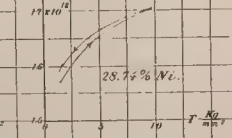


Fig. 24.

50.72% Ni.

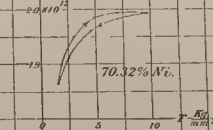


Fig. 35. Pure Nickel.

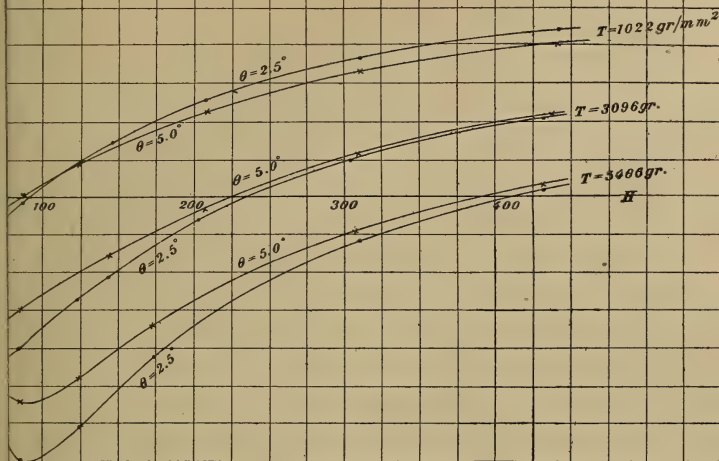


Fig. 32. 28.74% Ni.

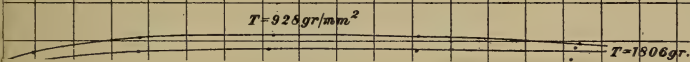


Fig. 26. Soft Iron.

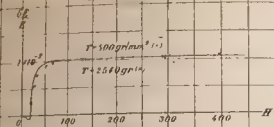


Fig. 27. Tungsten Steel.

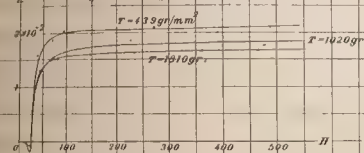


Fig. 28. Nickel.

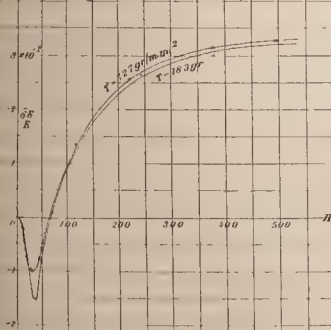


Fig. 30. Swedish Iron.

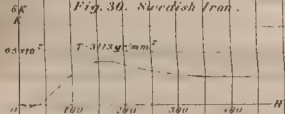


Fig. 31. Tungsten Steel.

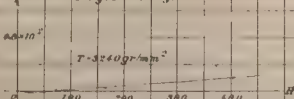


Fig. 29. Nickel.

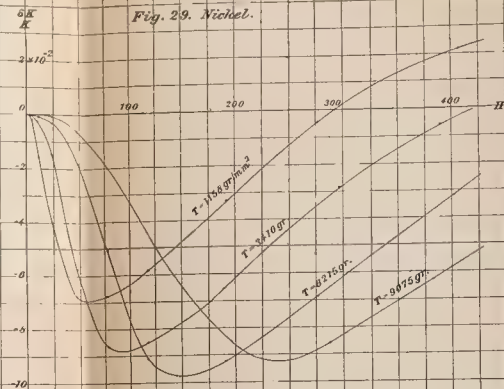


Fig. 33. 50.72% Ni.

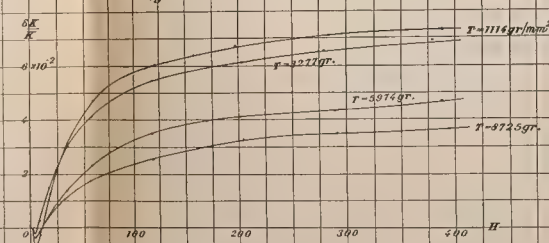


Fig. 34. 70.32% Ni.

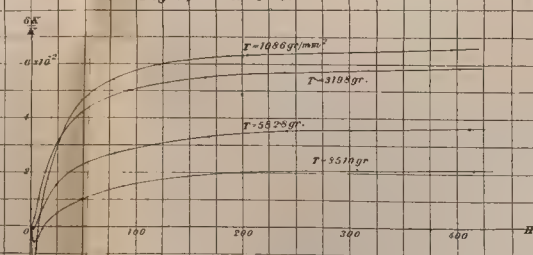


Fig. 35. Pure Nickel.

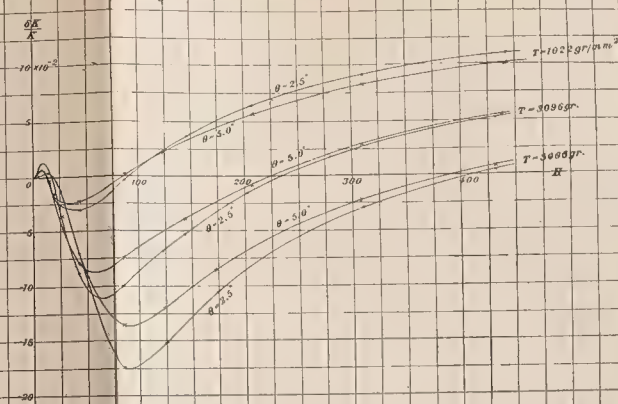


Fig. 32. 25.74% Ni.

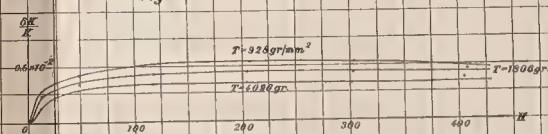


Fig. 36. Swedish Iron.

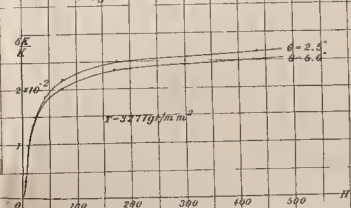


Fig. 37. Tungsten Steel.

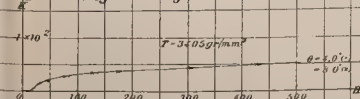




Fig 45. Swedish Iron.

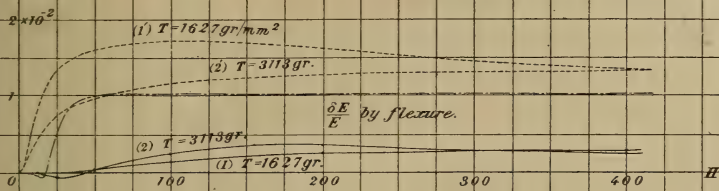


Fig 46. Tungsten Steel.

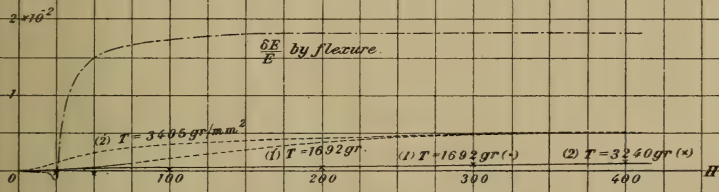


Fig. 38. 28.76% Ni.

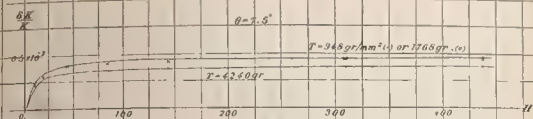


Fig. 39. 50.72% Ni.

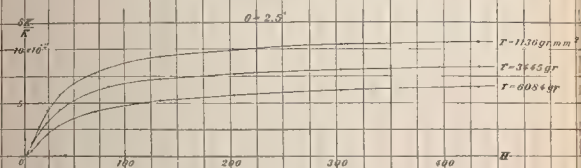


Fig. 40. 70.32% Ni.

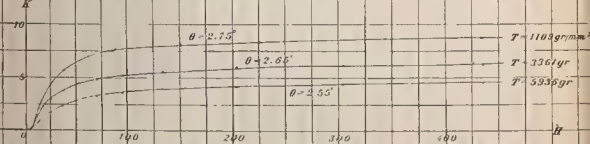


Fig. 41. Nickel.

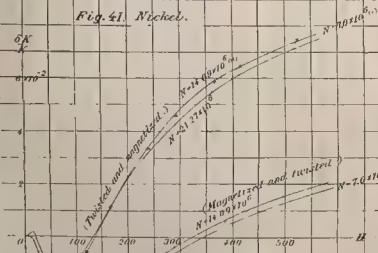


Fig. 42. Nickel.

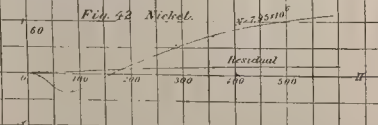


Fig. 43. Iron.

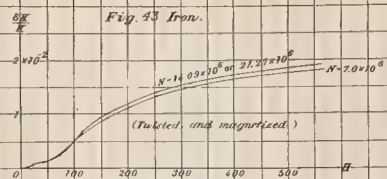


Fig. 44. Iron.

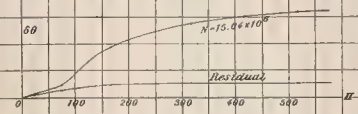


Fig. 48. 28.76% Ni.

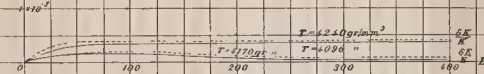


Fig. 49. 50.72% Ni.

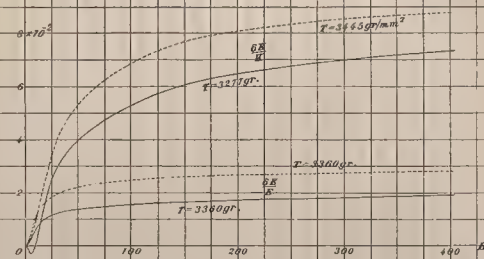


Fig. 45. Surdesh Iron.

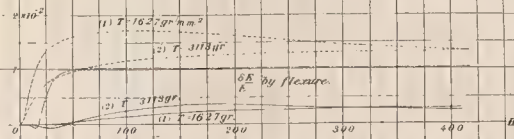


Fig. 46. Tungsten Steel.

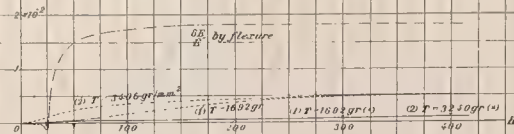


Fig. 47. Nickel.

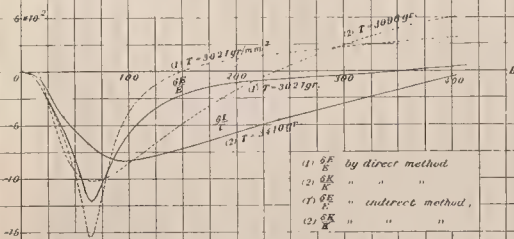


Fig. 50. 70.32 Ni.

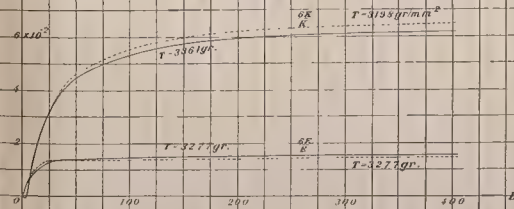


Fig. 54. Cast Cobalt.

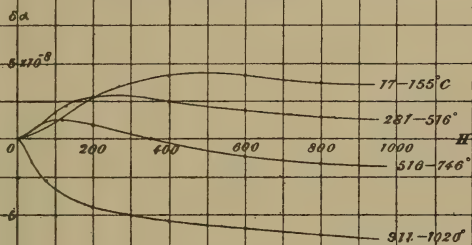


Fig. 55. Annealed Cobalt.

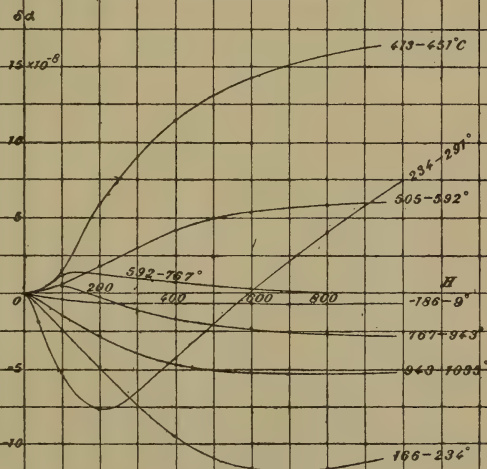


Fig. 51. Nickel.

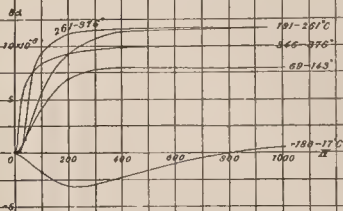


Fig. 52. Cast Cobalt.

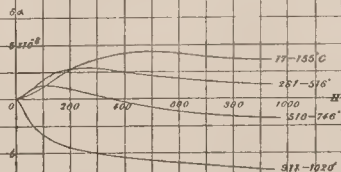


Fig. 53. Soft Iron.

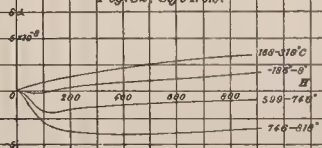


Fig. 54. Annealed Cobalt.

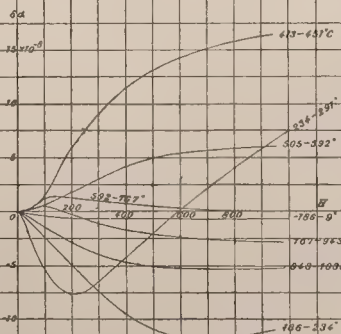
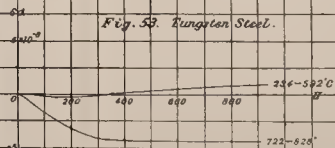
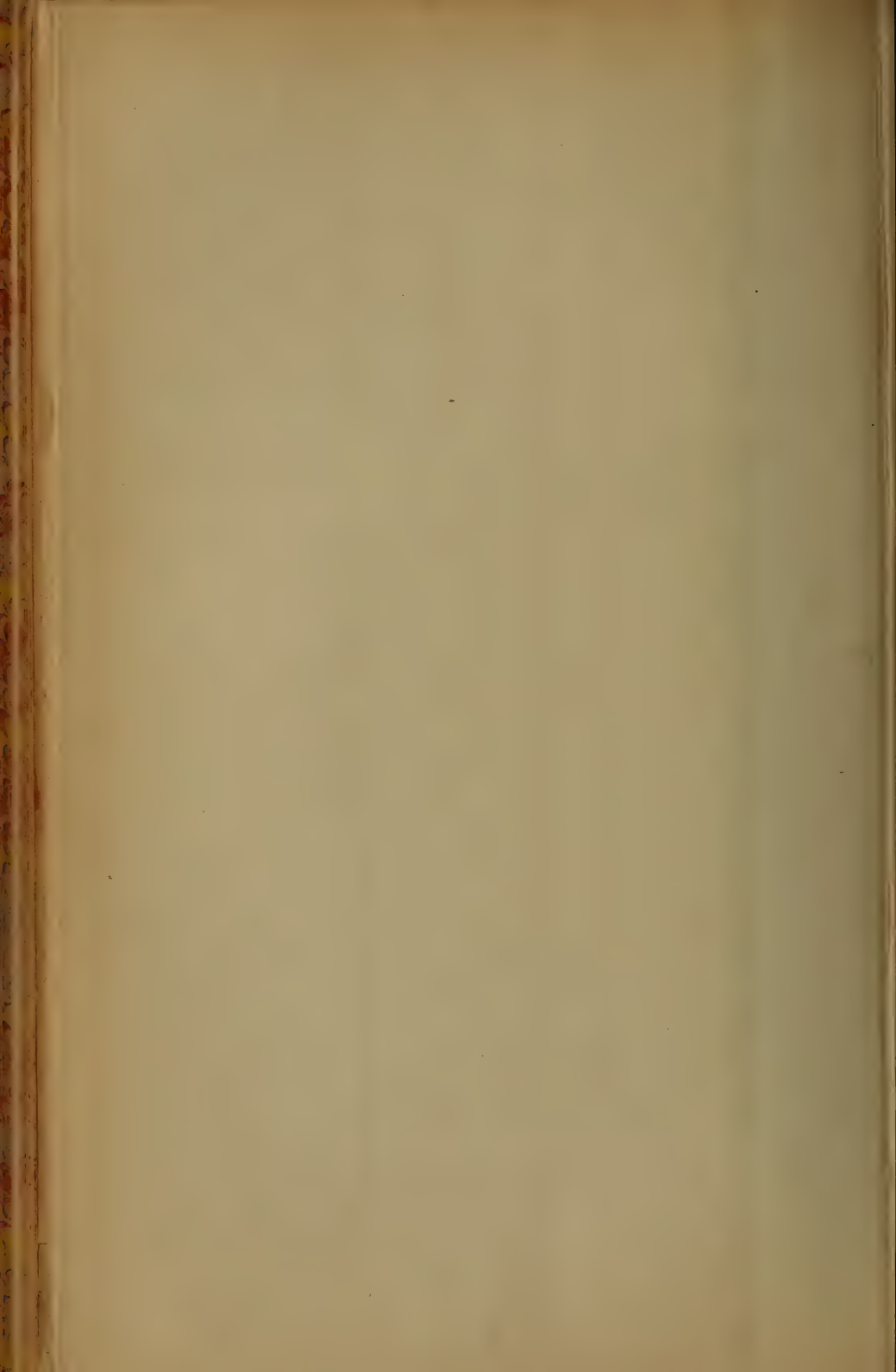
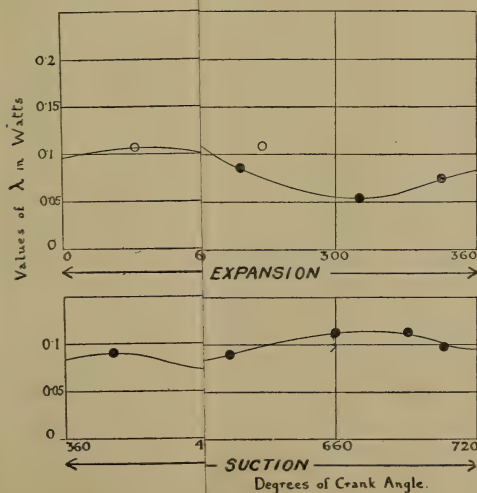
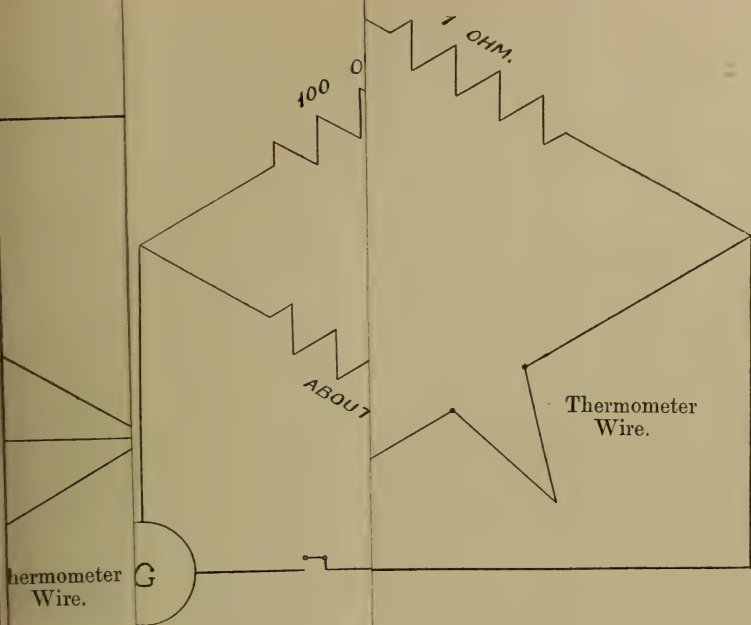


Fig. 55. Tungsten Steel.







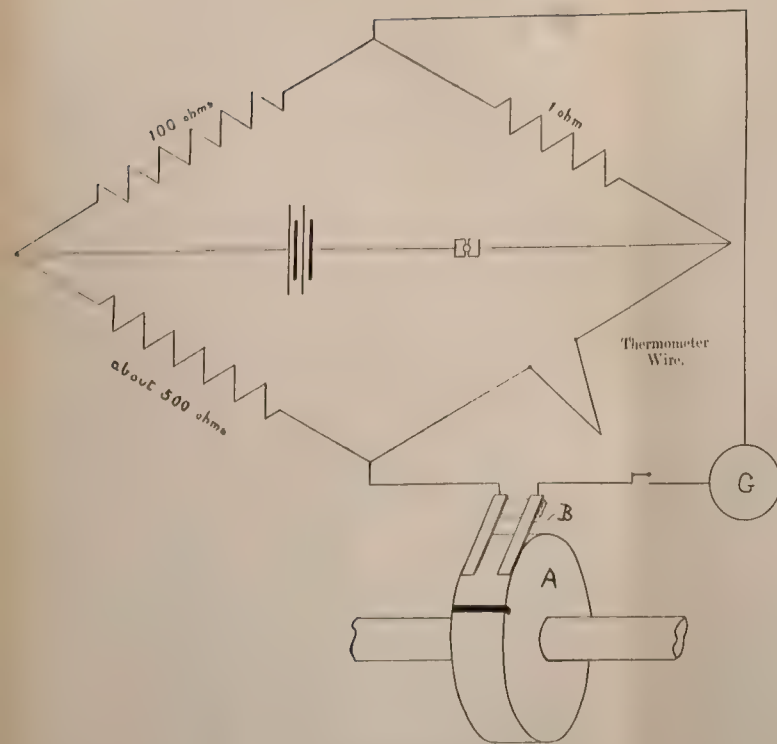


FIG. 2.

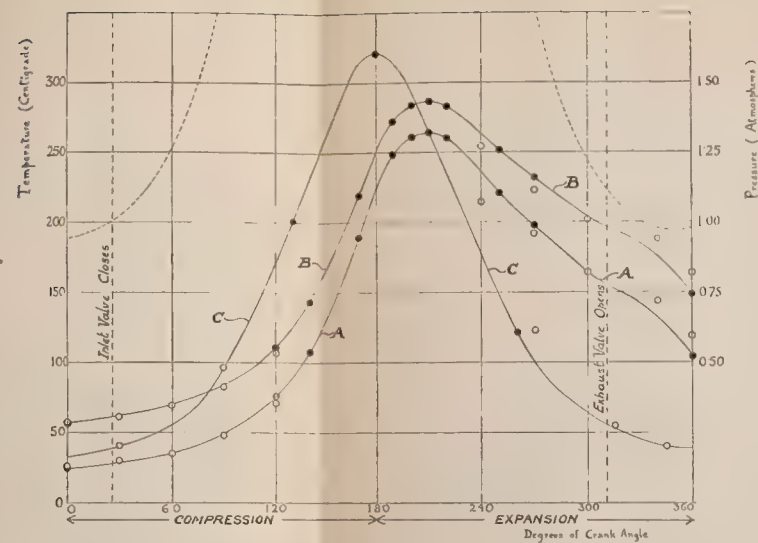


FIG. 2.

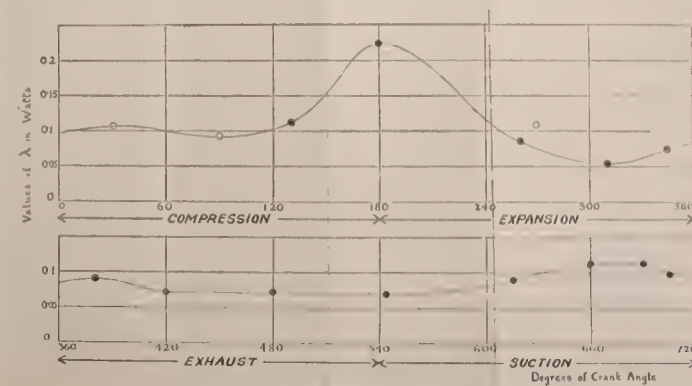
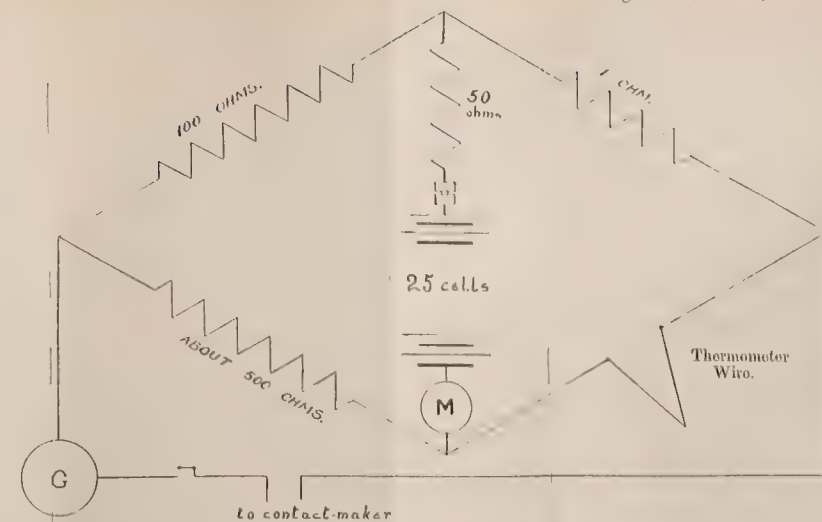
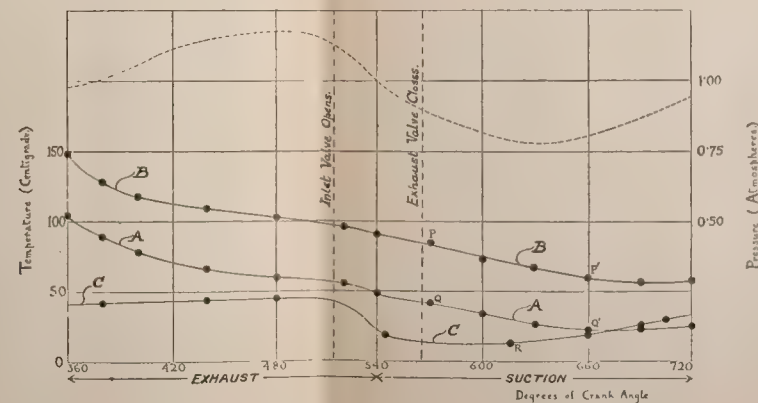


FIG. 4.

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

FEBRUARY 1907.

XI. *On the Electron Theory of Matter and on Radiation.*
By G. A. SCHOTT, B.A., B.Sc., University College of Wales,
Aberystwyth*.

§ 1. **O**NE of the most important problems of the Electron Theory of Matter is to account for the spectra emitted by the several elements; the solution of this problem, rather than of any other, seems likely to lead to the construction of a working model of the atom. By Electron Theory of Matter I mean any theory which assumes matter to consist of electrical charges, acting upon each other with electromagnetic forces only. For stability it is necessary to have both positive and negative charges, and one or both must be in more or less rapid motion. Since the atom is permanent, or very nearly so, the orbits of all the charges must be closed and of atomic dimensions; since it is electrically neutral, except when ionized, positive and negative charges must be present in equal amount. There is no need at present to distinguish between the two possible alternatives: (1) the positive electricity constitutes a sphere of uniform electrification of atomic size, the negative electricity exists as corpuscles (negative electrons) moving inside the positive sphere (J. J. Thomson); (2) both positive and negative electricity exist as discrete charges—electrons—moving in closed orbits of atomic size. The following discussion applies to both types of theory, except where special exception is made.

* Communicated by the Author.

Phil. Mag. S. 6. Vol. 13. No. 74. Feb. 1907.

P

In the present paper we shall study the radiation from a ring of electrons in a controlling field, both when it is in steady motion and when it is disturbed. The general case of a system of rings is much more complicated; it can be studied by the same method, but for the sake of simplicity it will not be considered here.

It may be said at once that a single ring cannot be made to account for spectrum series or bands. It follows that the single ring cannot serve as a model of an atom; nevertheless its study is useful because it throws much light on the conditions which such a model must satisfy in order to account for spectra. They are three in number: (1) the electromagnetic waves emitted by the disturbed ring and received by a stationary observer must be of sufficient intensity to give observable lines; (2) their frequencies must lie within the limits corresponding to the spectrum; (3) they must be given by a formula, such as that of Deslandres for bands, or those of Balmer, Rydberg, or Kayser and Runge for series, and this formula must be satisfied within very narrow limits for every line.

Although the last condition is much the most difficult to study, since it requires us to write down and solve more or less complicated frequency equations, it is the only one of the three on which to my knowledge any work has been done. In the present paper we shall only consider the first two conditions; it will be found that they can only be satisfied for a number of the waves emitted by a single ring too small to account for the lines of even one series. Hence it is useless in this case to study the third condition at all. In this investigation we shall require a number of experimental data, which are known only very roughly: for instance, the intensities of strong and weak spectrum-lines, the ionization in a gas giving a line spectrum, the time for which a free ion exists on the average; but the margin of error is so large, that a knowledge of the order of magnitude of these quantities is quite sufficient for our purpose. For instance, any one of these quantities may be estimated ten times too large or small without affecting the conclusions of § 14. Nevertheless the estimates of the energy radiated per second per ionized ring given in §§ 15, 16 agree so nearly with those of the energy radiated in spectrum lines given in § 10, that the ideal ring of § 11 in this respect is a satisfactory model of radiating systems, such as we find in flames or vacuum-tubes.

§ 2. A first difficulty in a theory of this type is to explain how it is that a system of electrons in orbital motion has a definite structure at all. This difficulty I have examined in

a previous communication*; it is there shown that, if the electron be supposed to expand, and n such electrons be moving in a circle at equal distances apart, any one of them is subject to two forces along the tangent to its orbit: (1) the reaction which the æther exerts on it in consequence of radiation; (2) a pull which the rest of the ring exerts on it as a result of the expansion. According as either force is in excess the electron is retarded or accelerated until its velocity reaches a perfectly definite equilibrium value; this value depends mainly on n ; changes in the field of force in which the ring moves, in its radius, in the mass of the electron, in its rate of expansion, produce no appreciable effect, unless they involve a change in the *order* of magnitude of the quantity which is changed.

We shall assume that the structure of the system is rendered definite by this cause.

§ 3. The following notation will be used:—

Charge of the electron = e electrostatic units,

Mass = m , velocity = v , velocity of light = C , $\frac{v}{C} = \beta$, radius of curvature of the orbit = ρ .

Occasionally we shall use a moving system of axes of ($\xi\eta\zeta$) the origin being at the electron, $O\xi$ along the tangent to the orbit, $O\eta$ towards the centre of curvature, $O\zeta$ along the binormal. The components of the mechanical force on the electron in these directions will be denoted by T , P , S .

The energy radiated per second will be denoted by R , for a single electron or for a group of electrons.

The following numerical values will be useful:—

$e = 3.4 \times 10^{-10}$ (J. J. Thomson).

$\frac{e}{Cm} = 1.88 \times 10^7$ for a slowly moving negative electron
(Kaufmann).

$\rho = 10^{-8}$ cm., or thereabouts.

Kinetic energy of α particle of radium = 5.9×10^{-6} erg
(Rutherford),

Kinetic energy of β particle of radium = 7×10^{-8} erg
(Rutherford),

both of course average estimates.

Limits of the spectrum between which series lines have been observed: $\lambda = 10^{-5}$ cm. and $\lambda = 8 \times 10^{-5}$ cm.

With a time of revolution equal to the period of these extreme rays in a circle of atomic radius, a body has velocities given by $\beta = .0063$ and $\beta = .0008$ respectively.

* Phil. Mag. [6] vol. xii. p. 21.

§ 4. We shall first show that stray electrons cannot exist in a permanent atom; all electrons must be arranged in groups or rings.

An electron loses energy by radiation whenever it has acceleration; in fact

$$R = \frac{2Ce^2}{3(1-\beta^2)^2} \left\{ \frac{\beta^4}{\rho^2} + \frac{\dot{\beta}^2}{C^2(1-\beta^2)} \right\}.$$

This expression was first given by Liénard; it has been confirmed by different methods by Abraham and by myself. Both terms in R are positive; its least value is therefore given by $\dot{\beta}=0$ or $\beta=\text{constant}$. We find in this case, with the numerical values of § 3,

$$R = 2.4 \times 10^7 \frac{\beta^4}{(1-\beta^2)^2} \text{ erg/sec.},$$

for an orbit of atomic size, and greater in proportion for a smaller orbit. Comparing this with the value of the kinetic energy of an α or β particle, we see that β is necessarily extremely small. Hence we may calculate its energy and the radius of the orbit as if it obeyed the ordinary laws of mechanics.

Its kinetic energy is $E = \frac{1}{2} C^2 m \beta^2$.

Let it move in a circle of radius ρ under the action of an opposite charge ve . If the charge be central we have

$$\frac{C^2 m \beta^2}{\rho} = \frac{ve^2}{\rho^2}.$$

If it be a sphere of uniform electrification of radius b ,

$$\frac{C^2 m \beta^2}{\rho} = \frac{ve^2 \rho}{b^3}.$$

In any case,

$$C^2 m \beta^2 \geq \frac{ve^2 \rho^2}{b^3}.$$

To the present approximation $R = \frac{2Ce^2\beta^4}{3\rho^2}$; hence

$$\frac{R}{E} = \frac{4e^2\beta^2}{3Cm\rho^2} \geq \frac{4ve^4}{3C^3m^2b^3} \geq \nu \cdot 2 \cdot 10^9.$$

In other words, a single electron cannot move in a circle of atomic radius for any appreciable time. It will either pass out of the atom, fall into one of the groups of electrons already existing in the atom, or, as its velocity diminishes

owing to radiation, fall into the centre of attraction and correspondingly reduce its strength. In every case we may treat the system as one of which all the electrons are arranged in groups. It remains to examine the radiation from a group of electrons; it is of two types: (1) that due to the permanent motion; (2) that due to disturbances produced by causes external to the group.

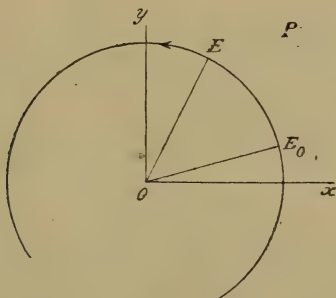
§ 5. J. J. Thomson* has shown that each of n equidistant electrons, moving uniformly in a circle, radiates energy at a very much slower rate than a single electron does for the same orbit and same velocity. He finds

$$R = \frac{2Ce^2\beta^2}{\rho^2} \frac{n^3(n+1)}{2n+1} \frac{(n\beta)^{2n}}{[2n]}.$$

This expression is only true for small values of $n\beta$. We require one which holds for all values of β less than unity.

Strictly speaking, in all that follows β must be less than unity by a small amount depending on the ratio of the radius of the electron to that of the orbit. For the negative electron this ratio is of the order 10^{-5} ; if $\beta = .99$ the results would be in error to about one part in one thousand, if $\beta = .999$ to one part in one hundred.

Let us consider the case of n equidistant electrons moving uniformly in a circle. Take the centre as origin, the axis



as Oz. The azimuth of the i th electron E_i may be taken to be $\omega t + \delta$, where ω is the angular velocity, so that $\beta = \omega\rho/C$.

The azimuth of the i th electron will be $\omega t + \delta + \frac{2\pi i}{n}$. Let

the polar coordinates, referred to Oz as axis and xOz as initial plane, of a point P be (r, θ, ϕ) ; and let the components along the radius, meridian, and parallel at P be, of the electric

* Phil. Mag. [6] vol. vi. p. 681.

force (P, Θ, Π), and of the magnetic force (Λ, M, N). At a point so distant that terms of order $\frac{\rho}{r^2}$ may be neglected in comparison with those of order $\frac{1}{r}$ I find

$$P = \Lambda = 0,$$

$$\Theta = N = \frac{2e\beta}{r\rho} n^2 \cot \theta \sum_{s=1}^{s=\infty} s J_{sn}(sn\beta \sin \theta) \sin sn \left[\omega \left(t - \frac{r}{C} \right) + \delta + \frac{\pi}{2} - \phi \right],$$

$$\Pi = -M = -\frac{2e\beta^2}{r\rho} n^2 \sum_{s=1}^{s=\infty} s J'_{sn}(sn\beta \sin \theta) \cos sn \left[\omega \left(t - \frac{r}{C} \right) + \delta + \frac{\pi}{2} - \phi \right].$$

By Poynting's Theorem we easily find

$$R = \frac{2Ce^2\beta}{\rho^2} n^2 \sum_{s=1}^{s=\infty} [sn\beta^2 J'_{2sn}(2sn\beta) - s^2 n^2 (1 - \beta^2) \int_0^\beta J_{2sn}(2sna) dx].$$

Using Duhamel's asymptotic value for the Bessel function

$$J_n(n\beta) = \frac{1}{\sqrt{2\pi n\gamma}} \exp. n \left(\gamma - \frac{1}{2} \log \frac{1+\gamma}{1-\gamma} \right), \quad \gamma = \sqrt{1-\beta^2}, \quad \beta < 1,$$

we see that the series converges, and that for large values of n we may take the first term only.

When $n\beta$ is small we may content ourselves with the lowest power; in that case we get J. J. Thomson's result. Generally we cannot neglect the higher powers and must write

$$R = \frac{Ce^2\beta^2}{\rho^2} \sqrt{\frac{n^3\gamma}{2\pi}} \exp. n \left(\gamma - \frac{1}{2} \log \frac{1+\gamma}{1-\gamma} \right)$$

approximately for n large.

For $n=1$ the series gives by direct summation

$$R = \frac{2Ce^2\beta^4}{3\rho^2(1-\beta^2)^2},$$

in agreement with the general result of § 4.

The reduction in the intensity of the radiation in the present case is clearly due to interference between the waves emitted by the several electrons of the ring. Each electron may be supposed to emit the same amount of energy as if it alone were present; but to absorb a great portion of the energy of the same type emitted by the remaining electrons of the ring. From this point of view, the reduction in the radiation from the electron is due to resonance. It is clear that the electron can in the same way absorb energy from any wave which passes it, provided the wave be properly attuned to the motion of the electron.

The interference, and consequent reduction in radiation, is by no means confined to the case of uniform circular motion. For example, n electrons moving in the same ellipse will interfere, if their eccentric angles increase at the same uniform rate, and are in arithmetic progression.

§ 6. It is a very general assumption that the D lines of sodium can be attributed to steady motion radiation of an electron moving in a circle, as for instance in the elementary explanation of the Zeeman effect. We shall proceed to show that this is not so.

If possible suppose the D lines to be due to the steady motion of a circle of n electrons.

To account for the frequency of the D lines we must have $n\beta = .0010$ (§§ 3, 5). Hence $n\beta$ is so small that J. J. Thomson's approximation suffices. By §§ 3, 5,

$$R = \frac{2Ce^2}{\rho^2} \frac{n(n+1)(n\beta)^{2n+2}}{2n+1}$$

$$= 7.2 \cdot 10^7 \cdot 10^{-6(n+1)} \frac{n(n+1)}{2n+1}.$$

The kinetic energy, E , of the ring is $\frac{1}{2}C^2mn\beta^2$. Hence

$$\frac{R}{E} = \frac{4e^2}{Cm\rho^2} \frac{n^2(n+1)(n\beta)^{2n}}{2n+1}$$

$$= 2.6 \cdot 10^{14} \cdot 10^{-6n} \frac{n^2(n+1)}{2n+1}.$$

These formulæ give the following values :—

$n=1,$	$2,$	$3,$
$R=2.4 \times 10^{-5},$	$3.6 \times 10^{-12},$	$1.7 \times 10^{-19},$
$R/E=8.7 \times 10^7,$	$26,$	$1.7 \times 10^{-6}.$

These numbers show that the whole energy would be dissipated by radiation for 1 electron in one millionth sec., for 2 in .04 sec., for 3 in one-fifth year.

E. Wiedemann* has measured the radiation from the D lines and found it to be 13.45×10^{10} erg per sec. per gram. Assuming the number of molecules in 1 c.c. of hydrogen at 0° C. and 760 mm. to be 4×10^{19} (J. J. Thomson), and the sodium molecule to be monatomic, this gives 7.7×10^{-12} erg per sec. per atom, or $R=3.9 \times 10^{-12}$ for each of the D lines. This is of the order of the second case above, $n=2$. In this case the atom can exist for at most .04 sec., whatever the

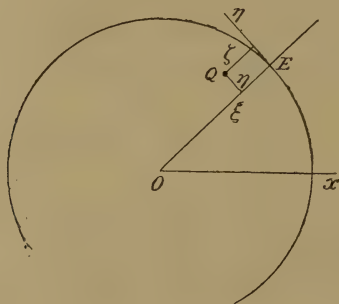
* Drude, *Optik*, p. 487.

conditions, for the radiation now considered is continually taking place. This is obviously impossible.

Apart from this it has long been known that spectrum lines vary very much in intensity and width with the conditions under which they are produced, which in itself is a sufficient proof that their energy is derived from external sources.

§ 7. We shall now consider the disturbances possible in a ring of n equidistant electrons in uniform circular motion, and the vibrations which it emits in consequence. The kinematics has been very fully treated by Maxwell*.

Let (ξ, η, ζ) be the displacements from steady motion of the i th electron, with reference to the moving axes of § 5.



We may write

$$(\xi, \eta, \zeta) = (A, B e^{\sqrt{-1}\alpha}, C) \exp \left[-\kappa t + \sqrt{-1} \left(q t - k \frac{2\pi i}{n} \right) \right],$$

where real parts only are to be taken.

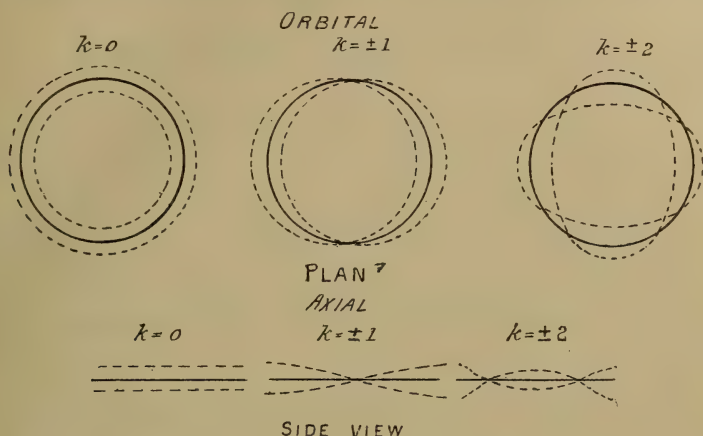
As we shall see, the axial vibration (ζ) is independent of the orbital vibration (ξ, η); hence, A, B, C, α may all be taken to be real. κ is a small quantity representing the damping; this we shall usually neglect. q is the relative frequency, as observed by an observer moving with the mean velocity ω of the ring. The electrons at any instant lie on a curve of k waves, with $2k$ nodes and loops; these waves travel round the ring with angular velocity q/k forward, their velocity relative to a stationary observer being $q/k + \omega$. Hence for the stationary observer the frequency is $q + k\omega$.

We may without loss of generality choose k to lie between $\pm \frac{n}{2}$. We shall call k the "class." The forms of the vibration for the chief classes are shown in the figures.

$k=0$ gives vibrations of the ring as a whole, with all the electrons in the same phase.

* Collected Papers, vol. i. pp. 321-323.

$k = \pm 1$, orbital, are to and fro vibrations in the plane of the ring; $k = \pm 1$, axial, are vibrations with one diameter fixed. In all cases positive values of k mean waves travelling forward, negative, backward.



We are not at present concerned with the forces which produce these vibrations, but only with the waves emitted, which we now proceed to examine.

§ 8. At a great distance from the ring I find

$$P = \Lambda = 0,$$

$$\begin{aligned} \Theta = N = & \frac{ne \cot \theta}{r \rho^2} \mathbf{A} \epsilon^{\sqrt{-1} q \left(t - \frac{r}{c} \right)} \sum_{s=-\infty}^{s=\infty} m l \beta J_m(l \beta \sin \theta) \epsilon^{\sqrt{-1} . m \psi} \\ & + \frac{ne}{r \rho^2} \cos \theta . \sqrt{-1} . \mathbf{B} \epsilon^{\sqrt{-1} \{ q \left(t - \frac{r}{c} \right) + a \}} \sum_{s=-\infty}^{s=\infty} l^2 \beta^2 J'_m(l \beta \sin \theta) \epsilon^{\sqrt{-1} . m \psi} \\ & + \frac{ne}{r \rho^2} \mathbf{C} \epsilon^{\sqrt{-1} . q \left(t - \frac{r}{c} \right)} \sum_{s=-\infty}^{s=\infty} \left(\frac{m \beta}{\sin \theta} - l \beta \sin \theta \right) l \beta J_m(l \beta \sin \theta) \epsilon^{\sqrt{-1} . m \psi}, \end{aligned}$$

$$\begin{aligned} \Pi = -M = & - \frac{ne}{r \rho^2} \sqrt{-1} . \mathbf{A} \epsilon^{\sqrt{-1} . q \left(t - \frac{r}{c} \right)} \sum_{s=-\infty}^{s=\infty} l^2 \beta^2 J'_m(l \beta \sin \theta) \epsilon^{\sqrt{-1} . m \psi} \\ & + \frac{ne}{r \rho^2} \mathbf{B} \epsilon^{\sqrt{-1} \{ q \left(t - \frac{r}{c} \right) + a \}} \sum_{s=-\infty}^{s=\infty} \left(\frac{m \beta}{\sin \theta} - l \beta^2 \sin \theta \right) l \beta J_m(l \beta \sin \theta) \epsilon^{\sqrt{-1} . m \psi} \\ & - \frac{ne \beta}{r \rho^2} \cos \theta \sqrt{-1} . \mathbf{C} \epsilon^{\sqrt{-1} . q \left(t - \frac{r}{c} \right)} \sum_{s=-\infty}^{s=\infty} l^2 \beta^2 J'_m(l \beta \sin \theta) . \epsilon^{\sqrt{-1} . m \psi}, \end{aligned}$$

where $m = k + sn$, $l = \frac{q}{\omega} + k + sn$, $\psi = \omega \left(t - \frac{r}{c} \right) + \delta + \frac{\pi}{2} - \phi$.

By using Poynting's Theorem we find for the radiation

$$\begin{aligned}
 R = & \frac{Ce^2n^2}{2\rho^4} \left\{ \mathbf{A}^2 \sum_{s=-\infty}^{s=\infty} l\beta [l^2\beta^2 J'_{2m}(2l\beta) - (m^2 - l^2\beta^2)l \int_0^\beta J_{2m}(2lx)dx] \right. \\
 & + \mathbf{B}^2 \sum_{s=-\infty}^{s=\infty} l\beta \left[\frac{1-\beta^2}{4} l^2\beta^2 J'_{2m}(2l\beta) + \frac{3+\beta^2}{8} l\beta J_{2m}(2l\beta) \right. \\
 & \quad \left. + \left\{ \frac{1+\beta^2}{2} (m^2 + l^2\beta^2) - 2ml\beta^2 - \frac{3+\beta^2}{8} \right\} l \int_0^\beta J_{2m}(2lx)dx \right] \\
 & - 2\mathbf{AB} \sin \alpha \sum_{s=-\infty}^{s=\infty} l\beta \left[\frac{m-l\beta^2}{2} l\beta J_{2m}(2l\beta) + \frac{m+l\beta^2}{2} l \int_0^\beta J_{2m}(2lx)dx \right] \\
 & + \mathbf{C}^2 \sum_{s=-\infty}^{s=\infty} l\beta \left[-\frac{1-\beta^2}{4} l^2\beta^2 J'_{2m}(2l\beta) + \frac{1+3\beta^2}{8} l\beta J_{2m}(2l\beta) \right. \\
 & \quad \left. + \left\{ \frac{1+\beta^2}{2} (m^2 + l^2\beta^2) - 2ml\beta^2 - \frac{1+3\beta^2}{8} \right\} l \int_0^\beta J_{2m}(2lx)dx \right] \Big\}.
 \end{aligned}$$

The expressions for the forces show that at a great distance from the ring the electric and magnetic forces are transverse and at right angles. They consist of an infinite series of harmonic terms, the frequency of the s th harmonic being $q + (k + sn)\omega$. Since negative values of s occur, there are in reality two distinct series, one with positive and the other with negative frequencies. The amplitude of the s th harmonic is of the order $J_m(l\beta)$, and the corresponding term in the radiation of order $J_{2m}(2l\beta)$. For small values of $l\beta$, such as correspond to waves of light, the orders are $(\frac{1}{2}l\beta)^\mu / \underline{\mu}$, and $(l\beta)^{2\mu} / \underline{2\mu}$, where $\mu = \text{Mod. } m$.

The fact that a single vibration of the ring, corresponding to one degree of freedom of the ring, gives rise to an infinite series of harmonics is due to the presence of the æther, which possesses an infinite number of degrees of freedom.

§ 9. Let λ be the wave-length of the vibration considered; then

$$l\beta = \frac{2\pi\rho}{\lambda}$$

For the extreme ultraviolet, $\lambda = 10^{-5}$ cm., $l\beta = \cdot 0063$; for the extreme red, $\lambda = 8 \times 10^{-5}$ cm., $l\beta = \cdot 0008$. In this case we need only retain the lowest power of $l\beta$, and the harmonic of greatest amplitude, given by $s=0$, $\mu = \text{Mod. } k$.

We find

$$\mu > 0$$

$$R = \frac{n^2(A^2 + B^2 \mp 2AB \sin \alpha + \beta^2 C^2)}{\rho^2} \frac{2\pi^2 C e^2}{\lambda^2} \frac{\mu(\mu+1)}{2\mu+1} \left(\frac{2\pi\rho}{\lambda}\right)^{2\mu},$$

where the sign $\mp$ refers to k ^{positive}
negative,

$$\mu = 0$$

$$R = \frac{n^2(\beta^2 B^2 + C^2)}{\rho^2} \frac{4\pi^2 C e^2}{3\lambda^2} \left(\frac{2\pi\rho}{\lambda}\right)^2.$$

In this case the terms involving A^2 , AB are of the next higher order.

Throughout the investigation terms of the orders $\frac{A^2}{\rho^2}$, $\frac{2\pi A^2}{\lambda\rho}$ have been neglected. Hence $\frac{A}{\rho} \dots$ must all be small.

The distance between consecutive electrons of the ring is $\frac{2\pi\rho}{n}$; except in very violent disturbances $A \dots$ can hardly be greater than a small fraction of this; that is $\frac{nA}{2\pi\rho}$ —call it σ —can hardly be greater than a small fraction.

The table gives values of R/σ^2 for three wave-lengths calculated with the numerical values of § 3.

Class, k =		0.	$\pm 1.$	$\pm 2.$	$\pm 3.$	$\pm 4.$
$\frac{R}{\sigma^2}$ for $\lambda =$	$6 \cdot 10^{-5}$ cm.	·00 58	·00029	$4 \cdot 8 \cdot 10^{-11}$	$2 \cdot 5 \cdot 10^{-18}$	$6 \cdot 4 \cdot 10^{-26}$
	$3 \cdot 5 \cdot 10^{-5}$ „	·0051	·0025	$1 \cdot 2 \cdot 10^{-9}$	$1 \cdot 9 \cdot 10^{-16}$	$1 \cdot 2 \cdot 10^{-23}$
	$2 \cdot 10^{-5}$ „	·046	·024	$3 \cdot 6 \cdot 10^{-8}$	$1 \cdot 7 \cdot 10^{-14}$	$3 \cdot 8 \cdot 10^{-21}$

§ 10. It is very frequently assumed that every disturbance of a system of electrons shows itself by a line in the spectrum of the system. We shall now show that for a single permanent ring of electrons this is not the case, because many of the vibrations emitted are far too weak to affect the photographic plate appreciably.

We must first form an estimate of the least amount of energy which will enable a vibration just to produce an impression on the plate. Bearing in mind that the photometric intensity of a spectrum-line is indicated by a scale-number, from 1 to 10, we see that we must find out how the energy of a weak line 1 compares with that of a strong line 10.

Each of the D lines is numbered 10, and their energy may for our purpose be taken as a measure of that of every other line 10.

Eder and Valenta* give the times of exposure required to produce distinct photographic impressions under the same conditions of certain ultraviolet lines; the following are examples:—

Mg lines, $\lambda = 2936, 2928, 2802$, all 10, 15 sec.

Mg lines, $\lambda = 3336 - 3329, 2782$, all 6, 90 sec.,

Zn lines, $\lambda = 2063 - 2024$, all 1, 1200 sec.

Assuming the photographic effect to be proportional to the product of intensity by time of exposure, we see that the intensities are as $1 : 1/6 : 1/80$, practically as $1 : (1.6)^{-4} : (1.6)^{-9}$.

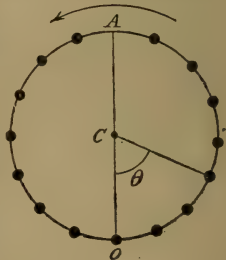
Kayser† proposes to use a scale due to Jewell in which the intensities are as $1 : 2^{-4} : 2^{-9}$. We shall not be far wrong if we take as our ratio, for 1 and 10, $1 : 400$. This makes the energy of the weakest observable line, $1, 10^{-14}$ erg per sec. per atom, the radiation in the D line being taken as $4 \cdot 10^{-12}$ (§ 6).

In all probability this energy is underestimated, probably very much so for the ultraviolet; Pflüger‡, from an investigation on the relative intensities of spectrum-lines by means of a thermopile, concludes that the extreme ultraviolet lines are enormously stronger than the visible lines, although their photographic effect is very much less. For example, he finds that the Mg line, $\lambda = 280 \mu\mu$, gives 970 scale-divisions deflexion, whilst the Mg line $\lambda = 383 \mu\mu$, of about the same photometric intensity, gives only 56 scale-divisions; this too although 280 is a single line, whilst 383 includes several neighbouring lines too near to separate.

§ 11. It is desirable that we should form some idea as to the values of σ to be expected for vibrations of different classes. For this purpose we shall consider the following problem:—

A ring of $n + 1$ equidistant electrons is in steady orbital motion. An electron is suddenly expelled; required to find the subsequent motion.

To fix the ideas suppose n odd. After the expulsion of the electron we are left



* *Beiträge zur Photochemie*, p. 44.

† *Spectroscopie*, vol. i. p. 646.

‡ Drude, *Annalen*, xiii. p. 890.

with a ring of n electrons, arranged as in the diagram, and revolving in the direction of the arrow.

This is not a possible state of steady motion for a ring of n electrons, because

- (1) they are not equidistant;
- (2) the angular velocity and radius correspond, not to n electrons, but to $n+1$.

When n is at all large, the angular velocity and radius are practically the same for n electrons as for $n+1$. The deviation (2) produces slight oscillations in velocity and radius about the values corresponding to n electrons; but they are very small compared with the oscillations due to (1). We shall for simplicity neglect them. Thus we treat the ring as one of n electrons initially displaced from the state of steady motion, with zero radial displacement and zero velocity.

For the angular displacement I find

$$\frac{2\pi}{n(n+1)} \sum (-1)^{k-1} q' \sin \left(qt - k \frac{2\pi i}{n} \right) \frac{1}{(q+q') \sin(k\pi/n)},$$

and therefore for the angular velocity,

$$\frac{2\pi}{n(n+1)} \sum (-1)^{k-1} qq' \cos \left(qt - k \frac{2\pi i}{n} \right) \frac{1}{(q+q') \sin(k\pi/n)},$$

where the summation is for all values of k between $\pm \frac{n}{2}$, excluding zero, and q, q' are the frequencies corresponding to classes $k, -k$. The effect of damping has been neglected, so that the amplitudes are approximate only for the initial stage of the disturbance.

It is to be remarked that, as in the case of Saturn's rings considered by Maxwell*, there are four frequencies for disturbances in the plane of the ring; for two of these the tangential displacement is a large multiple of the radial displacement, for the remaining two it is about twice as great; the values of q, q' belong to the first pair, because in our case the radial disturbance is negligible.

For vibrations of classes $\pm k$ we have initially

$$\begin{aligned} \sigma_0 &= \frac{q'}{(n+1)(q+q') \sin(k\pi/n)} \text{ for } k \\ &= \frac{q}{(n+1)(q+q') \sin(k\pi/n)} \text{ for } -k; \end{aligned}$$

* Collected Papers, vol. i. p. 321.

maximum kinetic energy

$$= \frac{\pi^2 m \rho^2 \cdot q^2 q'^2}{n(n+1)^2 (q+q')^2 \sin^2 (k\pi/n)} \text{ for } \pm k;$$

that is, $\pi^2 m \rho^2 q^2 \sigma_0^2/n$ for k , and $\pi^2 m \rho^2 q'^2 \sigma_0^2/n$ for $-k$. The value of σ_0 is clearly less than $\frac{1}{n \sin (k\pi/n)}$ for $\pm k$.

§ 12. The ideal problem of § 11 differs in two respects from an actual case of ionization of a ring. The expulsion of the electron cannot really be instantaneous, but must be more or less gradual; the disturbance produced will in reality be less violent than in the ideal case, the values of σ will be less, and probably the vibrations of higher classes will be weaker, not merely absolutely, but also relatively to those of lower classes. Again, the effect of damping on account of radiation has been neglected; in consequence the values of σ diminish with the time according to an exponential law, those of lower classes very rapidly, those of higher classes much more slowly. We must now consider the effect of damping on the radiation.

It has long been thought that the radiation from a gas, in a vacuum-tube or in a flame, is connected with the ionization of its atoms. Stark*, from his experiments on the Doppler effect of canal rays, concludes that the line-spectra are emitted by positive ions produced from atoms by the expulsion of a negative electron, and that band-spectra are emitted in the recombination of the positive ion and negative electron. We may compare a radiating gas to a system of a very large number of rings like that of § 11. Suppose that on the average a ring remains ionized for T seconds. Further, suppose that the energy of an ionized ring due to a vibration of class k and amplitude σ is $E = A\sigma^2$, and the corresponding radiation $R = B\sigma^2$. We have

$$\frac{dE}{dt} = -R,$$

whence $\frac{1}{E} \frac{dE}{dt} = -\frac{R}{E} = -\frac{B}{A}$, a constant.

Write $B/A = \gamma$; then

$$E = E_0 e^{-\gamma t}, \quad R = R_0 e^{-\gamma t},$$

where t is reckoned from the instant of ionization.

* *Physikalische Zeitschrift*, vi. p. 892; 'Nature,' lxxiii. pp. 78, 389, 533.

The mean radiation from the gas per second per free ion is

$$\frac{1}{T} \int_0^T R dt = R_0 \frac{1 - e^{-\gamma T}}{\gamma T}.$$

When the damping γ is very small, as for the vibrations of high class, the last factor is practically unity and the radiation is R_0 . When the damping is large, as for vibrations of low class, the last factor is nearly $\frac{1}{\gamma T}$, and the radiation is nearly $R_0/\gamma T$, that is E_0/T . Thus the relative intensity of waves of high class and small radiation depends mainly on the relative radiation, calculated in § 9. The relative intensity of waves of low class and large radiation depends mainly on their relative initial energy, calculated in § 11.

The damping, γ , is given by $\gamma = \frac{R_0}{E_0}$; for E_0 we may take the maximum kinetic energy of § 11, which is $\frac{\pi^2 m \rho^2 q^2}{n} \sigma_0^2$; thus $\gamma = \frac{n R_0 / \sigma_0^2}{\pi^2 m \rho^2 q^2}$. This may also be written

$$\gamma = \frac{n R_0 / \sigma^2}{\pi^2 C^2 m (q \beta / \omega)^2}.$$

§ 13. We shall now try to form an estimate of the values of γT for vibrations of different classes, in order to decide for which of these classes the intensity of the waves emitted is determined by the initial radiation, or by the initial energy.

In § 8 we saw that a vibration of class k produces a series of harmonic waves, whose frequencies are $q + k\omega + s n \omega$. The strongest of these, for k between $\pm \frac{n}{2}$, is given by $s=0$; this alone will be considered, because the others are extremely weak in proportion. If λ be the corresponding wave-length, we have

$$q + k\omega = \frac{2\pi C}{\lambda};$$

hence

$$\frac{q\beta}{\omega} + k\beta = \frac{2\pi\rho}{\lambda}.$$

For frequencies within the limits of the spectrum $\frac{2\pi\rho}{\lambda}$ is very small; hence $\frac{q\beta}{\omega}$ is very nearly equal to $-k\beta$. Therefore for waves of corresponding frequency we find

$$\gamma T = \frac{n T}{\pi^2 C^2 m \beta^2} \frac{R_0}{k^2 \sigma_0^2}.$$

The first factor depends only on the nature of the system of rings. The second factor depends on the class k of the vibration considered; its value can be found by means of the table of § 9.

With the numerical values of § 3 we get

$$\frac{1}{\pi^2 C^2 m} = 1.9 \cdot 10^5.$$

The value of $\frac{n}{\beta^2}$ can be calculated if we assume that the value of β is fixed by the condition of § 2, in which case it depends largely on the value adopted for the rate of expansion of the electron. If a be the radius of the electron, the table given previously (*loc. cit.* p. 24) shows that with $\frac{\dot{a}}{a} = 10^{-16}$, $\beta = .023$ for $n=10$, $.566$ for $n=100$, while with $\frac{\dot{a}}{a} = 10^{-46}$, $\beta = .00074$ for $n=10$, $.398$ for $n=100$. In these cases we have for n/β^2 the values $1.9 \cdot 10^3$ for $n=10$, $2.8 \cdot 10^2$ for $n=100$, with $\frac{\dot{a}}{a} = 10^{-16}$, and $1.9 \cdot 10^7$ for $n=10$, $6.3 \cdot 10^2$ for $n=100$, with $\frac{\dot{a}}{a} = 10^{-46}$. Except for small values of n , n/β^2 does not vary greatly, even for a small value of $\frac{\dot{a}}{a}$; we may take it to be 1000; with this value $\frac{n}{\pi^2 C^2 m \beta^2}$ is of order $2 \cdot 10^8$.

The value of T , the average life of a free ion, is unknown for the gas in a flame or a vacuum-tube, but it has been measured for other cases, *e. g.* gases ionized by Röntgen rays, where it has been found to be of the order of $1/2$ second*. In default of a better estimate we shall assume T to be of this order in the present case, and therefore $\frac{nT}{\pi^2 C^2 m \beta^2}$ of order 10^8 .

The table of values of R_0/σ_0^2 of § 9 now shows that for $k = \pm 3$ or more, γT is of order 10^{-7} or less; for $k = \pm 2$, γT is of order 1 for $\lambda = 2 \cdot 10^{-5}$ cm., and of order $.001$ for $\lambda = 6 \cdot 10^{-5}$ cm.; for $k=0$, ± 1 , γT is very large.

Thus the intensities of waves of classes ± 3 and upwards are determined by radiation, those of waves of classes 0 and ± 1 by the initial energy, those of classes ± 2 mainly, if not entirely, by the initial radiation.

In the former case the energy radiated per free ion per sec. by the gas is R_0 , in the latter it is $R_0/\gamma T = E_0/T$.

* J. J. Thomson, 'Conduction of Electricity through Gases,' p. 18.

§ 14. We shall now apply these results to decide whether our system of rings can account for spectrum-lines of the intensities of those found in nature. In § 6 we found that for a line of photometric intensity 10, *e. g.* the D lines of sodium, the energy radiated is of the order $4 \cdot 10^{-12}$ erg per sec. per atom; and in § 10 we estimated that for a line of photometric intensity 1, that is just observable, the radiation is of the order 10^{-14} erg per sec. per atom. These estimates are based on the assumption that every atom in the sodium flame at every instant shares in the radiation. If we assume that the radiation is due only to the ions, the radiation per sec. per ion is greater in proportion.

The ionization in a flame is difficult to estimate; Stark* calculates the ionization in the unstriated positive column of a vacuum-tube to be of the order 10^{-5} , and states that for a flame it is smaller. This value makes the radiation for a line 1 at least 10^{-9} erg per sec. per ion.

This is the least value that R_0 can have for any vibration of class ± 3 or upwards, if that vibration is to give an observable spectrum-line.

In order to calculate R_0 we use the table of values of R/σ^2 given in § 9. The values of σ_0 have been found in § 11; for class k σ_0 is of order $\frac{1}{k\pi}$, for small values of k and moderately large values of n .

Let us apply our results to the case $k = \pm 3$.

The value of R/σ^2 is $1 \cdot 7 \cdot 10^{-14}$ for $\lambda = 2 \cdot 10^{-5}$ cm. Hence the value of R is $2 \cdot 10^{-16}$.

This is only one five-millionth part of the value required for the faintest observable line.

For greater wave-lengths the value of R is less still.

Even if we had assumed all the atoms to be ionized the value of R would have been 50 times too small.

We conclude that a vibration of class ± 3 or upwards is far too weak to produce an observable spectrum-line.

Again, consider the vibrations of class ± 2 .

The values of R/σ^2 are for

$$\lambda = 2 \cdot 10^{-5} \text{ cm.}, \quad 3 \cdot 6 \cdot 10^{-8},$$

$$\lambda = 3 \cdot 5 \cdot 10^{-5} \text{ cm.}, \quad 1 \cdot 2 \cdot 10^{-9},$$

$$\lambda = 6 \cdot 10^{-5} \text{ cm.}, \quad 4 \cdot 8 \cdot 10^{-11}.$$

The corresponding values of γT are $\cdot 9, \cdot 03, \cdot 001$ respectively, those of $(1 - e^{-\gamma T})/\gamma T$ are $\cdot 66, 1, 1$ respectively. The corresponding radiations are respectively $6 \cdot 10^{-10}, 3 \cdot 10^{-11}, 1 \cdot 2 \cdot 10^{-12}$ erg per sec. per ion.

* *Die Elektrizität in Gasen*, p. 269.

We conclude that with an ionization 10^{-5} our system might give an extremely faint line in the ultraviolet, but could hardly give one in the visible spectrum. On the other hand, if all the atoms were ionized it could give a strong line in any part of the spectrum.

It is noteworthy that for these vibrations of classes ± 2 and upwards the brightest lines occur in the ultraviolet, where the value of R/σ^2 is greatest. This is because the initial radiation R_0 is here of most importance.

§ 15. From what has been said, it follows that observable spectrum-lines can only be produced by vibrations of classes 0, ± 1 , and under certain circumstances by those of classes ± 2 . We shall now consider the relative intensities of these lines.

For classes 0, ± 1 the damping is large; hence the radiation is approximately E_0/T (§ 13).

For E_0 we may take the maximum kinetic energy. For $k = \pm 1, \pm 2$ this is equal to $\frac{\pi^2 m \rho^2 q^2}{n} \sigma_0^2$, where

$$\sigma_0 = \frac{q'}{(q + q')(n + 1) \sin(k\pi/n)} \text{ for } k,$$

and

$$\frac{q}{(q + q')(n + 1) \sin(k\pi/n)} \text{ for } -k.$$

The radiation becomes $\frac{\pi^2 m \rho^2 q^2}{nT} \sigma_0^2$, which for a spectrum-line reduces to

$$\frac{\pi^2 C^2 m \beta^2}{nT} k^2 \sigma_0^2 = k^2 \sigma_0^2 \cdot 10^{-8} \text{ erg per sec. per ion (§ 14).}$$

For a moderately large value of n this becomes

$$10^{-9} \cdot \left\{ \frac{q'}{q + q'} \frac{k\pi/n}{\sin(k\pi/n)} \right\}^2 \text{ erg per sec. per ion for } k,$$

and

$$10^{-9} \cdot \left\{ \frac{q}{q + q'} \frac{k\pi/n}{\sin(k\pi/n)} \right\}^2 \text{ erg per sec. per ion for } -k.$$

These are of the order (10^{-9}) of the radiation for an observable line with the ionization (10^{-5}) calculated by Stark for a vacuum-tube. Their relative intensities depend mainly on the corresponding free frequencies q, q' .

§ 16. We have hitherto neglected to take into account the slight disturbance due to the fact that the initial velocity and radius of our ring are not exactly those necessary for the steady motion of a ring of n equidistant electrons. The differences are the same for each electron of the ring,

and consequently during the readjustment each electron moves in the same way, so far as these disturbances are concerned. In other words, the vibrations due to these differences are vibrations of class 0. Their amplitudes are of the order of the differences in velocity and radius, and therefore small; nevertheless they may be comparable with the mean amplitudes for classes ± 1 , ± 2 , because these are much reduced by radiation. We must therefore enquire whether these vibrations of class 0 are powerful enough to produce observable spectrum-lines.

Let $\Delta\beta$ be the excess of the value of β for $n+1$ above that for n electrons in steady motion. Then the energy E_0 is of the order $\frac{1}{4}C^2mn(\Delta\beta)^2$, and the radiation is of the order $\frac{C^2mn}{4T}(\Delta\beta)^2$, that is, of order $10^{-9} \cdot \left(\frac{n\Delta\beta}{2\beta}\right)^2$ erg per sec. per ion.

The velocity β is given by an equation of the form

$$\frac{nU}{\beta} = \frac{\rho^2 \psi(\beta)}{C} \frac{\dot{a}}{a^2}^*,$$

$$\text{where } U = 2 \sum_{s=1}^{s=\infty} \left[sn\beta^2 J'_{2sn}(2sn\beta) - s^2 n^2 (1-\beta^2) \int_0^\beta J_{2sn}(2snx) dx \right].$$

We find, as in § 6,

$$nU/\beta = \sqrt{\frac{n\gamma}{2\pi}} \cdot \exp. n \left(\gamma - \frac{1}{2} \log \frac{1+\gamma}{1-\gamma} \right), \quad \gamma = \sqrt{1-\beta^2}.$$

Hence β is given by

$$\sqrt{n\gamma} \cdot \exp. n \left(\gamma - \frac{1}{2} \log \frac{1+\gamma}{1-\gamma} \right) = \sqrt{2\pi} \frac{\rho^2 \psi(\beta)}{C} \frac{\dot{a}}{a^2}.$$

The right-hand member may also be written

$$\sqrt{2\pi} \cdot \frac{Cm\rho^2}{e} \frac{\dot{a}}{a^2}, \text{ where } m \text{ is the mass of the electron.}$$

The value of $\sqrt{2\pi} \frac{Cm\rho^2}{e^2}$ varies but slightly with β , that is with n ; its value is $4 \cdot 10^{-12}$. Thus the right-hand member is nearly constant and small when $\dot{a}/a$ is small. I find

$$\frac{n\Delta\beta}{2\beta} = \frac{-\log \left(10^{-12} \cdot \frac{\dot{a}}{a} \right)}{2n\gamma^2}, \text{ very approximately.}$$

* Schott, *loc. cit.* p. 23.

This gives the following results :—

n		10.	20.	50.	100.	1000.
$\frac{n\Delta\beta}{2\beta}$	$\frac{\dot{a}}{a} = 10^{-16}$	·60	·31	·15	·09	·03
	$\frac{\dot{a}}{a} = 10^{-46}$	1·25	·62	·26	·19	·04

These values show that the vibrations of class 0 for small values of n are sufficiently powerful to give an observable spectrum-line, even when the ionization is as small as that in a vacuum-tube. For rings of 20 electrons or so, these lines are about as bright as those of classes ± 1 , for larger rings they are probably weaker, for smaller rings stronger. The ratios of the intensities are not generally very large, but are comparable with those of Jewell's scale.

§ 17. We have found that of all the vibrations of our ring only those of classes 0, ± 1 , and occasionally ± 2 , can produce lines sufficiently strong to be observable, but it may happen that the frequency of any one of these lines is outside the limits of the spectrum. In § 13 we saw that

$$\frac{q\beta}{\omega} + k\beta = \frac{2\pi\rho}{\lambda},$$

where λ is the wave-length of the line. For the extreme red $\frac{2\pi\rho}{\lambda} \leq \cdot 0008$, for the extreme ultraviolet $\frac{2\pi\rho}{\lambda} \leq \cdot 0063$; these are upper limits, since ρ cannot exceed the radius of the atom, that is, 10^{-8} cm. For rings of from 10 to 20 or more electrons $\beta > \cdot 02$; hence for such rings

$$\frac{q}{\omega} + k < \cdot 04 \text{ for a line in the extreme red,}$$

$$\frac{q}{\omega} + k < \cdot 3 \text{ for a line in the extreme ultraviolet.}$$

Negative frequencies are possible; they represent lines of the same absolute frequency. As usual let $\left| \frac{q}{\omega} \right|$ represent the absolute value of $\frac{q}{\omega}$. Then we must have

$$\text{for positive } q : \left| \frac{q}{\omega} \right| < \cdot 04 - k, \cdot 3 - k,$$

$$\text{for negative } q : \left| \frac{q}{\omega} \right| > k - \cdot 04, k - \cdot 3,$$

for a red, ultraviolet line respectively.

Thus the only vibrations, which can give rise to observable spectrum-lines, are

for positive q : those of classes 0, -1 , and occasionally -2 .
for negative q : those of classes 0, $+1$, and occasionally $+2$.

That is, for a single ring of electrons, if we divide all the frequencies into groups, each group corresponding to vibrations of the same type, *e.g.* axial vibrations, orbital vibrations and so on, but of various classes, n in number, each group of frequencies can give rise only to two, or at most three, observable spectrum-lines.

§ 18. For example, Nagaoka* has discussed a system of particles illustrating spectra; he compares the free oscillations of a rotating ring with the vibrations giving rise to spectrum-lines, and finds certain analogies between the grouping of the lines in bands and series and that of the oscillations of the ring, when their frequencies are compared. The frequencies for the axial oscillations are given by a quadratic, different for each class: these two groups he considers analogous to the vibrations producing bands. The frequencies for the orbital oscillations are given by a quartic with a pair of real and a pair of imaginary roots; the two groups belonging to the real roots he considers analogous to the vibrations giving series. Each line of a band or series corresponds to a value of k (h in Nagaoka's notation), that is, to an oscillation of that class.

But we have just seen that any one group gives rise to at most three spectrum-lines. On the other hand, the Balmer series of hydrogen has 29 lines, and few series are known with less than 10. Even if we suppose all Nagaoka's 4 groups to combine to give a single series we can only get 12 lines at most; this is obviously inadequate. For bands the difficulty is still greater.

When we consider the whole series of waves emitted in one group, we find that the intensities differ little for the first 2 or 3 members, given by $k=0, \pm 1, \pm 2$ as the case may be, but after that diminish with very great rapidity, faster than the terms of a geometric progression whose ratio is one to one million. Nothing like this is found for series or for bands.

I think this difficulty is conclusive against Nagaoka's view; but apart from that the frequencies of the oscillations used by Nagaoka are the frequencies *relative to the ring* (n in his notation); the frequencies of the *waves* emitted by the ring and received by a *stationary* observer are different ($n + h\omega$ in his notation). The two sets only agree when the

* Phil. Mag. ser. 6, vol. vii. p. 445.

frequencies n are very large compared with the angular velocity ω , which is not the case. If it be true that the values of n crowd together when h becomes large, as Nagaoka supposes, it follows that the frequencies of the lines produced approach to coincidence with an arithmetic progression, with difference equal to ω . This corresponds neither to a band nor to a series, but to a set of lines of constant frequency difference.

§ 19. Nagaoka's model may be modified in two ways. (1) The controlling field, in which the ring moves, may be altered; for instance, the ring may be made stable as in J. J. Thomson's model*. A comparison of Nagaoka's equations (9), (12) with Thomson's equations (4), (3), and indeed with Maxwell's equations (14) (22)†, shows that in all these cases the frequency equations are fundamentally the same; we have the same six groups of vibrations, each of the same number n of classes, but the values of the frequencies are altered; this is true however we modify the controlling field. Hence in all these cases the system can give rise to observable spectrum-lines for at most three classes in each group, that is 18 lines in all.

(2) In the cases just considered the velocity has been supposed negligible compared with that of light, and therefore higher powers of β have been omitted. Let us remove this restriction.

The frequency equations reduce as before to a separate equation for axial and orbital vibrations, but each equation is transcendental and therefore has an infinite number of roots for each of the n classes. For stability all these must be real, if damping be neglected; if damping be taken into account they must be complex with positive imaginary part. Can each of these give rise to 3 spectrum-lines? If so, we have a sufficient number to account for series and bands.

§ 20. I have found the complete frequency equations for a ring rotating with large velocity in a given controlling field, but the results are very complicated; as an example I give the frequency equation for axial vibrations; in the notation of the present paper Nagaoka's equation (9) may be written

$$\nu - J = \frac{C^2 m \rho}{e^2} \left(\frac{q\beta}{\omega} \right)^2,$$

where the central positive charge is equal to ν of the negative charges, and

$$J = \sum_{i=1}^{i=n-1} \frac{\sin^2(k\pi i/n)}{4 \sin^3(\pi i/n)}.$$

* J. J. Thomson, *Phil. Mag.* ser. 6, vol. vii. p. 237.

† Collected Papers, vol. i. pp. 315, 316.

The corresponding complete equation is

$$\nu - (1 - \beta^2)^2 J - \frac{1}{2} \beta^2 (5 - \beta^2) H \\ = \left\{ \frac{C^2 m \rho}{e^2} + (1 + \beta^2)(K - 2H) \right\} \left(\frac{q\beta}{\omega} \right)^2 - 2\beta(1 - \beta^2) M \frac{q\beta}{\omega} + f(0, 0) - f(k, q),$$

where

$$K = \sum_{i=1}^{i=n-1} \frac{1}{4 \sin \pi i/n}, \quad H = \sum_{i=1}^{i=n-1} \frac{\sin^2(k\pi i/n)}{4 \sin(\pi i/n)}, \\ M = \sum_{i=1}^{i=n-1} \frac{\sin(2k\pi i/n) \cos(\pi i/n)}{8 \sin^2(\pi i/n)}, \\ f(k, q) = \sum_{s=-\infty}^{s=\infty} \frac{1}{16} \cot \frac{(2s-2k+1)\pi}{2n} \left[-(1 - \beta^2) l^2 \beta^2 J'_m(l\beta) \right. \\ \left. + (1 + 3\beta^2) l \beta J_m(l\beta) + \{ (1 + \beta^2)(m^2 + l^2 \beta^2) \right. \\ \left. - 4m l \beta^2 - 1 - 3\beta^2 \} \int_0^{l\beta} J_m(y) dy \right],$$

with $m = 2s + 1$, $l = \frac{2q}{\omega} + 2s + 1$.

With the numerical values of § 3 we find $\frac{C^2 m \rho}{e^2} = 47,000$ approximately. The constants ν , K , H , M are of order n ; for small values of k J is of the same order; small values of k alone are of importance for our purpose. Thus ν , K , H , M , and J are all small compared with $\frac{C^2 m \rho}{e^2}$ unless n be larger than is physically possible, since the distance between consecutive electrons must be a large multiple of their radius.

The largest terms of $f(k, q)$ are generally given by $m = \pm 1$, that is, by $s = 0$, $s = -1$; these are of the orders

$$\frac{1}{8} \cot \frac{(2k \mp 1)\pi}{2n} \left(\frac{q}{\omega} \pm \frac{1}{2} \right)^2 \beta^2 \text{ respectively.}$$

Unless $\frac{q\beta}{\omega}$ be very large, $f(k, q)$ is at most of the order $K \left(\frac{q\beta}{\omega} \right)^2$.

If it be expanded in a series of ascending powers of $\frac{q\beta}{\omega}$, the frequency equation takes the form

$$\nu - (1 - \beta^2)^2 J = \frac{C^2 m \rho}{e^2} \left(\frac{q\beta}{\omega} \right)^2 + a_0 + a_1 \frac{q\beta}{\omega} + a_2 \left(\frac{q\beta}{\omega} \right)^2 + a_3 \left(\frac{q\beta}{\omega} \right)^3 + \dots$$

where a , a_1 , a_2 , $a_3 \dots$ are functions of n and β , which

diminish with increasing index, and are small compared with $\frac{C^2 m \rho}{e^2}$. The equation is transcendental, and thus has an infinite number of roots; for stability all must be real. We shall, however, not concern ourselves with the question of stability.

When $\frac{q\beta}{\omega}$ is small enough we may neglect the higher powers in the series; we then obtain a quadratic, which is not very different from Nagaoka's equation, and has two roots of the same order of magnitude. For values of $k=0, \pm 1, \pm 2$, these correspond to observable spectrum-lines, six in number at most.

§ 21. When $\frac{q\beta}{\omega}$ is large, the equation reduces approximately to

$$0 = \frac{C^2 m \rho}{e^2} + a_3 \frac{q\beta}{\omega} + a_4 \left(\frac{q\beta}{\omega} \right)^2 + \dots$$

This equation has an infinite number of large roots. Can any of these give rise to observable spectrum-lines?

In order that they may do so the frequencies must fall within the proper limits, that is, $\frac{q\beta}{\omega} + k\beta$, which is equal to $\frac{2\pi\rho}{\lambda}$, must lie between $\cdot 0063$ and $\cdot 0008$. We find

$$\begin{aligned} a_3 = \frac{1}{6} \sum_{s=-\infty}^{s=\infty} \cot \frac{(2k-2s-1)\pi}{2n} [2J_{2s+1}\{(2s+1)\beta\} \\ + (1-\beta^2)(2s+1)\beta J'_{2s+1}(2s+1)\beta], \\ < \frac{\beta(3+\beta^2)}{6(1-\beta^2)} \cot \frac{\pi}{2n}. \end{aligned}$$

This is exceedingly small compared with $\frac{C^2 m \rho}{e^2}$; hence $\frac{q\beta}{\omega}$ is a very large number; it follows that $\frac{q\beta}{\omega} + k\beta$ cannot be small, for k is less than $\pm \frac{n}{2}$, and β is less than unity. The only exception might occur for n very large and β nearly unity. For $n=1000$, $\beta=\cdot 9$, we find

$$a_3 < 1800, \quad \left| \frac{q\beta}{\omega} \right| > 26.$$

In this case the distance between consecutive electrons is only 50 times the radius of an electron. Even when the

electrons are as crowded as this, the value of $k\beta$ is as great as 26, and that of k as great as 30, so that the line is far too weak to be observable.

We conclude that the large roots of the frequency equations of the ring cannot give observable spectrum-lines; so that the most general type of single ring cannot account either for spectrum-series or for bands.

§ 22. The argument of this paper may be summarized briefly as follows. After some preliminary work (§§ 1-10) an ideal radiating system is considered, consisting of a large number of mutually independent similar rings of electrons, each in orbital motion in a suitable controlling field, of which a fraction are ionized owing to the previous expulsion of an electron (§§ 11-12).

Each ion executes a number of vibrations; these can be arranged in six groups according to the frequency (q) relative to the rotating ring; each group includes n classes of vibrations, n being the number of electrons in the ionized ring (ion), and the class (k) giving the number of segments in which the ring vibrates (§ 7).

Each vibrating ion emits a corresponding number of waves (§ 8), of which the frequencies to a stationary observer are $(q+k\omega)$, ω being the angular velocity of rotation of the ring.

The intensities of these waves after the first two or three classes fall off with such great rapidity (§§ 9, 14), that waves of classes ± 3 and upwards are far too weak to give rise to observable spectrum-lines; waves of classes ± 1 and 0 give rise to lines whose intensities are of the same order of magnitude as those of spectrum-lines (§§ 14, 16); waves of classes ± 2 do so when the ionization is large (§ 14).

When the velocity of the electrons in steady motion is very small compared with that of light, there are six frequencies (q) for each class (k); hence the ring gives at most 18 observable lines (§§ 18, 19). When the velocity is comparable with that of light, the frequency equations are transcendental, and there are an infinite number of frequencies (q) for each class (k); but of these frequencies only six can give rise to observable lines, so that the maximum number for the ring is only 18 as before (§§ 20, 21), that is, too small to account for series or bands.

XII. *On our Perception of Sound Direction*.*

By Lord RAYLEIGH, O.M., Pres. R.S.†

IT is some thirty years ago since I executed a rather extensive series of experiments in order to ascertain more precisely what are the capabilities of the ears in estimating the direction of sounds ‡. It appeared that when the alternative was between right and left, the discrimination could be made with certainty and without moving the head, even although the sounds were pure tones. Nor was any difficulty introduced by the requirement that the ears should be stopped at the moment when the sounds commenced.

On the other hand, if the question was whether a sound were situated in front or behind the observer, no pronouncement could be made in the case of pure tones. The impossibility of distinguishing front and back carries with it further confusions relating to cases where the sound may be obliquely situated. But with sounds of other character and notably with the speaking voice, front and back could often be distinguished. It is understood, of course, that the head was kept still. A slight rotation, bringing a pure tone (originally situated exactly in front or exactly behind) to the right or the left, gives the information that was previously lacking.

The discrimination between right and left is usually supposed to be explicable by the greater intensity of sensation experienced by the ear which lies nearer to the sound. When the pitch is pretty high, there is no doubt that this explanation is adequate. A whistle of pitch f^{iv} , preferably blown from a gas-bag, is much better heard with the nearer than with the further ear. "A hiss is also heard very badly with the averted ear. This observation may be made by first listening with both ears to a steady hiss on the right or left, and then closing one ear. It makes but little difference when the further ear is closed, but a great difference when the nearer ear is closed. A similar observation may be made upon the sound of running water." In a modified form of

* This paper formed the substance of the Sidgwick lecture given at Cambridge on November 10, 1906, and (except the last two or three pages) was written before the delivery of the lecture. I have learned since from Dr. L. More that three years ago at Cincinnati he made experiments which led him to similar conclusions. It is to be hoped that Dr. More will publish an account of his work, the more as it was conducted on lines different from mine.

† Communicated by the Author.

‡ 'Nature,' xiv. p. 32 (1876); Phil. Mag. iii. p. 546 (1877); Phil. Mag. xiii. p. 340 (1882). 'Scientific Papers,' i. pp. 277, 314; ii. p. 98.

the experiment the ear, say the right, nearest to the falling water is stopped with the right hand. The comparatively feeble sound then heard may be much increased if the left hand be so held at a little distance out as to reflect the sound into the left ear. The effect remains conspicuous even when the hand is held out at full arm's length. Of course a reflector larger than the hand is still more effective.

The discrimination between the right and left situations of high sounds is thus easily explained upon the intensity theory; but this theory becomes less and less adequate as the pitch falls. At a frequency of 256 (middle $c=c'$) the difference of intensities at the two ears is far from conspicuous. At 128 it is barely perceptible. But although the difference of intensities is so small, the discrimination of right and left is as easy as before.

There is nothing surprising in the observation that sounds of low pitch are nearly as well heard with the further as with the nearer ear. When the wave-length amounts to several feet, it is not to be expected that the sound (originating at a distance) could be limited to one side of the head. The question is well illustrated by calculations relating to the incidence of plane waves upon a rigid spherical obstacle, and the results may conveniently be repeated here from 'Theory of Sound,' § 328.

$\frac{2\pi c}{\lambda}$	μ	$F+iG$	F^2+G^2
$\frac{1}{2}$	1	$\cdot522+\cdot139 i$	$\cdot294$
	-1	$\cdot159-\cdot484 i$	$\cdot260$
	0	$\cdot430-\cdot217 i$	$\cdot232$
1	1	$\cdot668+\cdot238 i$	$\cdot503$
	-1	$-\cdot440-\cdot303 i$	$\cdot285$
	0	$+\cdot322-\cdot365 i$	$\cdot237$
2	1	$\cdot797+\cdot234 i$	$\cdot690$
	-1	$\cdot250+\cdot506 i$	$\cdot318$
	0	$-\cdot154-\cdot577 i$	$\cdot356$

In this table $2\pi c$ is the circumference of the sphere, and λ is the wave-length of the sound. The symbol μ denotes the cosine of the angle at the centre of the sphere between the direction of the sound ($\mu=1$) and the point upon the

sphere at which the intensity is to be reckoned. $F + iG$ denotes the (complex) condensation, and $F^2 + G^2$ the intensity. In the present question of a sound situated say to the right of the observer, the intensity at the right ear corresponds to $\mu = +1$, and at the left ear to $\mu = -1$. In the case of the head the circumference ($2\pi c$) may be taken at about 2 feet or a little less, so that $2\pi c/\lambda = \frac{1}{2}$ corresponds about to middle c , or frequency 256. It will be seen that the difference of intensities for $\mu = \pm 1$ is only about 10 per cent. of the whole intensity.

For still smaller values of $2\pi c/\lambda$, *i. e.* in the present application for still graver notes, the difference of intensities may be adequately expressed by a very simple formula. It appears that

$$(F^2 + G^2)_{\mu=1} - (F^2 + G^2)_{\mu=-1} = \frac{3}{4} \left(\frac{2\pi c}{\lambda} \right)^4,$$

while at the same time the total value of $F^2 + G^2$ approximates to .25. A fall in pitch of an octave thus reduces the difference of intensities 16 times. At frequency 128 the difference would be decidedly less than one per cent. of the whole; and from this point onwards it is difficult to see how the difference could play any important part.

So far as I am aware no explanation of the above difficulty, emphasised in 1876, has been arrived at. A few months since I decided to repeat, and if possible to extend, the observations, commencing with frequency 128. Two forks of this pitch were mounted in the open air at a considerable distance apart, and were electrically maintained, one driving the other. In connexion with each was a resonator which could be put out of action by interposing (without contact) the blade of a knife or a piece of card. An observer with eyes closed, placed between the two forks and so turned as to have them upon his right and his left, could tell with certainty which resonator was in action. The ears may be open all the time; or, what is in some respects better, they may be closed while the changes at the resonators are being made, and afterwards opened simultaneously. If one ear be opened, the sound will appear to be on that side; but when the second ear is also opened, the sound assumes its correct position, whether or not this involves a reversal of the earlier judgement. When the sounds were in front and behind, instead of to the right and the left, several observers agreed that a discrimination between front and back could not be made.

In another method of experimenting a single resonator and fork (which need not be electrically maintained) suffice, but

more than one assistant may be required. The observer, either on his feet, or more conveniently seated upon a rotating stool, is turned round until he loses his bearings. There is no difficulty in this*, but some precautions are needed to prevent the bearing being afterwards recovered. The wind may act as a tell-tale. It is often necessary to cover the eyes with the hands as well as to close the eye-lids in order sufficiently to exclude the light. Until all is ready for a judgement, the ears are kept closed by pressure with the thumbs, and it is usually advisable to keep the thumbs in motion and thus to cause miscellaneous noises loud enough to drown any residue of the sound under observation. Pure tones of pitch 128 and 256 yielded by this method result in agreement with those already described.

The turn-table facilitates observation upon the question as to the relative loudness with which a sound is heard according as the source is on the same, or the opposite, side as the ear in use. In my own case I thought I could detect an advantage when the source was on the same side as the open ear, while others could detect no difference. This relates to pitch 128. At 256 the advantage is quite marked.

In considering whether the discrimination between right and left at pitch 128 can really be attributable to the small intensity-difference, it occurred to me that, if so, the judgement might perhaps be disturbed by the introduction of an obstacle, such as a piece of board, near the head of the observer and on the same side as the sound. But it was found that no mistakes could thus be induced, although in each trial the observer did not know whether the board were in position or not. Another circumstance, unfavourable to the intensity theory, may also be mentioned. It was found that the observer on the turn-table could sometimes decide between the right and the left before un-stopping his ears.

The next step in the investigation appeared to be the examination of pure tones of still graver pitch. A globe, such as are sold to demonstrate the combustion of phosphorus in oxygen gas, was sounded with the aid of a hydrogen flame†. Careful observation revealed little or no trace of overtones. The frequency was about 96 vibrations per second, thus by the interval of the Fourth graver than the 128 forks. At the temperature of the observations this would correspond to a wave-length of about 12 feet. If we use this value in the formula already given, we find for the proportional difference

* The process is aided by the illusion of a reverse rotation when the real rotation has stopped.

† *Phil. Mag.* vii. p. 149 (1879); ‘*Scientific Papers*,’ i. p. 407.

of intensities on the two sides of a sphere of 2 feet circumference only about 2 parts per thousand.

Observation in the open air showed that there was no difficulty whatever in deciding whether this low sound was on the right or the left. Several observers agreed that the discrimination was quite as easy as in the case of forks of pitch 128. On the other hand, as was to be expected, the front and back situations could not be discriminated.

At this stage a reconsideration of theoretical possibilities seemed called for. There could be no doubt but that relative intensities at the two ears play an important part in the localization of sound. Thus if a fork of whatever pitch be held close to one ear, it is heard much the louder by that ear, and is at once referred instinctively to that side of the head. It is impossible to doubt that this is a question of relative intensities. On the other hand, as we have seen, there are cases where this explanation breaks down. When a pure tone of low pitch is recognized as being on the right or the left, the only alternative to the intensity theory is to suppose that the judgement is founded upon the difference of *phases* at the two ears. But even if we admit, as for many years* I have been rather reluctant to do, that this difference of phase can be taken into account, we must, I think, limit our explanation upon these lines to the cases of not very high pitch. For what is the difference of phase at the two ears when a sound reaches the observer—say from the right? It is easy to see that the retardation of distance at the left ear is of the order of the semi-circumference of the head, say one foot. At this rate the retardation for middle c ($c' = 256$) is nearly one quarter of a period; for c'' (512) nearly half a period; for c''' (1024) nearly a whole period, and so on. Now it is certain that a phase-retardation of half a period affords no material for a decision that the source is on the right rather than on the left, seeing that there is no difference between a retardation and an acceleration of half a period. It is even more evident that a retardation of a whole period, or of any number of whole periods, would be of no avail. In the region of somewhat high pitch a judgement dependent upon phase would seem to be hardly possible, especially when we reflect that the phase-differences enter by degrees, rising from zero when a sound is directly in front or behind to a maximum or minimum in the extreme right and left positions.

As to whether there is any difficulty in localizing to the right or left a tone of pitch 512, an early observation,

* Conf. 'Theory of Sound,' § 385.

conducted with the aid of two forks and resonators of this pitch, gave an answer in the negative. The localization was as easy and as distinct at pitch 512 as at pitch 256. But it is quite possible that 512 is not sufficiently near the particular pitch for which the retardation would have the value of precisely half a period.

The calculations for a spherical obstacle already quoted give some information upon this point. From the values of $F + iG$ corresponding to $\mu = \pm 1$ for the case of $2\pi c/\lambda = 1$, it appears that there is approximate opposition of phase. A closer examination and comparison of the three cases shows that exact opposition will occur at a somewhat higher value, say $2\pi c/\lambda = 1.1$. Calculating from this and taking $2\pi c$ at 2 feet, I find at the temperature of the observations a pitch about a minor third *above* 512, as that corresponding to a phase-difference of half a period.

Naturally, in its application to the head, the calculation is not very trustworthy, and I thought it important to make sure by actual experiment that there is no pitch in this region for which the discrimination of right and left is at all uncertain. Tuning-forks not being available, I fell back upon "singing-flames," *i. e.* tubes, usually of metal and about 1 inch in diameter, maintained in vibration by hydrogen flames. In order to eliminate overtones, the tubes were provided near their centres with loosely-fitting rectangular blocks about two diameters long, held in position by the friction of attached springs. In this way and with the precaution of not sounding them more loudly than was necessary, the tones were, it is believed, sufficiently pure. Trials were made in the open air on many occasions, the pitch ranging in all from d' to g''' , and there was never the smallest suspicion of a difficulty in discriminating right and left. In the region from c'' to g'' of special interest, the pitch was varied by half semitones with the aid of sliding prolongations of the tube. During most of the observations the listener was placed upon the rotating stool and was in ignorance of the real position of the source. This precaution is of course desirable; but after a good deal of practice I found that I was able to trust the direct sensual impression. In the case of right or left the impression is always distinct and always correct; but in trying to discriminate front and back there is usually no distinct impression, and when there is it often turns out to be wrong.

We may fairly conclude that in this region of pitch (above $c'' = 512$) the discrimination of right and left is not made upon the evidence of phase-differences, or at any rate not upon

this evidence alone. And this conclusion leads to no difficulty; for, as has already been explained, the difference of intensities at the two ears gives adequate foundation for a judgement. It would seem that at high pitch, above c'' , the judgement is based upon intensities; but that at low pitch, at any rate below c (128), phase-differences must be appealed to.

It remained to confirm, if possible, the suggestion that not only are we capable of appreciating the phase-differences with which sounds of equal intensity may reach the two ears, but that such appreciation is the foundation of judgements as to the direction of sounds—in particular of the right and left effects. The obvious method is to conduct to the two ears separately two pure tones, nearly but not quite in unison. During the cycle, or beat, the phase-differences assume all possible values; and the mere recognition of the cycle is evidence of some appreciation of phase-differences. Experiments on these lines are not new. In 1877 Prof. S. P. Thompson* demonstrated “the existence of an interference in the perception of sounds by leading separately to the ears with india-rubber pipes the sounds of two tuning-forks struck in separate apartments, and tuned so as to ‘beat’ with one another—the ‘beats’ being very distinctly marked in the resulting sensation, although the two sources had had no opportunity of mingling externally, or of acting jointly, on any portion of the air-columns along which the sound travelled. The experiment succeeded even with vibrations of so little intensity as to be singly inaudible.” And in an observation of my own†, where tones supposed to be moderately pure were led to the ears with use of telephones, a nearly identical conclusion was reached. But although the cycle was recognized, in neither case apparently was there any suggestion of a right and left effect.

In repeating the experiment recently I was desirous of avoiding the use of telephones or tubes in contact with the ears, under which artificial conditions an instinctive judgement would perhaps be disturbed. It seemed that it might suffice to lead the sounds through tubes whose open ends were merely in close proximity one to each ear, an arrangement which has the advantage of allowing the relative intensities to be controlled by a slight lateral displacement of the head towards one or other source. Two forks of frequency 128, independently electrically maintained, were placed in different rooms. Associated with each was a suitably tuned resonator

* Phil. Mag. Nov. 1878.

† Phil. Mag. ii. p. 280 (1901); ‘Scientific Papers,’ iv. p. 553.

from the interior of which a composition (gas) pipe led the sound through a hole in a thick wall to the observer in a third room. With the aid of closed doors and various other precautions of an obvious character, the sounds were fairly well isolated. But each resonator emits a rather loud sound into its neighbourhood, a little of which might eventually reach the other. To eliminate this cause of disturbance more completely, a second resonator * of like pitch was employed in association with each fork, so situated that the phases of vibration in the two resonators were opposed. By a little adjustment it was possible to provide that but little sound radiated externally from the combination, though the internal vibrations might be as vigorous or more vigorous than when one resonator was employed alone. These arrangements were so successful that when one fork singly was in action, the sound was imperceptible from the tube belonging to the other, even though the open end were pressed firmly into the ear, by which the effect is enormously increased. The open ends of the two pipes may thus be regarded as sources of sound of constant intensity.

In the greater number of experiments the observer, leaning over a table for the sake of steadiness, placed his head between the pipes, which were at such a distance apart that one or two inches separated the open ends from the adjacent ears. At the very first trial on July 31, the period of the cycle being 5 seconds, Lady Rayleigh and I at once experienced a distinct right and left effect, the sound appearing to transfer itself alternately from the one side to the other. When the effect was at its best, the sound seemed to lie entirely on the one side or on the other.

The beat may be slowed down until it occupies 40 or even 70 seconds, thus giving opportunity for more leisurely observation. The position of the head should be so chosen that the right and left effects are equally distinct. Under these circumstances it is found that the sound seems to be predominantly on the right or on the left for almost the whole of the cycle, the transitions occupying only small fractions of the whole time. The observations may be made with the ears continuously open, or, as in some of the outdoor experiments, the ears may be opened and closed simultaneously at short intervals. It is perhaps better still, keeping the ears open, to close periodically with the thumbs the open ends of the tubes from which the sounds issue. When the tubes are closed, no sound is audible. It should be said that

* Three out of the four resonators consisted of "Winchester" bottles from which the necks had been removed.

although the best results require the position of the head to be carefully chosen, right and left effects are perceived through a considerable range. It is only necessary that the intensities be approximately equal.

These results are quite decisive, if we can assume that the sounds were sufficiently isolated—that nothing appreciable could pass from the open end of a pipe to the wrong ear. It was easy to verify that when one pipe and the opposite ear were closed, next to nothing could be heard; but it may perhaps be argued that the test is not delicate enough. The risk of error from this cause is diminished by approximating the open ends to their respective ears. Many experiments of this kind were made, but without influencing the results. Finally, short lengths of rubber tubing were provided, by means of which the ears could be connected almost air-tight with the pipes. In this case, to avoid being deafened, it was necessary to reduce the sounds by withdrawing the resonators from their respective forks. The right and left effect remained fully marked. Another argument to show that the effects can not be explained by sounds passing round the head will be mentioned presently.

A question of great importance still remains to be considered. Before the laboratory experiments can be accepted as explanatory of the discrimination of right and left when a single sound is given in the open, it is necessary to show that in the former the sensation of right (say) is associated with a phase-difference such that the vibration reaching the right ear *leads*. There was no difficulty in obtaining a decision. While one observer listens as described for right and left effects, a second observes the maxima and minima of the beat as heard by one ear situated symmetrically with respect to the two sources. In the case of sounds of higher pitch to be considered later some precaution is required here; but for the present sounds, corresponding to a wave-length of nearly 9 feet, there is no difficulty. Under good conditions the minimum, represented by a *silence*, is extremely well marked, and can often be signalled to within half a second. This signal, corresponding to *opposition* of phases, gives the required information to the first observer. If a signal for the maximum, representing phase-agreement, is desired, it is best made by halving the intervals between the silences.

The results can be stated without the slightest ambiguity. The *transitions* between right and left effects correspond to agreement and opposition of phase, not usually recognized by the first observer as maxima and minima of sound. When the vibration on the right is the quicker, the

sensation of right follows agreement of phase, and (what is better observed) the sensation of left follows opposition of phase. And similarly when the vibration on the left is the quicker, the left sensation follows agreement, and the right follows opposition of phase. The question which fork is vibrating the quicker is determined in the usual way by observing the effect upon the period of the beat of the addition or removal of a small load of wax. If for example the beat is *slowed* by loading the right fork, we may be sure that that fork was originally the quicker. A large number of comparisons of this kind have been made at various times, and in no case (at this pitch) has the rule been violated. It is not a little remarkable that by merely listening to right and left effects aided by signals giving the moment of phase-opposition, it is possible conversely to pronounce which fork is the quicker, although the difference of frequencies may not exceed $\cdot 02$ vibration per second.

It should not pass unnoticed that the laboratory experiments cover a wider field than the observations in the open. In the latter case, if the single sound of pitch 128 is in front or behind, there is agreement of phases at the two ears. As the position becomes more and more oblique, the phase-difference increases; but it can never exceed a moderate amount, about one-eighth of a period, which is attained when the position is precisely to the left or right. Phase-differences in the neighbourhood of half a period do not occur. From the laboratory experiments it appears that the right and left effects are not subject to this limitation; but that, for example, a right effect is experienced when the vibration reaching the right ear leads, whether the amount of the lead be small or whether it approaches the half period.

The right and left or, as I shall sometimes say for brevity, the *lateral*, sensations observable in this way are so conspicuous that I was curious to inquire how I had contrived to miss them in the earlier experiments with telephones already alluded to. The apparatus was the same as before. In the neighbourhood of the electro-magnet driving each fork (128) was placed a small coil of insulated wire whose circuit was completed through a telephone. The double wires connected to the telephones were passed through the perforated wall. In order to weaken the higher overtones, thick sheets of copper intervened between the electro-magnets and the coils. But when the telephones were held to the ears, the sounds were perceived to be of a more mixed character than I had expected; and I am forced to the conclusion that I must formerly have overestimated their approximation to the

character of pure tones. Although the cycle could be recognized, a distinct lateral effect was not perceived, and the failure was evidently connected with the composite character of the sounds. By loading the disks of the telephones with penny-pieces (attached at the centres with wax) the higher components could be better eliminated. It was then possible to fix the attention upon the fundamental tone and to recognize its transference from left to right during the cycle. But the effect was by no means so conspicuous as with the tubes, and might perhaps be missed by an unprepared observer*.

The subject now under consideration is illustrated by a curious observation accidentally made in the course of another inquiry. A large tuning-fork of frequency about 100, mounted upon a resonance-box, was under examination with a Quinke tube. This consisted of a piece of lead pipe more than one quarter of a wave-length long, one end of which was inserted into the resonance-box. At a distance of one eighth of a wave-length from the outer end a lateral tube was attached which communicated with one ear by means of an india-rubber prolongation. When the second ear was closed, it appeared to make no difference to the sound whether or not the outer end of the Quinke tube was closed with the thumb. But when the second ear was open, marked changes in the sound accompanied the opening and closing of the Quinke tube. On the view hitherto held, it would appear very paradoxical that a change not affecting the sound heard in either ear separately should be able to manifest itself so conspicuously. It was easily recognized that the alterations observed were of the nature of right and left effects, and that they could be explained by the local *reversal of phase* which accompanied the closing of the Quinke tube.

The conclusion, no longer to be resisted, that when a sound of low pitch reaches the two ears with approximately equal intensities but with a phase-difference of one quarter of a period, we are able so easily to distinguish at which ear the phase is in advance, must have far reaching consequences in the theory of audition. It seems no longer possible to hold that the vibratory character of sound terminates at the outer

* Subsequently by a much heavier loading (53 gms.) the telephone-plates were tuned approximately to pitch 128, as could be verified by tapping them with the finger. To find room for these extra loads, the ear-pieces of the telephones had to be modified. The sounds now heard were very approximately pure tones, and the lateral effects were as distinct as those observed when the sounds were conveyed through tubes. It is easy to understand that considerable complication must attend an accompaniment of octave and higher harmonics, which would transfer themselves from right to left more rapidly than the fundamental tone.

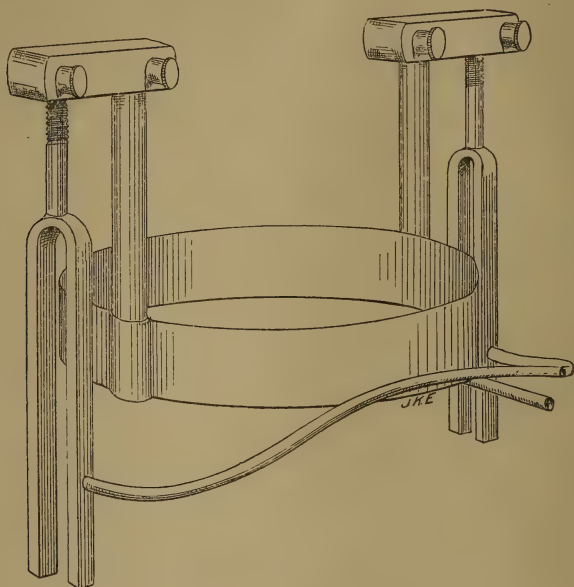
ends of the nerves along which the communication with the brain is established. On the contrary, the processes in the nerve must themselves be vibratory, not of course in the gross mechanical sense, but with preservation of the period and retaining the characteristic of phase—a view advocated by Rutherford, in opposition to Helmholtz, as long ago as 1886. And when we admit that phase-differences at the two ears of tones in unison are easily recognized, we may be inclined to go further and find less difficulty in supposing that phase-relations between a tone and its harmonics, presented to the *same ear*, are also recognizable.

The discrimination of right and left in the case of sounds of frequency 128 and lower, so difficult to understand on the intensity theory, is now satisfactorily attributed to the phase-differences at the two ears. The next observations relate to pure tones of pitch 256. Two large forks of this pitch were used, such as are commonly to be found in collections of acoustical apparatus. Tuned with wax so as to give beats of 3 or 4 seconds period, they may be held (after excitation) by their stalks one to each ear, preferably by an assistant. The sensation of transference from right to left was fully marked, and when the conditions were good, especially in respect of equality of intensities at the two ears, the whole of the sound seemed to come first from one side and then from the other. The method of holding in the fingers is satisfactory as regards the isolation of the sounds. Practically nothing of either fork can be heard by the further ear. But there is a little difficulty in maintaining quite constant the relative positions of forks and ears.

In another arrangement, which has certain advantages, the forks are mounted, stalks upwards, on a sort of crown, in such a fashion that the free ends of the forks are about opposite the ears. A sketch by Mr. Enock is reproduced in the figure (p. 226). If the crown be sufficiently large, it can be adapted to various sized heads with the aid of pads. At the back, attached to the crown, is a forked tube of brass, symmetrically shaped, whose open ends abut upon the faces of the forks. The short limb, forming the stalk, is prolonged by india-rubber tubing and so connected with the ear of the assistant observer. If the forks are vibrating equally, a well-defined silence marks the moment of phase-opposition. To excite and maintain the vibrations, a violin-bow is employed in the usual manner.

Very good observations may be made in this way if the vibrations of the forks are equally and sufficiently maintained. The assistant, listening through the forked tube, is able to give a sharp signal at the moment of phase-opposition. A

transition in respect of right and left effect occurs at this moment, and the sequence rule already stated defining whether the transition is from right to left or from left to right is found to be obeyed.



There are some advantages, of course, in an experimental arrangement allowing the sounds to be uniformly maintained. As in Helmholtz's vowel investigations, the 256 forks can be driven by the 128-interrupter-forks already employed, and in each case the frequency of the driven fork is the exact double of the frequency of the driving fork. The observations may be made in two ways. Either the 256 forks may themselves be brought close to the ears, leading wires being conveyed through the wall; or, what is on the whole preferable, the method employed for the 128 per second tones may be followed. In this case each 256 fork is associated with two resonators, vibrating in opposite phases, with one of which the pipe leading to the observation room is connected. The isolation was good, each sound being inaudible through the tube provided for the other.

Excellent results were obtained in this way. With good adjustment the transitions between right and left were sharply marked and the sequence rule already formulated was obeyed. It is again to be noted that right and left effects are observable in the neighbourhood of phase-opposition, a situation which

does not arise when one sound in the open influences both ears.

Unless the open ends are pretty close to the ears, the sound has more tendency to travel round the head to the wrong ear than was observed at pitch 128. This may raise a question whether, after all, the right and left effects may not be due to small differences of intensities at the two ears varying periodically. The best answer to this objection is to consider what would be the consequence of such invasions. Suppose that at one moment the vibration on the right is in advance by one quarter period, so that a full right sensation is being experienced. The retardation in travelling round the head will at this pitch be about one quarter period, so that the sound starting from the right in advance will on arrival at the left be in approximate phase-agreement with the principal sound there. On the other hand, the sound starting from the left, already a quarter period in arrear, will on arrival at the right be in approximate phase-opposition with the principal sound on the right. The effect of travel round the head is therefore to augment the sound on the left and to diminish the sound on the right. This would evidently tend to cause a sensation of sound on the *left*, and cannot therefore be the explanation of the observed sensation of a sound on the right. The same considerations will apply, if in less degree, to sounds of pitch 128, and to sounds somewhat higher in pitch than 256.

The next sounds to be experimented upon in order of pitch were from forks giving e' of 320 vibrations per second. These could not be driven from the 128 per second interrupters, and were merely held to the ears in the fingers of an assistant. The right and left effect was very marked, but there was a little difficulty at first in fixing the moment of phase-opposition. After a few trials it became sufficiently clear that the rule was the same as at the lower pitch, viz. that the quicker fork asserts itself after the maximum of the beat, corresponding to phase-agreement.

From this point, as the pitch rises, the observations become more difficult, partly no doubt on account of purely experimental complications, but also, I believe, because the effects are themselves less well marked. From two forks of pitch g' , electrically driven from the 128 interrupters and provided with resonators in connexion with pipes, fairly distinct right and left effects were obtained, but at first there were discrepancies as to which effect followed phase-opposition. These appear to have been due to faulty observation of the phase of opposition. As might have been anticipated, the moment

of silence representing phase-opposition varies with the position in the room of the ear of the assistant observer. To secure a satisfactory signal at the higher pitches, this position requires to be carefully chosen. In the later experiments a resonator was always employed, whose mouth was symmetrically situated with respect to the open ends of the pipes, which are the proximate sources of sound, connexion with the ear of the assistant observer being through a suitable rubber tube. After this there was no ambiguity, the rule of the lower pitch being uniformly followed. But when the open ends of the pipes were not very close to the ears, perhaps 2 inches distant, the right and left effects seemed to two observers (including myself) not only to be rather obscured, but to be concentrated into the neighbourhood of phase-opposition. A third observer, however, heard the right and left effects more strongly, and with less apparent concentration towards phase-opposition.

On the theory that passage of sound round the head had something to do with these complications, the open ends of the pipes were brought much closer to the ears, but without fitting air-tight, the resonators being re-adjusted so as to diminish the loudness. In this condition of things the two observers experienced the right and left effect more normally and without special concentration in the region of phase-opposition. But the observation is certainly more difficult than at lower pitches, and I believe that the effects are really less pronounced.

Experiments similarly conducted with forks of pitch c'' (512) gave results of the same character. When the open ends of the pipes were quite close to the ears, the right and left effect was pretty good and fairly distributed. In this three observers concurred. But a slight withdrawal of the pipes introduced confusion, the extent of which, however, appeared to vary with the observer. In all cases the right and left effects, when sufficiently marked to be observed, obeyed the sequence rule.

At pitch e'' (640) the results were not very different. The open ends of the pipes being close to the ears, but not fitted air-tight, only pretty good right and left effects could be observed, and these appeared to be crowded towards phase-opposition. The sequence rule was obeyed.

Finally, trials by the same method were made with forks of pitch g'' ($6 \times 128 = 768$). No particular difficulty was encountered in satisfying the necessary experimental conditions, but the results were of a nondescript character. Even when the open ends of the pipes were close to the ears,

I could not satisfy myself that I experienced any right and left effect. Another observer thought he heard a little. It seems clear that at any rate the limit was being approached.

It will be understood that some of these observations were not made without difficulty. Probably an experimenter new to the work would feel misgivings with respect to some even of the easier decisions. But all the more important results have the concurrence of at least three observers *. I regard it as established that up to pitch g' phase-differences are attended with marked lateral effects. They are probably the principal basis on which discriminations of right and left are founded—at any rate below e' (256).

As has already been suggested, it was reasonable to anticipate that phase-difference would cease to avail as an indicator at high pitch. Up to about e' the conditions are favourable. At this pitch the phase-difference at the ears affected by a distant sound increases from zero when the source is in front or behind to a maximum of a quarter period (in one or other direction) when the source is on the right or the left in the line of the ears. This is the phase-difference for which one would expect the lateral sensation to be most intense, so that up to this pitch the lateral sensation would keep step with the true lateralness of the source. At a point somewhat higher in pitch it would seem that complications must enter. The maximum sensation (corresponding to a phase-difference of a quarter period) would occur while the source was still in an oblique position, and the sensation of lateralness would diminish while the true lateralness was still increasing. At a pitch in the neighbourhood of e'' (640) the maximum phase-difference would rise to half a period, a phase-difference which could not give rise to lateral sensation at all. Thus, although there might be right and left sensations from sources obliquely situated, these sensations would fail when most needed, that is when the source is really in the line of the ears. In this case a perception of phase-differences would seem to do more harm than good. At a pitch a little higher, ambiguities of a misleading and dangerous kind would necessarily enter. For example, the same sensations might arise from a sound a little on the left and from another fully on the right.

On the whole it appears that the sensation of lateralness due to phase-difference disappears in the region of pitch where there would be danger of its becoming a misleading guide. It is not suggested that there is any precise numerical coincidence. If it were a question of calculating a pitch precisely, it might be necessary to look beyond the size of

* Lady Rayleigh, Mr. Enock, and myself.

modern adult heads to those of our ancestors, perhaps in a very distant past. It is fortunate that when difference of phase fails, difference of intensity comes to our aid. Perhaps it is not to be expected that we should recognize intuitively the very different foundations upon which our judgement rests in the two cases.

A rather difficult question arises as to whether in the laboratory experiments it is possible to distinguish the phases of agreement and of opposition. Not unnaturally perhaps, the apparent movements of the sound from right to left and back are liable to be interpreted as parts of a general movement of revolution, so that, for example, phase-agreement may correspond to the front and phase-opposition to the back position. In a particular case the question is as to the direction of the revolution, whether clockwise or counter-clockwise. With respect to this, my observers frequently disagreed, from which I am disposed to conclude that in these experiments phase-agreement and opposition are not definitely connected with front and back sensations.

At this point there seems to be some discrepancy with the observations of Prof. S. P. Thompson, who found* that "when two simple tones in unison reach the ears in opposite phases, the sensation of the sound is localized at the back of the head." In Prof. Thompson's most striking experiment a microphone is connected in series with a battery and two similar Bell telephones, one of the telephones being provided with a commutator by which the direction of the current through it can be reversed. When the current flows similarly through the telephones, a light tap near the microphone is heard in the ears; but when the current is reversed in one of them a sensation is experienced "only to be described as of some one tapping with a hammer *on the back of the skull from the inside.*" In some (rather inadequate) experiments I have not succeeded in repeating this observation.

The other branch of the subject, which I had hoped to treat in this paper, is the discrimination between the front and back position when a sound is observed in the open; but various obstacles have intervened to cause delay. Among these is the fact that (at 64 years of age) my own hearing has deteriorated. Thirty years ago it was only pure tones, or at any rate musical notes free from accompanying noises, that gave difficulty. Now, as I find to my surprise, I fail to discriminate, even in the case of human speech. It is to be

* Phil. Mag. November 1878, p. 391.

presumed that this failure is connected with obtuseness to sounds of high pitch, such as occur especially in the sibilants. For some years I have been aware that I could no longer hear as before many of the high notes from bird-calls, such as I employ with sensitive flames for imitating optical phenomena. If, as seems the only possible explanation, the discrimination of front and back depends upon an alteration of *quality* due to the external ears, it was to be expected that it would be concerned with the higher elements of the sound. In this matter it would not be surprising if individual differences manifested themselves, apart from deafness. A "paddle-box" formation of the external ear, if not ornamental, may have practical advantages.

My assistant, Mr. Enock, is able to make discriminations between front and back, though I think not so well as I used to be able to do. Experiments of this kind are easily tried on a lawn in the open, the observer closing his eyes and ears, with if necessary a movement of the thumbs over the latter to drown residual external sounds. At the moment of observation the ears are of course opened. In observing sounds from sources not conveniently moveable, such as the ticking of a clock, the rotating stool is useful.

As had been expected, Mr. Enock's judgement was liable to be upset by the operation of little reflecting flaps situated just outside the ears. The arrangement was that of Prof. Thompson's "*pseudophone*"*, whereby the reflectors, whose planes were at an angle of 45° with the line of the ears, could be rotated in a manner unknown to the observer about that line as axis. In my use of it the two reflectors were always adjusted symmetrically. Thus, if the reflectors were so turned as to send into the ears sounds from the front, no mistakes were made, as if the action were co-operative with the natural action of the external ears. On the other hand, if the collars carrying the reflecting flaps were turned through 180° so as to reflect into the ears sounds from *behind*, frequent mistakes ensued. I hope before long to be able to confirm and extend these observations.

In conclusion, I will remark that the facts now established have a possible practical application. In observing fog-signals at sea it is of course of great importance to be able to estimate the bearing. If a sound is of sufficiently long duration (5 or 6 seconds), it is best by turning the body or head to bring it apparently to the right and to the left, and to settle down into the position facing it, where no lateral effect remains. If, as for most fog-signals, the duration be

* Phil. Mag. November 1879.

decidedly less than this, it may be preferable to keep still; but we are then liable to serious errors, should the signal happen to come from nearly in front or nearly behind. A judgement that the signal is to the right or left may usually be trusted, but a judgement that it comes from in front or behind is emphatically to be distrusted. If, for example, the sound seems to come from a position 45° in front of full right, we must be prepared for the possibility that it is really situated 45° behind full right. A combination of 3 or 4 observers facing different ways offers advantages. A comparison of their judgements, attending only to what they think as to right and left and disregarding impressions as to front and back, should lead to a safe and fairly close estimate of direction.

Terling Place, Witham.

XIII. *The Behaviour of Iron under Small Periodic Magnetizing Forces.* By J. M. BALDWIN, M.A., B.Sc.*

[Plate VII.]

1. **T**HE object of the present paper, which was undertaken at the suggestion of Professor Lyle, is to investigate the relations which obtain under actual working conditions between the amplitudes and phases of the different harmonics of the magnetic induction produced in iron, and those of the magnetic intensity producing it, for low values of the induction; and from the harmonic expressions obtained for these quantities to deduce the loss of energy in the iron. The work which has been done previously on this subject has been performed under statical conditions, which are seldom realized in practice.

The induction produced in iron in weak magnetic fields has been measured by Baur† using the ballistic method (statical). For the range $H=0.38$ to $H=0.016$ (with corresponding limits of the induction $B=283$ and $B=3.3$) he found that the permeability μ could be represented closely by an expression of the form

$$\mu = a + bH,$$

where a and b are constants for the specimen. By extrapolation, μ would tend to a definite limit in exceedingly weak fields. This limit for the specimen examined was 185.

Later, Lord Rayleigh‡, working with the compensated magnetometer method (statical), showed that, for unannealed

* Communicated by the Physical Society: read October 26, 1906.

† C. Baur, Wied. Ann. xi. p. 394 (1880).

‡ Lord Rayleigh, Phil. Mag. [5] vol. xxiii. p. 225 (1887).

iron, the permeability over the range $H=0.04$ to $H=0.00004$ was appreciably constant, while for values of H greater than 0.04 the expression

$$\mu = a + bH$$

agreed well with experiment up to $H=1.2$. The value of a for hard iron was 90 or 100. Similar results held for steel; but for soft iron a complication arose due to the time-lag of the magnetization. The effect of this time-lag in soft iron has been further investigated by Ewing*, who found that up to $H=0.1$ any change of H instantaneously produced an induction proportional to the change in H , and that then the induction gradually changed (in the same direction as the instantaneous effect) for some seconds afterwards. In the experiments, a description of which is given later, it will be found that when a bundle of wires (in which the eddy-currents are small) is subjected to cyclic variations of magnetic intensity, the hysteresis loop is practically a straight line at the lowest inductions over the whole range of frequencies used (50 to 8.5). This would show that the amount of creeping of the induction that had time to occur was negligible for these frequencies. The iron, however, was not very soft.

2. The relation between the amplitude and phase of the harmonics of the intensity of an alternating magnetic field and of the corresponding magnetic induction produced in iron have been investigated for different frequencies by Lyle†; the amplitude of the first harmonic of the intensity varying from 0.4 to 5.67 for approximately sinusoidal waves, and to 14 for waves not sinusoidal. The total amount of energy lost in the iron per c.c. per cycle was also calculated from the Fourier expressions for the allied current and flux waves, and was found to be given with considerable accuracy within the limits $B=1000$ and $B=12,000$ (when the magnetizing current wave was approximately sinusoidal) by the formula

$$I = (a + bn)\mathfrak{B}^x,$$

where I is the total iron loss,

a, b, x are constants for any particular specimen,

n the frequency,

and $\mathfrak{B}$ the "effective induction," which is $\frac{\sqrt{2}}{2\pi n}$ times the square root of mean square of $\frac{dB}{dt}$.

* J. A. Ewing, Proc. Roy. Soc. vol. xlv. p. 269 (1889); Magnetic Induction in Iron and other Metals, 3rd edition, p. 127.

† T. R. Lyle, Phil. Mag. [6] vol. ix. p. 104 (1905).

From the total iron loss I , the sum of the static hysteresis and the calculated eddy-current loss was subtracted, and thus the kinetic hysteresis was obtained. This kinetic hysteresis increased with the induction and with the frequency.

3. In the present series of experiments, in which the magnetic intensity ranged from $H=1.5$ to $H=0.01$ and B from 600 to 2.5, the hysteresis areas could have been determined only for the higher inductions by the statical methods, but the wave-tracer of Professor Lyle* affords a convenient method of obtaining the total iron loss, when the magnetic field is produced by an alternating current of any desired frequency. The portion of this loss of energy due to eddy-currents set up in the iron can be approximately calculated, and subtracting this from the total iron loss, the loss due to hysteresis is obtained.

Two specimens were experimented upon, in one of which (a bundle of iron wires) the eddy-current loss was comparatively small, while in the other (a rod of Lowmoor iron 0.3 sq. cm. in cross section) the eddy-current loss was considerable. For the former specimen it was found that for very low inductions the lag of phase of the induction behind the magnetizing force was very small, and also that the permeability was practically uninfluenced by the frequency of the alternations. In the latter, where the eddy-currents are considerable, the resultant intensity inside the rod will no longer be in phase with the current in the solenoid, and so the resultant induction will always lag behind the magnetizing current.

Assuming that the induction at any point is in phase with the magnetic intensity at that point, as the experiments with the wire bundle show, and also that the permeability has a constant value for low inductions, equations can be obtained connecting the apparent permeability and the lag in phase of the induction behind the magnetizing current with the frequency.

Let OF (see figure) represent the resultant flux across the central section of the rod. The variation of the flux will produce an E.M.F. in a circuit round the rod in quadrature with OF and behind it in phase. This E.M.F. produces a current, which in turn produces a flux FM or IF (where $IF=FM$) along the rod. This flux will also, on the assumptions which have been made, be at right angles to OF . Then OI is the flux which would have passed along the rod if there had been no eddy-currents.

* T. R. Lyle, *Phil. Mag.* [6] vol. vi. p. 549 (1903).

If $OI = F$, $OF = F_r$, $FI = F_e$ it follows that
 $F = \mu H \alpha$, $F_r = \mu_r H \alpha$, $F_e = k F_r \omega$,

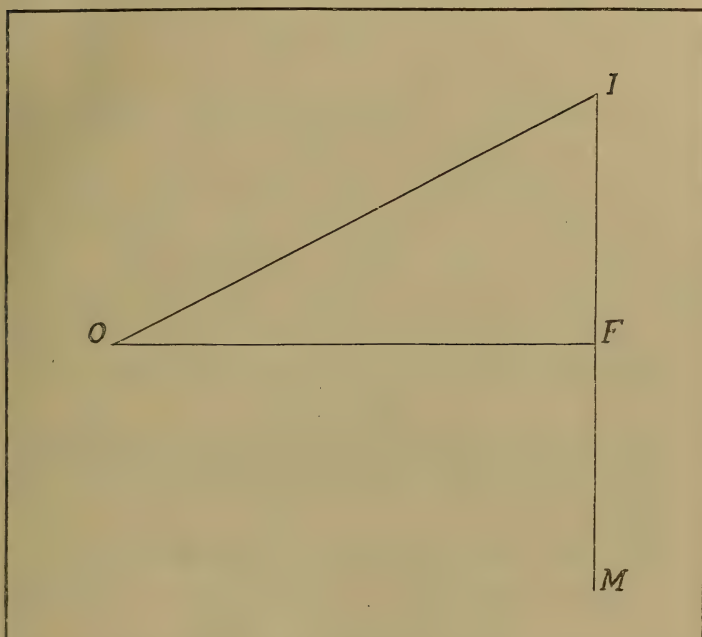
where μ is what the permeability would be if there were no eddy currents,

μ_r is the observed apparent value of the permeability,

α is the area of cross section of the rod,

ω is the frequency multiplied by 2π ,

and k is a constant.



Then writing θ for the angle FOI , the lag in phase of the induction behind the magnetizing current,

$$\cos \theta = \frac{F_r}{F} = \frac{\mu_r H \alpha}{\mu H \alpha} = \frac{\mu_r}{\mu},$$

and

$$\tan \theta = \frac{F_e}{F_r} = \frac{k F_r \omega}{F_r} = k \omega.$$

In these equations ω , μ_r , θ are observed variable quantities, while μ and k are constants for the specimen on the assumptions given above, and these assumptions can be tested by noting the agreement between the values of the quantities found from the equations.

4. The arrangement of apparatus was practically identical with that already described by Lyle *, the iron ring and the coils on it being replaced by a long solenoid, through which the specimen (a long straight rod or bundle of wires) passed. Round the centre of the specimen a small secondary coil was wound, by means of which the flux was determined. The speed was regulated by altering the number of cells in the armature circuit of the rotary converter and by altering slightly the resistance in the field-magnet circuit. The reducing factor of the galvanometer was determined by connecting a Clark cell to the galvanometer, with a megohm in series, the current balance being used only to adjust the current to the required value.

The procedure in any one experiment has also been described. In every case a full wave was taken, the ordinates being 12° in phase apart. In the reductions, no notice was taken of the seventh harmonic, which was always small and could not have affected the results, its influence on the amplitude being quite negligible, and its influence on the phase being practically eliminated by taking the phase of the first harmonic from four ordinates on each side of the zero ordinate of the curve. In every case the third component † was plotted, and from it the ninth harmonic was taken, thus leaving the third harmonic.

The wave forms having been analysed, the results were reduced to absolute measure by applying to them the proper factors ‡ to reduce them to magnetic intensity and induction respectively in the form

$$H = H_1 [\sin \omega t + h_3 \sin_3(\omega t - \phi_3) + h_5 \sin 5(\omega t - \phi_5)]$$

$$B = B_1 [\sin(\omega t - \theta_1) + b_3 \sin 3(\omega t - \theta_3) + b_5 \sin 5(\omega t - \theta_5)]$$

the harmonics of order higher than the fifth being neglected.

From these the value of the iron loss per c.c. per cycle $(I = \frac{1}{4\pi} \int H dB)$ is

$$I = \frac{H_1 B_1}{4} \{ \sin \theta_1 + 3h_3 b_3 \sin 3(\theta_3 - \phi_3) + 5h_5 b_5 \sin 5(\theta_5 - \phi_5) \}.$$

To obtain the amount of this energy loss which is due to hysteresis, the eddy-current loss has been calculated from

* T. R. Lyle, *Phil. Mag.* [6] vol. ix. p. 104 (1905).

† T. R. Lyle, *Proc. Roy. Soc. Victoria*, vol. xvii. pt. 2, p. 394 (Feb. 1905); *Phil. Mag.* [6] vol. xi. p. 25 (1906).

‡ T. R. Lyle, *Phil. Mag.* [6] vol. vi. p. 549 (1903).

the approximate formula given by Searle and Bedford * on the assumptions that μ is constant and that $\frac{dH}{dt}$ does not vary over the section

$$\frac{dX}{dt} = \frac{A}{8\pi\rho} \cdot \left(\frac{dB}{dt}\right)^2;$$

where X is the eddy-current loss per c.c.,

A the area of section of the rod or wire,

and ρ the specific resistance of the iron ;

$$\therefore E = \frac{A}{8\pi\rho} \int_0^T \left(\frac{dB}{dt}\right)^2 dt,$$

where T is the period of the alternations,

and E is the eddy-current loss per c.c. per cycle.

Substitute for B and perform the integration,

$$\begin{aligned} E &= \frac{A}{8\pi\rho} B_1^2 \omega \cdot 2\pi^2 [1 + 9b_3^2 + 25b_5^2] \\ &= \frac{\pi A}{8\rho T} \mathfrak{B}^2, \end{aligned}$$

where $\mathfrak{B}^2 = B_1^2 [1 + 9b_3^2 + 25b_5^2]$. ($\mathfrak{B}$ is the quantity which has been referred to before as the "effective induction.")

E having been calculated, the hysteresis U is given by

$$U = I - D.$$

5. The first specimen experimented upon was a bundle of iron wires of low permeability, which were taped together to form a long circular cylinder, the details of which are given below. A coil was wound on the centre of the bundle for measuring the induction produced in the iron. Four different frequencies were employed, namely, 50, 28, 15.5, and 8.5.

Details of Iron Wire Bundle.

Number of wires	46.
Mean diameter of wires	0.0789 cm.
Sectional area of iron	0.2250 sq. cm.
Length of wires	307 cm.
Specific resistance at 12°·5 C.	1.366×10^4 .
Turns on central coil	400.

Details of Magnetizing Solenoid.

Length of solenoid	170 cm.
Internal diameter of windings.	1.6 cm. q.p.
Constant of solenoid	$H = 53.37$ C.

* G. F. C. Searle and T. G. Bedford, Phil. Trans. cxviii. p. 33, 1902, App. 1.

The results obtained with this specimen are given in the following tables (I. to IV.). The meanings of the different symbols in the tables are given by

T = period,

$$H = H_1 [\sin \omega t + h_3 \sin 3(\omega t - \phi_3) + h_5 \sin 5(\omega t - \phi_5)],$$

$$B = B_1 [\sin(\omega t - \theta) + b_3 \sin 3(\omega t - \theta - \psi_3) + b_5 \sin 5(\omega t - \theta - \psi_5)],$$

$$\mu_0 = \frac{B_1}{H_1}, \quad \mu = \frac{B_{\max.}}{H_{\max.}},$$

$$\omega \mathfrak{B} = \sqrt{2} \cdot \text{R.M.S.} \left(\frac{dB}{dt} \right);$$

I = Iron loss per c.c. per cycle,

E = Calculated eddy-current loss per c.c. per cycle,

$$I = \sigma \mathfrak{B}^{1.6}.$$

Statical Permeability for different Inductions.

B	11.3	44.8	111.7	171.3	281.8	421.2	603.0
H	.0747	.2452	.5324	.7474	1.0664	1.3903	1.7119
μ	152.0	182.8	209.5	229.2	264.3	303.0	352.2

A number of the results contained in the tables are shown graphically by means of curves. The wave-form of the magnetizing current is in every case approximately sinusoidal, and so the results obtained may be taken as typical of a series of induction-waves produced by magnetizing currents of similar wave-form.

The most important of the characteristics are μ_0 and θ , which connect the first harmonic of the induction with that of the magnetic intensity. If the magnetizing current and the flux-waves are assumed sinusoidal, then they will be given by expressions of the form

$$H = H_0 \sin \omega t,$$

$$B = B_0 \sin(\omega t - \theta),$$

$$B_0 = \mu_0 H_0;$$

and from these, since the total iron loss per c.c. per cycle is

$$I = \frac{1}{4\pi} \int H dB,$$

it follows that

$$I = \frac{H_0 B_0}{4} \sin \theta.$$

Thus when μ_0 and θ are given in terms of B_0 , the behaviour of the iron is determined.

TABLE I.—Wire Bundle. T = .020.

T.	H ₁ .	h ₃ .	φ ₃ .	h ₃ .	φ ₃ .	B ₁ .	θ.	h ₃ .	ψ ₃ .	b ₃ .	ψ ₃ .	μ ₀ .	B _{max.}	H _{max.}	μ.	Σ.	I.	E.	I-E.	σ.
.02017	1.659	.0344	59.63	.0009	61.66	528.3	14.25	.0532	30.20	.0078	40.43	318.5	533.2	1.720	310.0	535.4	53.09	376	49.33	.002286
.02019	1.289	.0384	63.54	.0000		362.9	12.44	.0467	36.24	.0100	41.42	281.5	369.5	1.340	275.8	366.9	24.73	176	22.97	.001950
.02015	.998	.0436	65.01	.0020	59.64	253.6	10.94	.0442	42.80	.0097	43.57	254.1	260.6	1.040	250.6	256.1	11.80	.86	10.94	.001654
.02016	.679	.0489	66.61	.0026	61.17	150.7	8.16	.0437	50.62	.0067	47.95	221.8	156.4	.7060	221.5	152.1	3.571	.303	3.268	.001152
.02016	.4532	.0514	67.73	.0014	70.73	92.02	6.22	.0459	56.26	.0104	51.14	203.0	95.62	.4724	202.4	93.01	1.104	.113	.991	.000782
.02025	.2216	.0547	68.07	.0082	73.38	39.36	3.80	.0487	60.90	.0094	63.78	177.6	41.40	.2314	178.9	39.82	1.410	.0207	.1203	.000388
.02008	.0736	.0523	63.44	.0109	82.32	12.18	1.59	.0482	64.10	.0103	75.90	165.4	12.67	.0764	165.8	12.32	.00590	.00200	.00390	.000106
.02007	.0148	.0576	69.41	.0113	86.85	2.36	1.27	.0481	65.84	.0134	82.77	158.8	2.46	.0155	153.8	2.39	.00018	.00007	.00011	.000045

TABLE II.—Wire Bundle. T = .036.

T.	H ₁ .	h ₃ .	φ ₃ .	h ₃ .	φ ₃ .	B ₁ .	θ.	h ₃ .	ψ ₃ .	b ₃ .	ψ ₃ .	μ ₀ .	B _{max.}	H _{max.}	μ.	Σ.	I.	E.	I-E.	σ.
.03575	1.666	.0352	61.37	.0026	48.28	558.9	14.97	.0616	28.51	.0137	39.07	335.5	563.2	1.724	326.7	569.7	58.94	2.45	56.49	.002298
.03595	1.287	.0373	60.62	.0048	63.39	374.8	12.79	.0477	33.42	.0124	40.59	291.2	377.5	1.347	280.2	379.4	26.22	1.08	25.14	.001959
.03614	.9943	.0369	62.17	.0044	62.01	257.2	10.67	.0426	37.50	.0127	44.29	258.6	261.6	1.041	251.3	259.8	11.62	.50	11.12	.001591
.03592	.6847	.0386	61.83	.0029	61.16	158.4	8.35	.0438	41.46	.0093	48.63	231.3	163.0	.7130	228.6	159.9	3.856	.192	3.664	.001148
.03576	.4745	.0389	60.81	.0050	57.30	97.23	6.29	.0412	47.45	.0054	51.01	204.9	101.4	.4975	203.8	98.01	1.242	.073	1.169	.000809
.03567	.2550	.0385	61.38	.0019	54.72	47.10	3.99	.0389	51.70	.0061	50.31	184.7	48.98	.2647	185.0	47.45	.2048	.0170	.1878	.000426
.03558	.0858	.0379	61.94	.0030	22.01	14.31	2.60	.0362	57.18	.0040	61.99	166.7	14.93	.0893	167.2	14.40	.01379	.00157	.01222	.000193
.03559	.0179	.0395	60.54	.0038	65.02	2.81	0.35	.0296	57.37	.0043	60.38	157.0	2.92	.0184	158.8	2.82	.00007	.00006	.00001	.000013

TABLE III.—Wire Bundle. T = 0.65.

T.	H ₁ .	h ₃ .	φ ₃ .	h ₅ .	φ ₅ .	B ₁ .	θ.	b ₃ .	ψ ₃ .	h ₅ .	ψ ₅ .	μ ₀ .	B _{max} .	H _{max} .	μ.	β.	L.	E.	I-E.	σ.
·0656	1.681	·0327	⁰ 58.97	·0001	⁰ 55.30	575.2	14.43	·0619	⁰ 27.18	·0101	⁰ 38.47	342.2	575.2	1.733	331.9	585.8	59.10	1.38	57.72	·002304
·0656	1.290	·0375	59.71	·0026	67.92	375.2	12.36	·0531	34.35	·0096	43.02	290.9	382.6	1.337	286.2	380.4	25.45	·58	24.87	·001894
·0655	·9804	·0373	62.03	·0009	64.15	254.4	10.42	·0476	38.47	·0091	45.70	259.5	260.8	1.019	255.9	257.2	11.06	·27	10.79	·001539
·0657	·7056	·0419	61.80	·0008	22.11	164.6	8.40	·0454	42.89	·0041	43.35	233.2	170.4	·7380	231.5	166.1	4.152	·110	4.042	·001163
·0656	·4597	·0431	62.18	·0027	9.71	94.20	6.09	·0410	47.65	·0035	38.26	204.9	98.17	·4823	203.6	94.93	1.123	·036	1.087	·000770
·0656	·2608	·0399	62.63	·0071	67.04	48.54	4.07	·0392	56.68	·0065	58.37	186.1	50.24	·2742	183.2	48.90	·2225	·0096	·2129	·000441
·0651	·0962	·0364	60.44	·0107	68.30	16.12	1.94	·0336	57.39	·0097	65.38	167.6	16.82	·1006	167.2	16.22	·01299	·00106	·01193	·000151
·0648	·0229	·0380	60.66	·0122	68.83	3.64	1.09	·0295	58.18	·0125	68.67	158.8	3.78	·0241	157.2	3.66	·00039	·00005	·00034	·000049

TABLE IV.—Wire Bundle. T = 1.17.

T.	H ₁ .	h ₃ .	φ ₃ .	h ₅ .	φ ₅ .	B ₁ .	θ.	b ₃ .	ψ ₃ .	b ₅ .	ψ ₅ .	μ ₀ .	B _{max} .	H _{max} .	μ.	β.	L.	E.	I-E.	σ.
·1150	1.985	·0416	⁰ 61.75	·0027	⁰ 68.80	852.8	18.20	·0734	⁰ 26.45	·0175	⁰ 33.60	429.7	843.1	2.072	406.9	876.5	129.1	1.8	127.3	·002527
·1146	1.005	·0371	60.73	·0034	63.48	261.4	10.06	·0421	37.73	·0073	40.60	260.1	265.8	1.044	254.6	263.6	11.28	·16	11.12	·001509
·1163	·7130	·0417	61.95	·0020	58.83	163.7	8.07	·0409	43.32	·0073	62.76	229.6	168.3	·7418	226.9	165.0	4.009	·062	3.947	·001133
·1173	·4656	·0438	61.06	·0033	68.74	95.40	5.92	·0418	48.57	·0086	51.37	204.9	99.72	·4871	204.7	96.24	1.124	·021	1.103	·000754
·1174	·2576	·0448	61.40	·0067	70.00	47.44	3.94	·0390	52.91	·0081	66.61	184.2	49.65	·2718	182.7	47.81	·2059	·0051	·2028	·000423
·1196	·1014	·0359	61.46	·0090	70.60	16.88	1.56	·0392	56.82	·0023	66.93	166.5	17.50	·1063	164.6	17.00	·01138	·00063	·01075	·000122
·1206	·0216	·0368	63.36	·0074	67.78	3.47	0.62	·0309	55.28	·0064	45.33	160.5	3.61	·0226	159.7	3.49	·00023	·00003	—	·00026

The relation between μ_0 and B_1 for the different series is shown in Pl. VII. fig. 1 (upper curve). It is seen that the value of μ_0 is very little affected by the frequency, the four sets of points lying on practically the same curve for values of B_1 up to 300 or so ; but beyond this the curves seem to separate, the value of μ_0 becoming smaller as the frequency is increased. On the same diagram are plotted the points for the statical permeability, and these also lie on the same curve.

μ_0 decreases as the induction is lowered, this decrease becoming more rapid as the value of B_1 becomes smaller, and the value of μ_0 appears to tend to a definite limit of about 157 for very low values of B_1 . This limit is better shown by plotting μ_0 against H_1 . If this be done, it appears that up to about $H=1.0$ the relation between μ_0 and H_1 can be well represented by a straight line, as Baur and others have already shown for the statical permeability.

The values of θ for the four series are plotted against B_1 in fig. 1 (lower curve), and here, although there is not so close an agreement between the four series as there was for the μ_0-B_1 curves, yet the points all lie near a mean curve. The value of θ is seen to decrease rapidly as B_1 diminishes, and the rate of increase becomes more and more rapid as the induction becomes lower, the value of θ being about 1° when B_1 is as low as 2, and there seems no reasonable doubt that in the absence of eddy-currents θ would become exceedingly small for very low values of B_1 .

The curves for b_3 (the ratio of the amplitude of the third harmonic of the flux-wave to that of the first) seem to show that this quantity decreases slowly as B_1 decreases, and finally takes up a value of about $\cdot 030$ for the weakest fields used. The behaviour of b_3 for the period $T=\cdot 020$ is somewhat different, for there, in the weaker fields, b_3 increases. It will be seen from Table I. that for this series h_3 also increases in the weaker fields, and has a considerably higher value than in the other series, though whether this is the cause or an effect of the increase of b_3 cannot be said. ψ_3 , the phase-lag of the third harmonic of the induction-wave behind the first, shows the same general behaviour for all four series ; gradually increasing as the induction decreases, it appears to tend to a value of about 60° for exceedingly low values of B_1 .

Of b_5 , all that can be said is that it has a value somewhere below $\cdot 01$, and of ψ_5 that it increases as the induction diminishes ; but little dependence can be placed on the

values of b_5 and ψ_5 for the lower inductions. This arises from the small value of b_5 . The amplitude of the galvanometer deflexion was seldom more than 20 mm., so that if b_5 were .01, the amplitude of the fifth harmonic would be only .2 mm., and a slight unsteadiness of the readings from any cause might greatly affect the values of both b_5 and ψ_5 .

In fig. 2 (lower curve) the hysteresis loss is plotted against the maximum value of B for the different experiments of the four series. To avoid confusion, several points at the lower inductions have been omitted, but in every case all points that have been left out lie on the curve. It is seen that here again the four series of points lie on practically the same curve; that is, the hysteresis loss for the specimen under examination is practically independent of the frequency, provided the maximum induction does not exceed 600. This being the case, it would be impossible with the present apparatus to detect with certainty a loss due to kinetic hysteresis at these low inductions.

On the same diagram (fig. 2, upper curve) is shown the variation of Steinmetz's coefficient σ^* ($I = \sigma B^{1.6}$), with the maximum value of the induction. The curves show that σ decreases rapidly as the induction diminishes, tending towards the value zero at very low inductions. Further, σ increases with frequency, the increase being due to the increased eddy-current loss, for if the eddy-current loss is deducted, the points all lie on one curve.

6. A further series of experiments was performed on a long rod of Lowmoor iron 0.3 sq. cm. in cross section, in which the eddy-current effect would be considerable. As for the former specimen, four different frequencies were employed, namely, 55, 28, 15, and 8.5.

Details of Lowmoor Iron Rod.

A straight cylindrical rod of circular section.

Area of section	.2999 sq. cm.
Length	360 cm.
Specific resistance at 12°·5 C...	1.119×10^4 .
Number of turns on central coil.	400.
Magnetizing solenoid as before.	

The results obtained with this specimen are given in Tables V. to VIII.

* In calculating σ , the effective induction $\mathfrak{B}$ has been used, not the maximum induction $B_{\max}$.

TABLE V.—Lowmoor Iron Rod. T = .018.

T.	H ₁ .	h ₃ .	φ ₃ .	h ₅ .	φ ₅ .	B ₁ .	θ.	b ₃ .	ψ ₃ .	b ₅ .	ψ ₅ .	μ ₀ .	B _{max} .	H _{max} .	μ.	g.	I.	E.	σ.
·01822	1·687	·0261	76°21	·0029	13°32	312·0	47·97	·0108	18°13	·0033	29°46	184·9	308·5	1·724	178·9	312·2	97·70	106·8	·009972
·01791	·9890	·0346	81·12	·0066	18·75	160·4	45·81	·0146	38·46	·0030	50·61	162·2	161·5	1·007	160·4	160·6	28·45	28·73	·008412
·01804	·4604	·0332	79·41	·0078	78·72	67·12	40·65	·0157	48·06	·0029	54·66	145·8	67·63	·4717	143·5	67·20	5·039	4·998	·006006
·01802	·0763	·0327	80·76	·0128	24·99	9·51	33·92	·0182	61·51	·0032	63·07	124·7	9·68	·0784	123·4	9·53	·1014	·1006	·002751

TABLE VI.—Lowmoor Iron Rod. T = .036.

T.	H ₁ .	h ₃ .	φ ₃ .	h ₅ .	φ ₅ .	B ₁ .	θ.	b ₃ .	ψ ₃ .	b ₅ .	ψ ₅ .	μ ₀ .	B _{max} .	H _{max} .	μ.	g.	I.	E.	σ.
·03598	1·666	·0442	77°69	·0043	10°37	436·5	43°62	·0230	41°30	·0039	40°41	262·0	440·5	1·730	254·6	473·6	125·85	106·22	·007484
·03607	1·295	·0442	79·23	·0051	10·82	315·3	41·06	·0209	49·94	·0024	48·98	243·5	318·4	1·329	239·6	316·0	67·23	55·23	·006731
·03609	·9935	·0499	79·54	·0071	18·99	229·2	39·16	·0235	52·87	·0039	43·56	230·7	232·6	1·026	226·7	229·8	36·07	29·21	·006012
·03616	·6930	·0541	79·84	·0097	18·76	148·0	35·47	·0262	54·12	·0030	57·18	213·6	151·7	·7204	210·6	148·5	14·93	12·17	·005004
·03625	·4780	·0551	79·35	·0108	19·60	94·72	33·00	·0284	59·89	·0043	60·19	198·2	97·50	·4968	196·3	95·08	6·200	4·978	·004241
·03632	·2412	·0577	80·15	·0121	21·47	42·89	27·79	·0340	63·71	·0072	68·03	177·8	44·33	·2483	178·5	43·15	1·214	1·023	·002939
·03653	·0803	·0569	80·69	·0129	18·95	12·40	21·86	·0353	71·04	·0067	72·52	154·5	12·88	·0825	156·0	12·48	·0835	·0850	·001648
·03642	·0237	·0602	82·20	·0134	18·29	3·47	20·19	·0385	75·37	·0086	82·49	146·6	3·63	·0245	147·8	3·49	·00718	·00669	·000972

TABLE VII.—Lowmoor Iron Rod. T = .065.

T.	H ₁ .	h ₃ .	φ ₃ .	h ₅ .	φ ₅ .	B ₁ .	θ.	b ₃ .	ψ ₃ .	b ₅ .	ψ ₅ .	μ ₀ .	B _{max} .	H _{max} .	μ.	B.	I.	E.	σ.
.0640	1.682	.0295	61.73	.0022	44.77	580.5	38.34	.0318	27.72	.0033	20.97	345.1	576.6	1.731	333.1	583.3	151.55	105.05	.005690
.0654	.9839	.0398	62.98	.0026	59.75	279.9	32.04	.0348	36.10	.0030	47.18	284.5	282.2	1.018	277.2	281.4	36.60	23.96	.004411
.0662	.4548	.0410	61.49	.0012	44.57	107.0	24.50	.0386	42.48	.0017	60.15	235.3	110.0	.4721	233.0	107.7	5.061	3.467	.002886
.0652	.0799	.0396	62.36	.0040	62.87	13.58	14.16	.0377	54.66	.0020	19.63	170.0	14.02	.0833	168.4	13.67	.0668	.0567	.001017

TABLE VIII.—Lowmoor Iron Rod. T = .118.

T.	H ₁ .	h ₃ .	φ ₃ .	h ₅ .	φ ₅ .	B ₁ .	θ.	b ₃ .	ψ ₃ .	b ₅ .	ψ ₅ .	μ ₀ .	B _{max} .	H _{max} .	μ.	B.	I.	E.	σ.
.1166	1.601	.0428	60.70	.0030	49.79	660.8	32.66	.0463	32.72	.0056	31.16	412.7	668.9	1.677	398.9	667.3	143.2	75.88	.004335
.1178	1.218	.0408	60.83	.0041	47.26	444.5	28.46	.0455	36.30	.0033	37.51	365.0	453.0	1.265	358.1	448.8	64.68	33.97	.003694
.1176	.9902	.0419	61.21	.0026	49.64	332.4	25.47	.0417	36.29	.0057	37.59	335.7	339.3	1.029	329.7	335.1	35.41	18.97	.003228
.1183	.6943	.0459	61.25	.0044	43.22	202.9	22.11	.0459	40.80	.0058	48.47	292.2	208.4	.7244	287.7	204.9	13.22	7.05	.002647
.1182	.4995	.0464	60.00	.0037	25.20	131.4	19.42	.0459	41.19	.0036	57.07	263.1	134.7	.5207	258.7	132.6	5.457	2.956	.002192
.1180	.2292	.0480	58.32	.0058	42.66	48.79	13.65	.0464	49.76	.0012	12.41	212.9	50.48	.2406	209.8	49.27	.6645	.4085	.001301
.1178	.0755	.0501	57.98	.0048	56.13	13.18	7.23	.0450	53.03	.0023	45.51	174.6	13.78	.0797	173.0	13.30	.0315	.0298	.000502
.1182	.0149	.0474	57.84	.0067	64.44	2.37	5.64	.0525	62.37	.0074	13.64	158.8	2.47	.0157	157.5	2.40	.00090	.00097	.000222

Statcal Permeability for different Inductions.

B	12.9	52.1	144.2	228.0	389.3	608.5	920.0
H	.0747	.2450	.5334	.7467	1.0671	1.3903	1.7097
μ	172.6	212.5	270.3	305.4	364.8	437.7	538.1

From the tables it appears that the wave form of the magnetizing current is nearly sinusoidal, and so the induction may be considered as that produced by magnetizing currents of similar wave-form.

The relation between μ_0 and B_1 is exhibited graphically in fig. 3 (upper curves). These curves show that for any one frequency the value of μ_0 diminishes as the induction is decreased, the decrease becoming more and more rapid as the induction gets slower. If μ_0 is plotted against H_1 , however, it is found that, as for the bundle of wires, the portions of the curves for low values of H_1 are approximately straight lines, but the range over which this is true is much more limited for the rod than for the wire bundle, extending in the former only to about $H = 0.3$. The curves appear to approach a definite limit for each particular frequency when the value of the induction becomes very low. It is seen also that the frequency has a marked effect on the value of μ_0 , an increase in frequency being accompanied by a decrease in μ_0 , the decrease being more marked for the higher frequencies.

On the same diagram is shown the relation between the statcal permeability μ and the induction.

θ also decreases as the induction is decreased, the rate of decrease becoming more and more rapid as the induction becomes lower (fig. 4). For each frequency, θ seems to approach a definite value when the induction is very low. Also an increase in the frequency is accompanied by an increase in θ .

The limiting values of θ and μ_0 obtained from the curves satisfy fairly well the equations obtained in § 3. The values used have been

θ	$\overset{\circ}{5}$	$\overset{\circ}{11.8}$	$\overset{\circ}{19.6}$	$\overset{\circ}{32}$
μ_r	155	150	142	119
ω	53.2	96.4	172.6	349.1

Substituting these values in the equations

$$\mu = \mu_r \sec \theta, \quad k = \frac{1}{\omega} \tan \theta,$$

the values obtained for μ are

	155.3	153.2	150.7	140.3,
and for k	.00165	.00217	.00206	.00179,

which must be considered a fairly satisfactory agreement.

It appears then that the experiments agree fairly well with the assumptions that at very low inductions the permeability tends to a definite limiting value, and that the lag in the phase of the induction at any point behind that of the magnetic intensity at that point is zero.

As regards the third harmonics, b_3 appears to have an approximately constant value for each series, its value decreasing as the frequency is increased, while ψ_3 increases as the induction decreases, tending to a value of about 60° in all cases for very low inductions.

In fig. 3 (lower curves) is shown the relation between the total iron loss and the induction, and the curves here show the marked effect of the eddy currents in increasing the loss of energy in such a rod as that at present being discussed. It will be noticed in Table V. that the calculated eddy-current loss is greater than the total iron loss. This eddy-current loss was calculated from a formula (§ 4) obtained on the assumptions that μ was constant and that $\frac{dH}{dt}$ was constant over the section. It appears then that for a thick rod such as the present one these assumptions depart considerably from the truth, and the question arises how far the results calculated on these assumptions for any particular specimen are in error. Owing to the uncertainty of the eddy-current loss, no attempt has been made to recognize kinetic hysteresis in the rod.

In fig. 5, Steinmetz's coefficient σ is plotted against $B_{\max}$. σ is seen to decrease rapidly as the induction is diminished, and for very low inductions takes a very low value not very different from zero.

In fig. 6 the hysteresis loops are shown for four of the experiments, the upper ones being the 1st and 7th of Table II. and the lower ones the 1st and 7th of Table VI., the frequency for all four being very nearly 28. Each pair

shows strikingly the effect of the value of θ on the iron loss. In the upper figure the one curve is for the highest induction ($B_{\max} = 563$), where the value of θ was $14^{\circ}97$; the other (for which each of the scales is 20 times that of the first) is for $B_{\max} = 14.9$, the corresponding value of θ being 2.60 . In the lower figure the values are $B_{\max} = 440$, $\theta = 43^{\circ}62$, and $B_{\max} = 12.9$, $\theta = 12^{\circ}40$. The marked effect of the eddy currents is seen by comparing the two curves A (for which the value of $H_{\max}$ is the same), and also the two curves B, with one another.

7. The main results obtained in this paper are as follows, the value of the induction being below 600.

In a specimen in which eddy-currents were very small :—

(1) μ_0 diminishes with B_1 , and tends to a finite value for very small values of B_1 , which for the specimen was about 157. For a considerable range at the lower inductions μ_0 can be represented closely as a linear function of H_1 .

(2) θ diminishes rapidly as B_1 diminishes, and tends to a very small limiting value for very low values of the induction.

(3) b_3 decreases slowly as B_1 diminishes, and approaches a value about .030 for the weakest fields used.

(4) ψ_3 gradually increases as the induction decreases and tends to a value of about 60° for exceeding low values of B_1 .

(5) Change of frequency has practically no influence on μ_0 and θ .

(6) The hysteresis loss was practically independent of the frequency, and so the loss due to kinetic hysteresis was too small to detect with certainty.

(7) Steinmetz's coefficient σ decreases rapidly as the induction is decreased, and has a value not very different from zero for very low values of B_1 .

In a specimen in which eddy currents were considerable,

(1) μ_0 diminishes with B_1 and tends to a finite value for very small values of B_1 for any particular frequency. The value for slow speeds was about 156.

(2) For a limited range at low inductions, μ_0 can be represented closely as a linear function of H_1 .

(3) θ diminishes as B_1 diminishes, and tends to a definite limiting value at very low values of the induction for each particular frequency.

(4) The limiting values of μ_0 and θ are consistent with the assumption that the induction at any point is in phase with the magnetic intensity at that point.

(5) b_3 is fairly constant for any particular frequency, its value diminishing as the frequency is increased.

(6) ψ_3 increases as the induction decreases, and tends to a limiting value of about 60° when B_1 is zero.

(7) Increase of frequency largely increases the total iron loss, but, owing to the uncertainty as to the eddy-current loss, it could not be said what part of this increased iron loss was due to an increase of hysteresis.

(8) For such a rod, the formula for the eddy-current loss

$$E = \frac{\pi A}{4\rho} \frac{\mathfrak{B}^2}{T}$$

calculated on the assumptions that μ is constant and $\frac{dH}{dt}$ constant over the section gives results which in some cases must be too large.

The work described in this paper has been performed in the Physical Laboratory of the Melbourne University, and I have to record my thanks to Professor Lyle for his kindness in placing the resources of his laboratory, and in particular his wave-tracer, at my disposal, and for his help in preparing this paper for publication.

XIV. *The Ionization of the Atmosphere over the Ocean.*

By A. S. EVE, M.A., McGill University, Montreal*.

IT is important to ascertain if the ionization of the atmosphere over the ocean is equal to the ionization of the atmosphere over the land. If the radium emanation in the atmosphere, and the penetrating radiation due to radium in the earth, are together the main causes of the ionization of the atmosphere near the surface of the earth, then it might be expected that the ionization over the land would be greater than that over the sea. For experimental evidence at present obtainable indicates that radium is present in sea-water to a markedly less degree than in the sedimentary rocks on land. And, since radium emanation decays to half value in four days, the wind is unable to transport the emanation from land to places in mid-ocean before the activity is materially decreased.

Unfortunately, but few observations of the ionization of the atmosphere have been made in mid-ocean. However, A. Boltzmann† made daily observations with an Ebert's instrument during a voyage from Dover to New York, Aug. 21-31, 1904. He found the number of ions present

* Communicated by Professor E. Rutherford.

† *Physikalische Zeitschrift*, 6 Jahrgang, No. 5.

per c.c. to be $n_+ = 1150$, $n_- = 800$, ratio 1.5. These values do not differ materially from those which obtain in Germany, or Lapland *, or Canada, and they indicate that the ionizations at sea and on land are approximately the same in magnitude.

The present paper contains an account of observations of the ionization of the atmosphere made from June 21 to 30, 1906, on board the s.s. "Athenia," Donaldson Line, sailing from Montreal to Liverpool. On this voyage the weather was bright and very clear, and there was neither fog nor rain. The barometer was fairly steady, and anticyclonic conditions prevailed with northerly winds. The sea was moderate to smooth, the motion of the ship was not great, and at no time was there much spray on deck or in the air. There were few passengers and there was much deck room. The captain and officers were kind and obliging. It will be seen then that the whole conditions were exceptionally favourable. It was possible to select positions for observations, screened from the sun, so that the wind blew fresh from the sea, unmixed with any air from the ventilators of the steamer.

The instrument used was the later pattern designed by Ebert, made by Günther and Tegetmeyer, of Brunswick. A small turbine driven by clockwork draws air through a vertical cylinder in which is a charged rod connected to an electroscope. The amount of electricity per cubic metre of air is determined from the fall of potential indicated by the electroscope. Assuming that the charge of an ion is 3.4×10^{-10} , it is easy to deduce the number of ions present per c.c. The instrument gave complete satisfaction, and there was no trouble about insulation. It may be noted that the key for winding the clockwork wore out on the voyage, but the chief engineer readily found a substitute for it. With care it was possible to obtain accurate readings even in a moderate sea, in spite of the motion of the ship. The natural leak of the electroscope has remained small throughout. The only difficulty is that the aluminium leaves are displaced by convection currents when there is a sudden change of temperature. For example, on altering the temperature of a room in Montreal from 61° F. to 20° F. the leaves have opened from 19.7 to 25.8 divisions, indicating an apparent change of voltage from 157 to 178. After a few minutes the leaves returned to their original position, showing that the movement was due to convection currents. It is, therefore, desirable to place the instrument for half an hour or so in the position of observation before beginning to take readings.

* G. C. Simpson found in Lapland, during 1903-1904, for the average values, $n_+ = 1120$, $n_- = 972$, ratio 1.2.

Before the voyage on the "Athenia" commenced, readings were taken on three successive days at Montreal, on the roof of my house, on the College Campus, and on the top of Mount Royal. The values obtained averaged $n_+ = 370$, $n_- = 367$. The results are extraordinarily small, being about one third or one quarter of those usually obtained at Montreal. The temperature was about 70° F. and there was a conspicuous heat-haze over the country, and the low readings were probably due to this haze. On the first day of the voyage similar low values were found on the River St. Lawrence, 25 miles above Quebec, about 100 miles from Montreal. It may be stated that the mean of several readings taken at Montreal during the winter and spring of 1905-6 are $n_+ = 1400$, $n_- = 1234$. But on Sept. 15, 1906, when all the weather conditions appeared similar to those on the sea voyage, I found at Outremont Golf Links, away from the smoke of Montreal, $n_+ = 768$, $n_- = 717$.

On the voyage in the Gulf of St. Lawrence the means of the readings were $n_+ = 761$, $n_- = 743$. But on the open sea the average values were $n_+ = 975$, $n_- = 783$, ratio 1.24. These figures are in fairly good agreement with those found by A. Boltzmann, namely, $n_+ = 1150$, $n_- = 800$, ratio 1.4. As the vessel approached the British Isles larger results were obtained, $n_+ = 1210$, $n_- = 1045$. And again in the Irish Sea, near the Isle of Man, the observations gave $n_+ = 1273$, $n_- = 872$. Here the atmosphere was extremely clear, so that the tops of the masts of distant vessels could be seen whilst the hulls and lower parts of the masts were hidden by the curvature of the earth. Such clear weather is usually accompanied by high ionization, just as the haze at Montreal probably caused the unusually low readings there. In consequence of these meteorological conditions the number of ions per c.c. near the British Isles was nearly three times as great as in the basin of the river St. Lawrence.

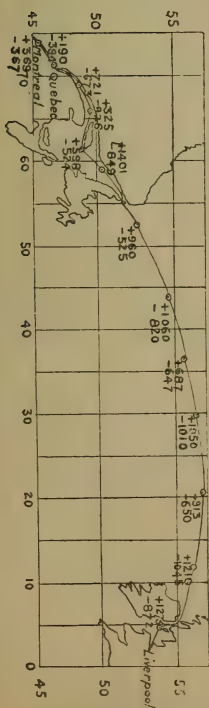
The whole series of observations are given in tabular form, and are also exhibited in a chart. Each observation consisted of two positive and two negative readings, but the means of these are alone recorded. Too much weight must not be attached to any particular reading, because all observers find that the number of ions present per c.c. is often subject to rapid and apparently capricious variations.

The results have been given in ions per c.c., but they can be reduced to electrostatic units per cubic metre by multiplying by 3.4×10^{-4} .

June 1906. S.S. "Athenia." Montreal to Liverpool.

Date.	n_+	n_-	Time.	Place.	Bar.	Therm.	Wind dir.	Wind vel.	Weather	Sun	Sea	Remarks.
18.....	310	271	6 p.m.	Montreal.	...	70°	S.E.	8	hazy.	bright.		Roof of house.
19.....	410	533	11 a.m.	Montreal.	30"	70	S.E.	4	hazy.	bright.		Shade of tree, College Campus.
20.....	336	296	5 p.m.	Montreal.	...	71	S.E.	10	hazy.	bright.		Shade of tree, top of Mount Royal, 800 ft.
21.....	183	388	2 p.m.	River St. Lawrence.	30.1	68	S.	10	hazy.	bright.		25 miles up river from Quebec.
22.....	721	673	8 a.m.	Gulf of St. Lawrence.	30.1	58	N.E.	15	clear.	bright.		Off Metise.
22.....	325	926	3.30 p.m.	Gulf of St. Lawrence.	30.2	50	N.E.	10	clear.	bright.		Some spray in air.
23.....	598	524	8 a.m.	Gulf of St. Lawrence.	30.4	50	...	0	clear.	bright.	smooth.	Off Anticosti Island.
23.....	1401	849	3 p.m.	Gulf of St. Lawrence.	30.4	50	...	0	v. clear.	bright.	smooth.	Water cold, icebergs.
24.....	960	525	2.30 p.m.	Straits of Belle Isle.	30.4	50	...	0	v. clear.	bright.	swell	Water two or three icebergs.
25.....	1060	820	4 p.m.	52° lat.	30.2	50	N.N.W.	15	v. clear.	bright.	moderate.	Water 46°, some spray.
26.....	686	647	2.30 p.m.	55° lat.	30.2	50	N.N.W.	25	clear.	cloudy.	rough.	White horses, spray in air.
27.....	1050	1010	11 a.m.	56° lat.	30.3	50	N.E.	20	clear.	cloudy.	moderate.	Little spray.
28.....	913	650	11 a.m.	56° lat.	30.4	50	N.	10	clear.	cloudy.	calm.	
29.....	1210	1045	11 a.m.	56° lat.	...	50	N.N.E.	20	v. clear.	cloudy.	slight.	
30.....	1273	872	10 a.m.	54° lat.	...	54	N.W.	15	v. clear.	bright.	calm.	Near Isle of Man.

Average at Montreal and Quebec	n_+	n_-	Ratio.
" in Gulf	323	372	.87
" at sea	761	743	1.02
A. Boltzmann, Dover to New York.	975	783	1.24
August 21-31, 1904.	1150	800	1.44



These experiments, and those of A. Boltzmann, indicate that the ionization over the ocean is about the same as that over the land.

The ionization in the air over land is generally attributed to radium emanation and its successive products in the atmosphere, and to the penetrating rays from the active matter in the earth. I am not aware that any other adequate cause has been suggested which will account for the production of positive and negative ions near the earth's surface, but there may be causes with which we are not at present acquainted.

Various experimenters have found that the relationship $q = \alpha n^2$ does not hold good, probably owing to the presence of dust and impurities in the air. Thus Mache and Rimmer* find $\alpha = 4.6 \times 10^{-6}$ for the air in a cave, and $\alpha = 2.1 \times 10^{-6}$ for surface air. Whilst Townsend, McClung, and Langevin find for dust-free air $\alpha = 1.1 \times 10^{-6}$. But supposing that we take any of these values as approximately correct for the lower layers of the atmosphere, then, since n is about 1000, we find $q = 1.1$ to 4.6 . So that if a few ions per c.c. per second are produced by active matter in the earth or air, the ionization of the atmosphere over the land surface can be satisfactorily explained. But Cooke has proved that the penetrating radiation at Montreal produces about 4 ions per c.c. per second in a brass vessel, and I have deduced from this value an estimate of the amount of radium near the earth's surface†, which is in fairly good agreement with the amount determined by Strutt by more direct methods. Again, an attempt to measure the ionization due to the α rays from the active matter in the air leads to an estimate of a few ions per c.c. per second‡. Hence the ionization of air over the land may be satisfactorily explained by the active matter in the earth and atmosphere.

A consideration of the production of ions in air over the ocean presents several difficulties. The following is a list of possible known causes, but there may be others with which we are not at present acquainted:—

- (1) Emanation, and its successive products, rising from radium contained in sea-water.
- (2) Penetrating radiation due to active matter in sea-water.

* *Phys. Zeit.* Sept. 15, 1906.

† *Phil. Mag.* Sept. 1906.

‡ *Phil. Mag.* July 1905.

- (3) Emanation, and its successive products, rising from radium in the land, and carried by wind to mid-ocean.
- (4) Charged particles due to the action of wind on the waves.
- (5) Charged particles due to the change of area occasioned by waves or ripples.

Before considering these it is desirable to obtain an estimate of the amount of radium in sea-water. Now Strutt has found, as an approximate determination, that sea-salt contains 15×10^{-12} grams of radium per gram of sea-salt, whilst his average value for 17 specimens of sedimentary rocks is 2.2×10^{-12} grams of radium per gram of rock. So that such rocks are about 15 times as active as sea-salt. But sea-salt forms about one-thirtieth part of the total constituents of sea-water, and we may therefore deduce that on the average a gram of rock contains about 450 times as much radium as a gram of sea-water.

The writer also has measured the amount of radium both in sea-salt and sea-water. The method employed was similar to that described by Strutt; and the standard used was a solution of radium bromide prepared by Dr. Boltwood, and of such quality that radium bromide gave per gram a heating effect of 110 gram-calories an hour. Four hundred grams of Omaha sea-salt, guaranteed pure by the importers, gave 2.04×10^{-14} grams of radium. This value is one-seventh of that obtained by Strutt. It appears then that 1 gram of sea-water should contain about 6×10^{-13} grams of radium. Again, Mr. C. H. T. Newton of Edinburgh University was good enough to collect some sea-water in mid-Atlantic between Montreal and Glasgow, and to forward it to me sealed in clean glass bottles. This water was boiled down, and in due course the emanation from 2500 c.c. of sea-water was collected and examined in an electroscope. The amount measured was very small and is that due, per gram of sea-water, to about 3×10^{-16} grams of radium. From these two experiments it appears safe to conclude that sea-water does not contain more than 6×10^{-16} grams of radium per gram of water, or only a three-thousandth part of the average determined by Strutt in various sedimentary rocks. It seems clear that the uranium denuded from the land and carried to sea by rivers sinks to the bottom of the ocean and ultimately gives rise to the supply of radium found in sedimentary rocks. And this view is supported by the results which Strutt found

for igneous and sedimentary rocks, because the average amount of radium present is approximately the same in the two cases.

If the amount of radium present in the sea is so small, it is certain that the amount of emanation rising from the ocean must be much less than that from the land.

Similar conclusions may be arrived at with respect to the penetrating radiation due to the active matter in the ocean. The coefficient of absorption of the γ rays of radium by water* equals $\cdot 034$, and since rocks are about 2.7 times as heavy as water, the coefficient of absorption by rock is about $\cdot 092$. In other words, the γ rays are reduced to half value after penetrating 7.5 cms. of earth, or 20.4 cms. of water. But if there is 500 times as much radium in the average rock as in sea-water, we should find the penetrating radiation over the ocean to be $\frac{2.7}{500}$ times that over land. But

Cooke has shown that the penetrating rays at Montreal produce 4 or 5 ions per c.c. per second. Hence at sea the penetrating radiation should be 135 times as small and produce only $\cdot 03$ ions per c.c. per second. It may be noted that a crucial test of the character of the penetrating radiation could be made by proving such radiation entirely absent, or very minute, above the surface of the sea or of a large lake a few metres deep, at a distance of one or two kilometres from land.

Again, the emanation arising from the radium in the land must be carried by wind over the ocean. Now it is known that cyclones sometimes cross the Atlantic in four or five days, but in steady weather, when anticyclonic conditions occur, the transportation of the emanation must proceed more slowly; the radium emanation decays to half value in four days. Hence it would be expected that the ionization over the ocean would be much less than on land. It is true that Greenland is only 300 miles north of the course of the "Athenia," but the results of Boltzmann were obtained at more than a thousand miles from any large area of land.

Mache and Rimmer† found $q=120$ for the air in caves at Vienna; and in free air q has been found to vary from 9 to 38‡. But if α is equal to 1.1×10^{-6} , and $n=1000$, we have seen that a value of $q=1$ is sufficient to account for the ionization observed over the sea. It must be remembered

* McClelland, *Phil. Mag.* July 1904.

† *Phys. Zeit.* 15 Sept. 1906.

‡ Schuster, *Proc. Manchester Phil. Soc.*, Nov. 12, 1904.

too that the air over the sea is free from dust, and that recombination of the ions may, therefore, take place more slowly than on shore. Thus the diminution in the activity of the emanation may be more than compensated for owing to the slower rate of recombination of ions.

Hence the wind-borne emanation rising from the land may be sufficient to account for the ionization observed over the ocean.

There are further possible causes of electrification to be considered. Thus Lenard has shown that drops of water shaken against metal acquire a positive charge, and the surrounding air has a negative charge. The electrification of the atmosphere in the neighbourhood of waterfalls is well known. It is probable that the wind breaking the crest of waves at sea produces a similar result. At the end of this paper (Note III.) I have recorded some effects of fine spray in the neighbourhood of an Ebert's instrument causing a rapid discharge of the electroscope, whether charged positively or negatively. This effect was certainly not due to defective insulation. Holmgren has also proved that when the area of water is rapidly changing, owing to the formation of ripples, the air receives a negative charge. In the present state of our knowledge it is not possible to assign values to these causes. But during the voyage of the "*Athenia*," in calm weather, when spray was absent, the number of ions was approximately 1000 per c.c., and it therefore seems probable that radioactive matter is the main cause of ionization, and that the ions recombine at a slow rate. Furthermore, it is likely that the greater part of the emanation at sea is carried by wind from land, whilst a very small quantity arises from the sea. Nevertheless, it must be admitted that the ionization observed in mid-ocean is in excess of that which would be predicted from theoretical considerations, and the question should receive further experimental tests whenever opportunities occur. Measurements should be made in mid-ocean of the quantity of active deposit obtained on a negatively charged wire. Elster and Geitel found low values for the active deposit at Juist, an island in the North Sea. Simpson obtained similar results in Norway when the wind blew from the land towards Hammerfest. These results bear out the view that emanation does not arise freely from sea-water. It must also be remembered that the Ebert's apparatus has limitations, for Langevin has proved that a large number of ions in the atmosphere move so slowly in an electric field that they escape detection by that means.

One or two notes are added on some results obtained by the Ebert instrument.

I.

*The Ionization of the Air in the Physics Building,
McGill University.*

It may be of interest to note the effect of the presence of radioactive matter and emanation on an Ebert instrument. Observations were made in various rooms in the Physics Building. The supply of radium is kept mainly in the basement. For comparison, readings were taken out of doors, and in the Chemistry Building into which radium has not been introduced.

The effect both of the direct rays and of the emanation of radium in the Physics Building is well marked.

March 17, 1906.

Place.	Time.	n_{+} .	n_{-} .
Chemistry Building	9.30 A.M.	1760	1550
Physics Building, basement	11.00 "	6660	6300
" " top floor	11.20 "	3660	2740
" " room, top floor	11.40 "		2680
" " Lecture Theatre, 2nd floor...	12.15 P.M.		3640
Out of doors, on College Campus	2.30 "	1690	1642
Physics Building, Workshop, ground floor	3.00 "	7160	4900

II.

The Effect of Smoke.

On another occasion a bee-keeper's smoker, consisting of a pair of bellows, with a chamber for burning tobacco, rags, or paper, was used to blow dense clouds of smoke around the Ebert's apparatus.

	n_{+} .	n_{-} .
No smoke	1778	1590
With smoke	1270	1000
No smoke	2060	

In this, as in other experiments, it was clear that the presence of smoke decreased the number of ions detected by the apparatus. A few puffs of tobacco-smoke blown into an electroscope will decrease the natural leak as much as 35 per cent. The ionization current due to .3 mg. of radium bromide placed near an electroscope was decreased from 4.2 to .7 scale-divisions per minute by the introduction of a small quantity of tobacco-smoke. These effects are similar to those first observed with an electrometer by Owens. It is clear that the presence of smoke, haze, or fog may cause a diminution in the number of ions observable by an Ebert apparatus.

III.

On the Effect of Spray.

It was natural to expect that an artificially created mist would also diminish the number of ions detected by the instrument. But under the conditions of the following experiment the reverse was the case. A hand-sprayer was obtained such as is used to sprinkle roses with fluids in order to destroy aphids. A horizontal current of air is driven by a condenser-pump towards the orifice of a small tube leading down to the fluid employed. In this experiment water was used and driven off in an exceedingly fine mist in all directions around, but not towards, the Ebert instrument. The electrification in the air, both positive and negative, was greatly increased by the spray.

March 18, 1906.	n_+	n_-
No spray	1640	1770
With spray	7070	9300

March 20, 1906.	n_+	
No spray	1390	
With spray	5350	

When the hand-sprayer was used without water there was

no increase in the electrification. The increase, when water was present, was probably due to the friction between the air and water. McLennan observed the high negative potential of the spray from Niagara Falls, and found it sufficient to maintain an insulated wire at 7000 volts. When the wind breaks the tops of the waves at sea and forms "white horses," electrification may be caused which will increase the apparent number of ions observed at sea.

Summary.

- I. The ionization of the atmosphere over the North Atlantic Ocean appears to be approximately the same in magnitude as in Europe or North America.
- II. The amount of radium contained in sea-water is very minute, and is $1/500$ to $1/2000$ part of the average amount determined by Strutt in various sedimentary and igneous rocks.
- III. Specimens of sea-water obtained from mid-Atlantic, and a sample of sea-salt, indicate that 1 gram of sea-water contains about 5×10^{-16} grams of radium.
- IV. The emanation from the radium in sea-water, and the penetrating radiation from the active matter contained in it, are insufficient to account for the ionization observed over the ocean.
- V. Emanation arising from radium on land, and carried to sea by the wind, is the only known cause which will account for the ionization effects observed over the ocean. The ionization observed is larger than would be anticipated from such a cause, but it is possible that the rate of recombination of ions over the sea may be less than over the land.

I am grateful to Professor Rutherford for helpful discussion and suggestions. I wish also to thank Mr. C. H. T. Newton, and the captain and officers of s.s. "Athenia" and of s.s. "Hibernian" for their kind assistance.

Montreal, December 1906.

XV. *A General Solution of the Electromagnetic Relations.*

By J. W. NICHOLSON, D.Sc., B.A.; Isaac Newton Student and Scholar of Trinity College, Cambridge*.

THE only general mode of solution of the differential equations of wave propagation in electromagnetic theory, which has led to a result directly useful for definite physical problems, has been that of Prof. Lamb†. This method is concerned with the determination of a vector (u, v, w) simultaneously satisfying

$$\begin{aligned} (V^2 + k^2)(u, v, w) &= 0 \\ \text{and} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

(u, v, w) being Cartesian components. The general expression found for this vector is somewhat cumbersome, and though well adapted for the solution of many problems, it frequently leads to tedious work. In this paper, an alternative solution will be developed by a different process. It is of a form more suited to certain classes of physical problems connected with spheres. Although it is not really distinct from that of Prof. Lamb, the equivalence of the two is not easy to prove, involving some laborious applications of reduction formulæ. The circuital relations, and not the equation of wave motion, form the starting point of the investigation. Instead of the Cartesian components of the vectors, we treat the components along the three main spherical polar directions at any point.

Let (x, y, z) , (a, b, c) be the components of electric and magnetic force along the directions of (r, θ, ϕ) increasing at any point. We may suppose the magnetic induction and force to be identical, without loss of generality. The space elements in the three principal directions are

$$(dr, \quad r d\theta, \quad r \sin \theta d\omega).$$

Since the work done in taking a unit magnetic pole round any circuit, divided by 4π , is the total current crossing that circuit, we have

$$\begin{aligned} &\int (a dr + br d\theta + cr \sin \theta d\omega) \\ &= 4\pi \iint (ur^2 \sin \theta d\theta d\omega + vr \sin \theta dr d\omega + wr d\theta dr) \\ &= \frac{1}{V^2} \frac{d}{dt} \iint (xr^2 \sin \theta d\theta d\omega + yr \sin \theta d\omega dr + zr d\theta dr) \end{aligned}$$

under the conditions supposed.

* Communicated by the Author.

† Proc. Lond. Math. Soc. xiii. p. 51 (1882).

Or $(\dot{x} r^2 \sin \theta, \dot{y} r \sin \theta, \dot{z} r) = V^2 \text{curl} (a, br, cr \sin \theta),$ (2)

the differentiations in the operation "curl" being

$$\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial \omega}.$$

In a similar manner, since the electromotive force in the circuit is the rate of decrease of the total magnetic induction across it,

$$(\dot{a} r^2 \sin \theta, \dot{b} r \sin \theta, \dot{c} r) = - \text{curl} (x, yr, zr \sin \theta). \quad (3)$$

We have further, as necessary for the existence of these relations, the solenoidal conditions

$$\text{div} (x r^2 \sin \theta, yr \sin \theta, zr) = 0 \quad \dots \quad (4)$$

$$\text{div} (a r^2 \sin \theta, br \sin \theta, cr) = 0, \quad \dots \quad (5)$$

the differentiations in "div" being $\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial \omega}.$

Thus

$$-r \sin \theta \frac{\partial c}{\partial \omega} = \frac{\partial}{\partial r} (a r^2 \sin \theta) + r \frac{\partial}{\partial \theta} (b \sin \theta)$$

or

$$\frac{\partial c}{\partial \omega} = - \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r^2 a) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (b \sin \theta) \right\}. \quad \dots \quad (6)$$

Now by certain relations embodied in the circuital formulæ,

$$\begin{aligned} \frac{\dot{a} r^2 \sin \theta}{V^2} &= - \frac{1}{V^2} \frac{\partial}{\partial \theta} (\dot{z} r \sin \theta) + \frac{1}{V^2} \frac{\partial}{\partial \omega} (\dot{y} r) \\ &= - \frac{\partial}{\partial \theta} \cdot \sin \theta \left\{ \frac{\partial}{\partial r} (br) - \frac{\partial a}{\partial \theta} \right\} \\ &\quad + \frac{\partial}{\partial \omega} \cdot \frac{1}{\sin \theta} \left\{ \frac{\partial a}{\partial \omega} - \frac{\partial}{\partial r} (cr \sin \theta) \right\}, \end{aligned}$$

But

$$\frac{1}{V^2} \frac{\partial^2}{\partial t^2} = -k^2, \quad \dots \quad (7)$$

if $2\pi/k$ is the wave-length.

Thus with the help of (6),

$$\begin{aligned} k^2 a r^2 \sin \theta &= \frac{\partial}{\partial \theta} \cdot \sin \theta \frac{\partial (br)}{\partial r} - \frac{\partial}{\partial \theta} \cdot \sin \theta \frac{\partial a}{\partial \theta} - \frac{1}{\sin \theta} \frac{\partial^2 a}{\partial \omega^2} \\ &\quad - \frac{\partial}{\partial r} \cdot r \sin \theta \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r^2 a) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (b \sin \theta) \right\}. \end{aligned}$$

The terms involving b are seen to vanish, and therefore

$$\frac{\partial^2}{\partial r^2} (r^2 a) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial a}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 a}{\partial \omega^2} + k^2 r^2 a = 0. \quad (8)$$

Now a , as a function of ω , may be expanded in a Fourier series. The coefficient $u^{(m)}$ of the term involving $\cos (m\omega + \epsilon^{(m)})$ satisfies

$$\frac{\partial^2}{\partial r^2} (r^2 u^{(m)}) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u^{(m)}}{\partial \theta} \right) - \frac{m^2 u^{(m)}}{\sin^2 \theta} + k^2 r^2 u^{(m)} = 0.$$

Writing $u^{(m)} = \Sigma u_n^m P_n^m (\cos \theta),$

where P_n^m denotes the associated harmonic of degree n and order m .

Then
$$\frac{\partial^2}{\partial r^2} (r^2 u_n^m) + (k^2 r^2 - n(n+1)) u_n^m = 0,$$

whose solution, finite (unless $n=0$) at the origin, is

$$u_n^m = A r^{-\frac{3}{2}} \frac{J}{n+\frac{1}{2}} (kr)$$

and the corresponding value of a becomes

$$a = A r^{-\frac{3}{2}} \frac{J}{n+\frac{1}{2}} (kr) \cdot P_n^m (\cos \theta) (\cos (m\omega + \epsilon)). \quad (9)$$

More generally, we must add, to the associated harmonic, an arbitrary multiplier of the type $Q_n^m (\cos \theta)$, and to the Bessel function, a multiple of $J_{-n-\frac{1}{2}} (kr)$. For convenience, these are ignored, for since they have the same differential properties as their primary types, their contribution to any formula may be at once inserted. The electrical component x must, by symmetry, satisfy the same differential equation as a , and we thus write

$$\left. \begin{aligned} a &= A r^{-\frac{3}{2}} \frac{J}{n+\frac{1}{2}} (kr) P_n^m (\cos \theta) \cos (m\omega + \epsilon) \\ x &= B r^{-\frac{3}{2}} \frac{J}{n+\frac{1}{2}} (kr) P_n^m (\cos \theta) \cos (m\omega + \delta) \end{aligned} \right\}, \quad (10)$$

where (A, B) bear no necessary relation to one another. They may be supposed to include a time factor $\exp. i k \nu t$, and a summation to be made for all integral values of m and n .

The remaining components now require calculation, and are somewhat more difficult to obtain.

We have

$$\frac{\partial}{\partial \theta} (c \sin \theta) - \frac{\partial b}{\partial \omega} = r \dot{x} \sin \theta = \frac{\imath k r x \sin \theta}{V} \quad \dots \quad (11)$$

and by the solenoidal relation,

$$\frac{\partial}{\partial \theta} (b \sin \theta) + \frac{\partial c}{\partial \omega} = -\frac{\sin \theta}{r} \frac{\partial}{\partial r} (a r^2). \quad \dots \quad (12)$$

Thus (b, c) each consist of two portions of different forms, one derived from a , and the other from x .

Performing the operations $\left(\frac{\partial}{\partial \omega}, \frac{\partial}{\partial \theta} \cdot \sin \theta \right)$ on (11) and (12), and subtracting, we obtain, since $\frac{\partial^2}{\partial \omega^2} \equiv -m^2$,

$$\begin{aligned} \frac{\partial}{\partial \theta} \cdot \sin \theta \frac{\partial}{\partial \theta} (b \sin \theta) - m^2 b = & -\frac{1}{r} \frac{\partial^2}{\partial r \partial \theta} (a r^2 \sin^2 \theta) \\ & - \frac{\imath k r}{V} \sin \theta \frac{\partial x}{\partial \omega} \quad \dots \quad (13) \end{aligned}$$

Put $b = b_1 + b_2$, where b_1 and b_2 are the portions to which a and x respectively give rise.

The factors involving r in b_1 and b_2 are respectively

$$\frac{1}{r} \frac{d}{dr} \cdot r^{\frac{1}{2}} J \quad \text{and} \quad r^{-\frac{1}{2}} J,$$

where J is briefly written for $J_{n+\frac{1}{2}}(kr)$.

Making the substitutions

$$\left. \begin{aligned} b_1 = & -\frac{A}{r} \frac{d}{dr} \cdot r^{\frac{1}{2}} J \cdot \delta_1 \cos(m\omega + \epsilon) \\ b_2 = & \frac{\imath k m}{V} B \cdot \frac{J}{\sqrt{r}} \cdot \delta_2 \sin(m\omega + \delta) \end{aligned} \right\} \quad \dots \quad (14)$$

where $(\delta_1 \delta_2)$ are functions of θ only, we obtain

$$\frac{d}{d\theta} \cdot \sin \theta \frac{d}{d\theta} (\delta_1 \sin \theta) - m^2 \delta_1 = \frac{d}{d\theta} \cdot \{ \sin^2 \theta P_n^m(\cos \theta) \} \quad (15)$$

$$\frac{d}{d\theta} \cdot \sin \theta \frac{d}{d\theta} (\delta_2 \sin \theta) - m^2 \delta_2 = \sin \theta P_n^m(\cos \theta) \quad \dots \quad (16)$$

and deduce at once,

$$\delta_2 = -\frac{1}{n \cdot n + 1} \cdot \frac{P_n^m(\cos \theta)}{\sin \theta} \quad \dots \quad (17)$$

If $\mu = \cos \theta$, the equation for δ_1 becomes

$$\frac{d}{d\mu} \cdot (1 - \mu^2) \frac{d}{d\mu} (\delta_1 \sin \theta) - \frac{m^2}{1 - \mu^2} \cdot \delta_1 \sin \theta = - \frac{d}{d\mu} \cdot \{1 - \mu^2 \cdot P_n^m(\mu)\} \quad (18)$$

Now

$$\mu P_n^m = \frac{1}{2n+1} (n-m+1 \cdot P_{n+1}^m + n+m \cdot P_{n-1}^m).$$

Also

$$\begin{aligned} 1 - \mu^2 \cdot \frac{dP_n^m}{d\mu} &= n+m \cdot P_{n-1}^m - n\mu P_n^m \\ &= \frac{1}{2n+1} \{n+1 \cdot n+m \cdot P_{n-1}^m - n \cdot n-m+1 P_{n+1}^m\}. \end{aligned}$$

Thus on reduction

$$2n+1 \cdot \frac{d}{d\mu} \cdot (1 - \mu^2 \cdot P_n^m) = n-1 \cdot n+m \cdot P_{n-1}^m - n+2 \cdot n-m+1 P_{n+1}^m.$$

The substitution

$$\delta_1 \sin \theta = p P_{n-1}^m + q P_{n+1}^m$$

is therefore suggested, giving

$$\begin{aligned} \frac{d}{d\mu} \cdot (1 - \mu^2) \frac{d}{d\mu} (\delta_1 \sin \theta) - \frac{m^2}{1 - \mu^2} \cdot \delta_1 \sin \theta &= -n \cdot n-1 \cdot p P_{n-1}^m \\ &\quad - n+1 \cdot n+2 \cdot q P_{n+1}^m. \end{aligned}$$

The equation (15) is therefore satisfied if

$$p = \frac{n+m}{n \cdot 2n+1}, \quad q = - \frac{n-m+1}{n+1 \cdot 2n+1},$$

and thus,

$$\begin{aligned} \delta_1 &= \frac{1}{2n+1 \cdot \sin \theta} \left\{ \frac{n+m}{n} P_{n-1}^m - \frac{n-m+1}{n+1} P_{n+1}^m \right\} \\ &= \frac{\sin \theta}{n \cdot n+1} \frac{dP_n^m}{d\mu} \cdot \dots \dots \dots (19) \end{aligned}$$

on reduction.

Thus

$$\begin{aligned} b &= b_1 + b_2 \\ &= - \frac{A \sin \theta}{rn \cdot n+1} \frac{d}{dr} \cdot r^{\frac{1}{2}} J \cdot \frac{dP_n^m}{d\mu} \cos (m\omega + \epsilon) \\ &\quad - \frac{\mu k m B r^{-\frac{1}{2}}}{V n \cdot n+1 \cdot \sin \theta} \cdot J \cdot P_n^m(\mu) \sin (m\omega + \epsilon). \quad (20) \end{aligned}$$

Again,

$$\frac{\partial c}{\partial \omega} = -\frac{\partial}{\partial \theta} (b \sin \theta) - \frac{\sin \theta}{r} \frac{\partial}{\partial r} (ar^2).$$

The portion of $\frac{\partial c}{\partial \omega}$ involving the constant A has the value

$$\frac{A}{r} \frac{d}{dr} (r^{\frac{1}{2}} J) \cdot \Theta \cos (m\omega + \epsilon),$$

where

$$\begin{aligned} n \cdot n + 1 \cdot \Theta &= \frac{d}{d\theta} \cdot \left(1 - \mu^2 \frac{dP_n^m}{d\mu} \right) - n \cdot n + 1 \cdot P_n^m \sin \theta \\ &= -\frac{m^2}{\sin \theta} P_n^m. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial c}{\partial \omega} &= -\frac{m^2}{n \cdot n + 1} \cdot \frac{A}{r} \cdot \frac{d}{dr} \cdot r^{\frac{1}{2}} J \cdot \frac{P_n^m(\mu)}{\sin \theta} \cos (m\omega + \epsilon) \\ &\quad - \frac{\kappa k m B r^{-\frac{3}{2}}}{V n \cdot n + 1} \cdot J \sin \theta \frac{dP_n^m}{d\mu} \sin (m\omega + \delta), \quad \dots \quad (21) \end{aligned}$$

from which c follows at once.

The calculation of y and z is now immediate.

Writing

$$\left. \begin{aligned} A &= n \cdot n + 1 \cdot C_n^m \\ B &= \frac{V}{\kappa k} n \cdot n + 1 \cdot D_n^m \end{aligned} \right\} \dots \dots \dots (22)$$

where (C_n^m, D_n^m) are arbitrary, the components of magnetic force become

$$\begin{aligned} a &= n \cdot n + 1 \cdot C_n^m r^{-\frac{3}{2}} J \cdot P_n^m(\mu) \cos (m\omega + \epsilon_n^m) \\ b &= -C_n^m r^{-1} \frac{d}{dr} (r^{\frac{1}{2}} J) \cdot \sin \theta \frac{dP_n^m}{d\mu} \cos (m\omega + \epsilon_n^m) \\ &\quad - m D_n^m r^{-\frac{1}{2}} J \cdot \frac{P_n^m}{\sin \theta} \sin (m\omega + \delta_n^m) \\ c &= D_n^m r^{-\frac{1}{2}} J \cdot \sin \theta \frac{dP_n^m}{d\mu} \cos (m\omega + \delta_n^m) \\ &\quad - C_n^m \cdot r^{-1} \frac{d}{dr} (r^{\frac{1}{2}} J) \cdot \frac{P_n^m}{\sin \theta} \cdot \sin (m\omega + \epsilon_n^m); \quad \dots \quad (23) \end{aligned}$$

while those of electric force are given by

$$\begin{aligned}\frac{ikx}{V} &= n \cdot n + 1 \cdot D_n^m r^{-\frac{3}{2}} J \cdot P_n^m(\mu) \cos(m\omega + \delta_n^m) \\ \frac{iky}{V} &= -m C_n^m r^{-\frac{1}{2}} J \cdot \frac{P_n^m}{\sin \theta} \cdot \sin(m\omega + \epsilon_n^m) \\ &\quad - D_n^m r^{-1} \frac{d}{dr} (r^{\frac{1}{2}} J) \sin \theta \frac{dP_n^m}{d\mu} \cos(m\omega + \delta_n^m) \\ \frac{ikz}{V} &= C_n^m r^{-\frac{1}{2}} J \cdot \sin \theta \frac{dP_n^m}{d\mu} \cos(m\omega + \epsilon_n^m) \\ &\quad - m D_n^m r^{-1} \frac{d}{dr} (r J) \cdot \frac{P_n^m}{\sin \theta} \sin(m\omega + \delta_n^m), \quad . \quad . \quad (24)\end{aligned}$$

where J may now be taken as $J(kr) + \lambda_n J(kr)$.
 $\qquad\qquad\qquad n+\frac{1}{2} \qquad\qquad\qquad -n-\frac{1}{2}$

When, in (x, y, z) , or (a, b, c) , we add to P_n^m an arbitrary multiple of the associated harmonic of type Q_n^m , and perform a summation for all integral values of m and n , we obtain the most general solution of the circuital relations corresponding to wave motion. In this form, the results are more appropriate to electromagnetic problems involving spheres, than those ordinarily given, their main utility lying in the fact that they give the radial and transversal components directly. For example, the period equations of free vibration of the space between two conducting spheres may be written down at once, and are seen to be independent of the question as to whether the vibrations are symmetrical* or tesseral.

XVI. *Diurnal Periodicity of the Spontaneous Ionization of Air and other Gases in Closed Vessels.* By ALEXANDER WOOD, B.A., D.Sc. (Glas.), Lecturer, Emmanuel College, Cambridge; and NORMAN R. CAMPBELL, M.A., Fellow of Trinity College, Cambridge†.

IF a stream of air be drawn through a metal vessel which is subsequently sealed up, the spontaneous ionization in the vessel undergoes a progressive change which has been investigated by several observers. Since the conclusions reached were various and sometimes conflicting, a new and careful series of experiments was undertaken. It was soon found that the phenomenon was more complicated than had

* Phil. Mag. May 1906; Macdonald, 'Electric Waves,' chap. 3.

† Communicated by Prof. J. J. Thomson.

at first been supposed; for, imposed upon the progressive change of the ionization, which may continue for some weeks, there is a periodic diurnal variation of the ionization. It was considered advisable to investigate this latter change more completely, abandoning for the present the original purpose of the research.

The existence of such diurnal variations has been suspected by several observers. Both of the authors*, when conducting separate experiments on spontaneous ionization, have been troubled by inexplicable variations in the current through their vessels; the current seemed smallest in the afternoon. Jaffé† also noticed the variations, and investigated them in some detail, concluding that there was a minimum at 2 P.M. Borgmann‡ states that the minimum of ionization is at 3 P.M. The first three observers had all worked at Cambridge, but since their experiences agree with those of Borgmann, it is probable that the effects are not due to local causes. A similar diurnal periodicity of the amount of radioactive emanation in the atmosphere has been observed by Dike§. His curve is obtained from the observations on six days only, so that it is impossible to attach much importance to its details, but its principal maximum (1 A.M.) agrees fairly well with one of the maxima of the ionization in our vessels.

As it was desired to vary the gas in which the ionization was to be measured as well as the metal of which the vessel was made, the first difficulty to be solved was the construction of a vessel which would hold hydrogen or coal-gas for several weeks. In the end the following device was adopted. The vessel, which had a capacity of about 6000 ccs., was placed on a brass plate and covered with a glass bell-jar. The bell-jar was cemented down to the plate with marine glue, and could then be exhausted and filled with any required gas. As the experimental vessel was thus placed in an atmosphere of the gas, no precautions were necessary to have it absolutely gas-tight. An insulated electrode passed through the brass plate up into the vessel, and the vessel was kept charged to a constant high potential in the usual way. The arrangement was found to hold a good vacuum for a considerable time. In order to obtain continuous readings of the ionization in the vessel, a device employed by Bronson|| was used. The

* N. R. Campbell, *Phil. Mag.* April 1905, p. 534; A. Wood, *ibid.* p. 562.

† G. Jaffé, *Phil. Mag.* Oct. 1904, p. 556.

‡ J. J. Borgmann, see 'Science Abstracts,' 1904, No. 1580.

§ 'Terrestrial Magnetism,' vol. xi. No. 3, p. 128.

|| H. Bronson, *Phil. Mag.* Jan. 1906, p. 143. See also C. Nordman, *Comptes Rendus*, cxxxviii. pp. 1418-1420 & 1596-1599.

electrode was connected to one pair of quadrants of a quadrant electrometer, and also to earth through a vessel, containing air ionized by uranium, which acted as a high resistance. The other pair of quadrants of the electrometer were connected to earth, and in this way the readings of the electrometer were, for small variations, directly proportional to the ionization. During the day the spot of light from the mirror was read on a scale in the ordinary way—the value of the scale readings being found by comparison with a potentiometer. At night the spot was projected on to a clock-driven drum covered with bromide-paper, and the initial and final readings of the potential being known from the scale readings, the intermediate values could be found by measurement of the trace.

It was thus possible to experiment with any gas in a vessel of any metal, and at first a great many combinations were tried. A zinc vessel was filled successively with hydrogen, coal-gas, carbon dioxide, and air; a lead vessel with carbon dioxide, coal-gas, and air; and a tin vessel with air. As the apparatus took some time to adjust itself, it was found to be impossible to study the initial changes in the ionization, but a record of the subsequent changes extending over several days, and in some cases over several weeks, was obtained. Curves have been drawn to represent these changes, and it will be seen that they exhibit two main features. In most cases there is a gradual increase of the ionization with the time. This had been previously observed by Burton and McLennan*, but whereas they found the increase to be rapid at first, then to fall away, and finally to cease altogether, leaving the ionization at a steady value, we found a fairly steady increase, which was maintained over a period of ten days or more. It is just possible that this difference is to be explained in some way by the differences in size of the vessels used.

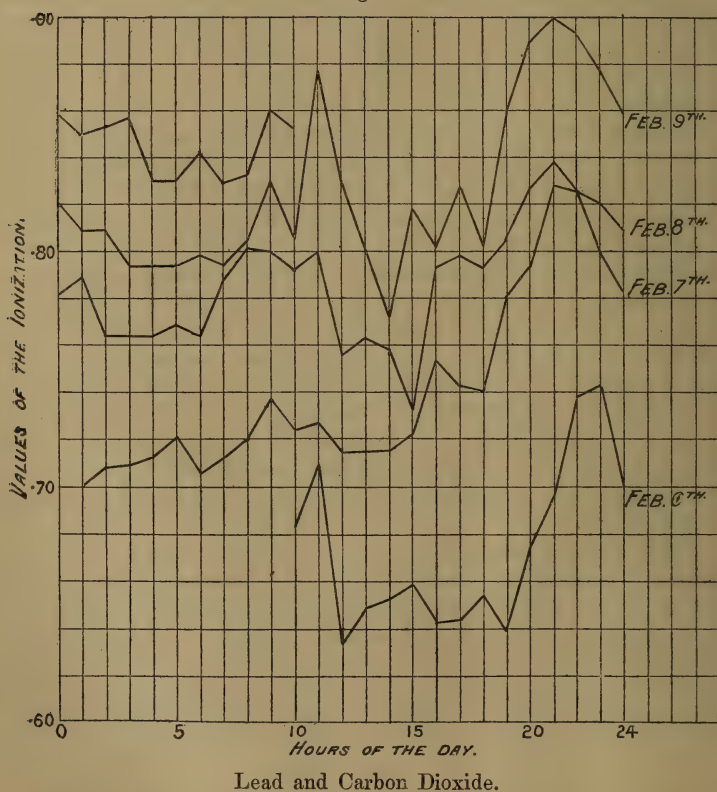
This increase occurred in the case of lead with each of the gases tried, and so would seem to be characteristic of the metal rather than of the gas. The curves representing the change of ionization with time are shown for lead and carbon dioxide in fig. 1, and for lead and coal-gas in fig. 2. The abscissæ are hours of the day, and the ordinates are the corresponding values of the ionization. Each separate curve represents one day. It will be seen that the curves lie successively above one another and that they differ by approximately equal intervals. In the case of zinc, however, no such increase has been found except in the single case

* *Phys. Review*, iv. 1903.

when the vessel contained coal-gas. Coal-gas is so uncertain in composition that little importance is attached to this result. Air was enclosed in the zinc vessel on the 10th of August this year, and its ionization measured on our scale was found to be 0.513. It was left enclosed until the 12th of October, and the ionization was then 0.521. The behaviour of the tin was similar to that of the lead, and showed a gradual increase in the ionization extending over the whole period of the investigation.

By far the most interesting feature of the curves, however, is their periodic character. This is quite well marked in the curves for lead, indeed it was in the curves for lead and carbon dioxide that it was first noticed. It will be seen that

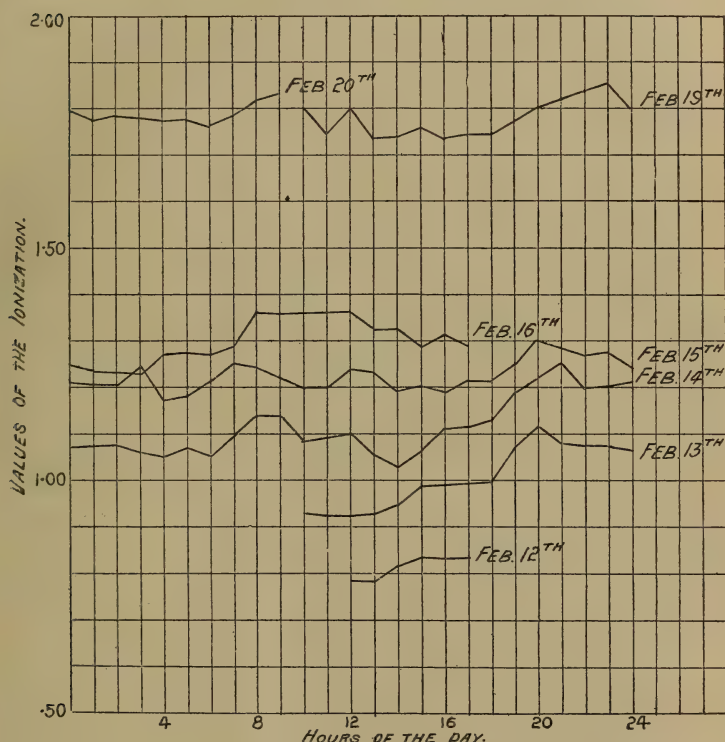
Fig. 1.



in fig. 1 each curve shows a very marked maximum at a time varying from 21 hours (9 P.M.) to 23 hours (11 P.M.) and a

minimum between 12 h. and 15 h. Not quite so well defined but still quite obvious are the second maximum occurring between 8 h. and 11 h., and the second minimum occurring between 2 h. and 6 h. The curves for lead and carbon dioxide thus show a double daily period. Looking now at the curves for lead and coal-gas (fig. 2), we find that

Fig. 2.

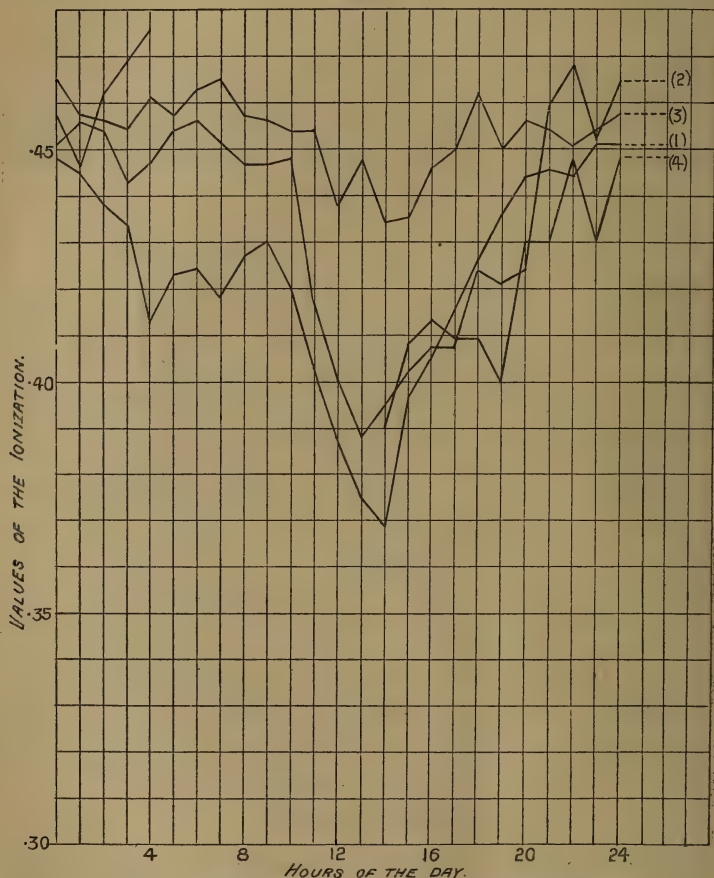


Lead and Coal-Gas.

the scale to which they are drawn makes them appear more regular, but the same features are apparent. Once more each curve shows a well marked maximum between 20 h. and 23 h., and a minimum between 12 h. and 16 h. Another maximum may be observed between 7 h. and 10 h., and a second minimum between 3 h. and 6 h. From this and other similar evidence it is concluded that the periodic variation of the ionization in the vessel is independent of the nature of the gas enclosed.

The next point of importance was to determine whether the periodicity was peculiar to lead vessels. Fig. 3 shows a set of curves drawn for zinc and carbon dioxide similar to those already given for lead. Here again the periodicity is quite evident, and the above noted maxima and minima can

Fig. 3.

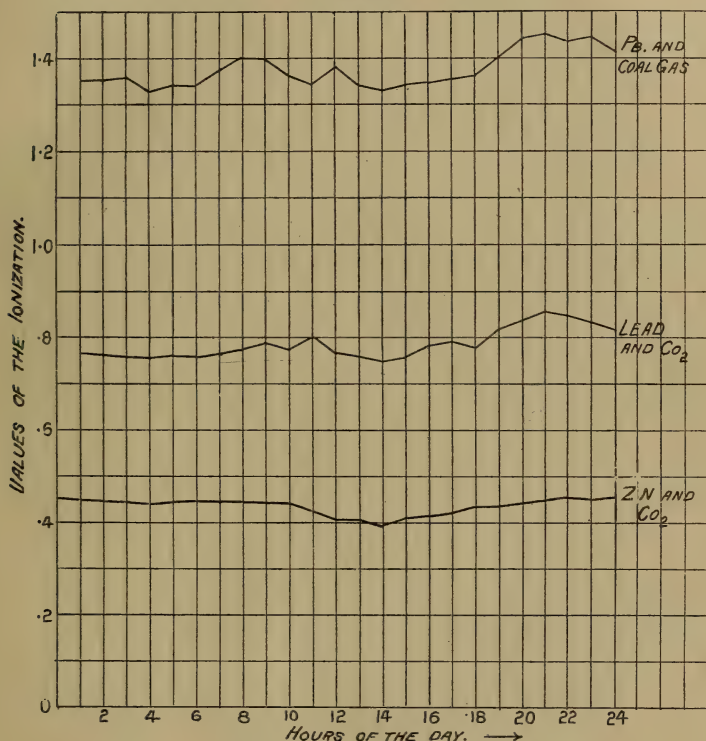


Curves for Zinc and Carbon Dioxide. March 5-8.

all be recognized. One very marked difference in this diagram is the frequent intersection of the curves representing successive days. This is due to the fact that little or no permanent increase in the ionization occurred. Several other similar sets were obtained for zinc, the gases used including coal-gas, air, and hydrogen. All the sets showed the same

periodic variation; and hence we are forced to conclude that this periodicity is independent not only of the gas enclosed in the vessel but also of the metal of which the vessel is made. Mean curves for the three different sets of curves already given were drawn and are represented in fig. 4.

Fig. 4.

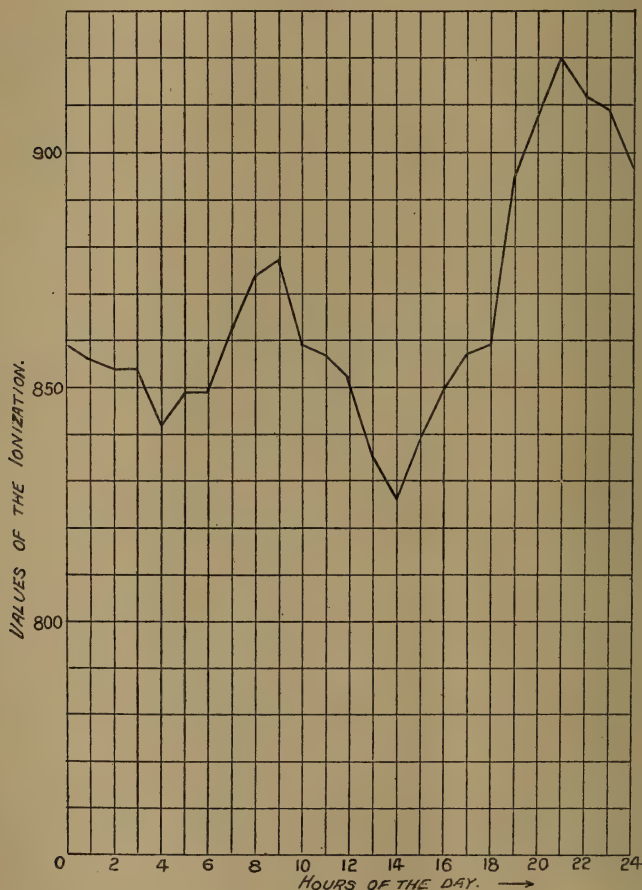


This brings out the similarity still more clearly, and is sufficient justification for drawing a single mean curve to represent the three sets of results. This is shown in fig. 5 (p. 272).

For subsequent results it will be more convenient to draw our curves in a somewhat different way. It was assumed that the permanent increase of ionization which took place from day to day with lead and with zinc (when coal-gas was enclosed) was uniform throughout each day. Its value was found by comparing the mean values of the ionization on each of two consecutive days. The corresponding day's

values of the ionization were then corrected for the rise, and the excess or defect of the value of the ionization for each hour of the day from the mean value for the day was plotted. In this way all the later curves were obtained.

Fig. 5.



Mean of Curves in Fig. 4.

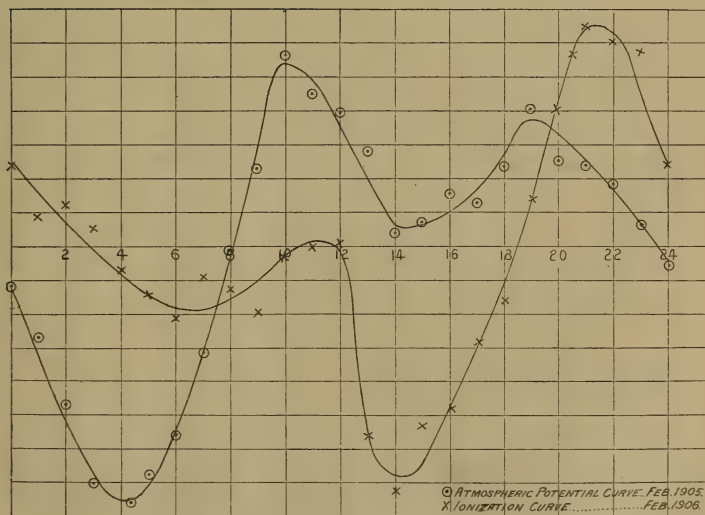
The periodic character of the variation was at first attributed to temperature as the most probable cause, and one of Callendar's patent recording thermometers, kindly put at our disposal by the Cambridge Scientific Instrument Company, was installed alongside of the apparatus. It was found

to be impossible to connect the temperature curve with the ionization curve in any way. The temperature curve had only one maximum and one minimum, so that if there be any connexion between the two curves the periodicity of the ionization must depend on some differential coefficient of the temperature. Then, again, the nature of the temperature curve depended entirely on the hot-water pipes, and was quite different on Saturdays and Sundays from the rest of the week, whereas we were unable to detect any change whatever in the character of our curves on these days.

The next quantity, variations of which were measured with a view to connecting them with the variations of the ionization, was atmospheric potential. It was at once evident that there was a close similarity between the curves representing the simultaneous variation of the two quantities. For some time, the atmospheric potential recorder at the Observatory here was run with the kind assistance of Mr. A. R. Hinks, but in the end it was found to be unnecessary, as the very full data published of the observations at Kew, when used instead, brought out the same similarity and at the same time showed that it could not be due to local causes.

In figs. 6 to 10 the ionization curves and the atmospheric

Fig. 6.



potential curves for the same month are plotted together. There is no connexion between the scales representing the

values of the two quantities. They are so chosen as to give nearly the same range in the two cases.

Fig. 7.

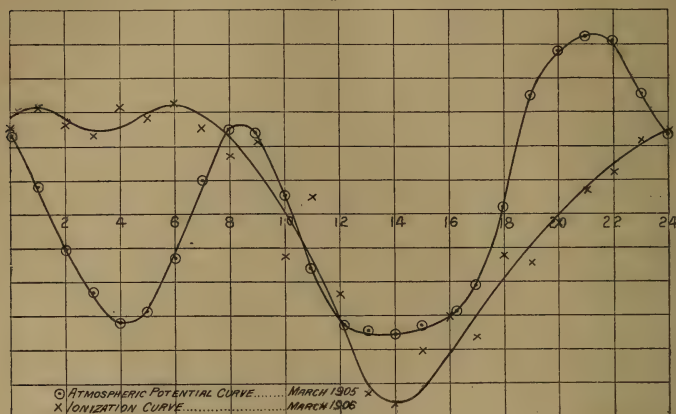
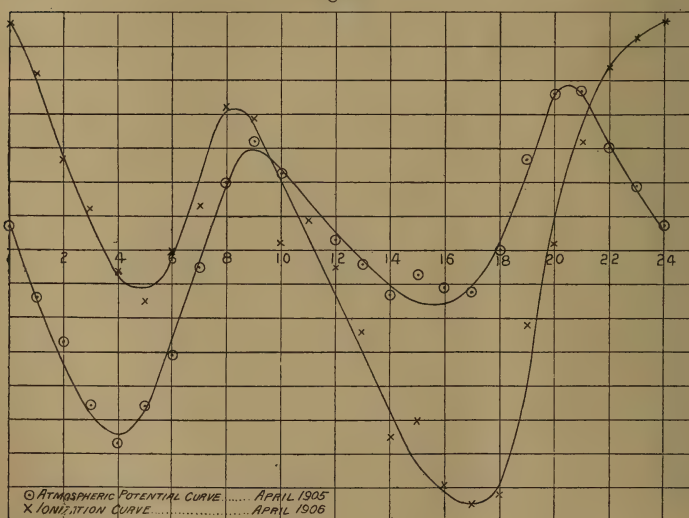


Fig. 8.



Lead was found to give a somewhat greater mean daily range than zinc, but the smaller total ionization of the zinc made the proportional range greater for zinc than for lead. In nine successive days in August the value of this ratio was 0.12, 0.09, 0.17, 0.14, 0.18, 0.16, 0.12, 0.14, 0.09; this gives a mean of 0.12, so that the variation in the ionization in the course of one day amounts to about 12 per cent. of

the whole. On comparing corresponding curves in figs. 6 to 10, it will be seen that the general forms of the curves are the same and that the maxima and minima correspond very closely. In order to make it quite clear that this does not

Fig. 9.

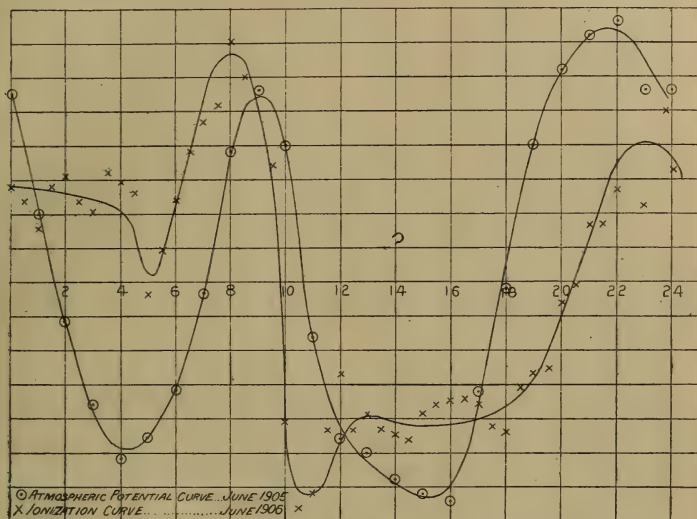
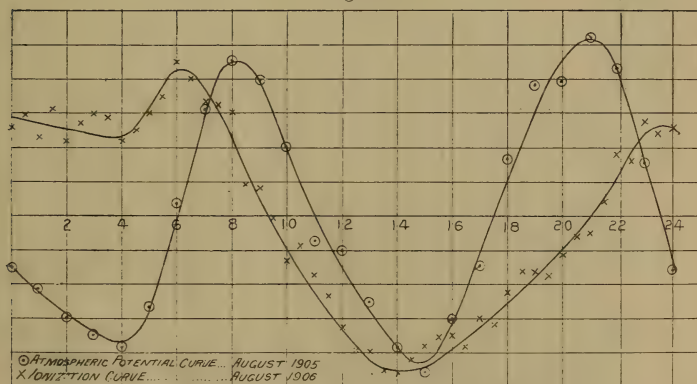


Fig. 10.



depend on the drawing of the curves, a table is given of the times at which actual maxima and minima points were observed. The times at which corresponding maxima and minima occurred are placed alongside one another in the same vertical column, the times for the atmospheric potential curves being always placed first.

Month.	1st Min.	1st Max.	2nd Min.	2nd Max.
Feb.	4 h. 6 h.	10 h. 12 h.	14 h. 14 h.	19 h. 21 h.
March	4 h. 3 h.	8 h. 6 h.	14 h. 14 h.	21 h. 21 h.
April	4 h. 5 h.	9 h. 8 h.	14 h. 17 h.	21 h. 24 h.
June	4 h. 5 h.	9 h. 8 h.	16 h. 10½ h.	22 h. 22½ h.
August	4 h. 4 h.	8 h. 6 h.	15 h. 14 h.	21 h. 23 h.
Means.....	4 h. 4½ h.	9 h. 8 h.	14 h. 14 h.	21 h. 23 h.

The close agreement of all these figures, and especially of the means, seems to be sufficient to establish a connexion between the two quantities involved. It may be noted that the atmospheric potential curves represent observations in fine weather only, while our observations were taken quite irrespective of weather conditions. So far as we could tell the weather conditions were entirely without effect on the ionization.

An attempt was made to analyse these curves into Fourier series without success. The curves were smoothed out and account was taken of the first four components, but no constant results were obtained. The apparatus has recently been improved, and in future months it is hoped that observations may be taken more continuously. In this way smoother curves may be obtained, and these may possibly analyse.

Briefly, then, the paper seeks to establish the following points:—

- (1) The ionization in a closed vessel undergoes a permanent increase for a considerable time after the gas contained in it has been enclosed, but this increase depends on the nature of the vessel, being comparatively large for lead and tin vessels and negligible for zinc vessels.
- (2) Superimposed on this variation of the ionization is a periodic variation having two maxima and two minima each twenty-four hours.
- (3) In their main features the curves representing this periodic change are the same as the curves representing the variations of atmospheric potential, and it is hoped in a later paper to establish a connexion between the two variations. For a possible connexion the reader may be referred to Richardson's letter to 'Nature'*

The authors welcome another opportunity for expressing their indebtedness to Professor Thomson.

* 'Nature,' April 26, 1906, p. 607.

XVII. *On Spark Potentials in Liquid Dielectrics.*

By Dr. P. E. SHAW*.

IN June of last year I read a paper before the Physical Society of London on the above subject; it was published in the *Phil. Mag.* for October. Since then Mr. R. F. Earhart has produced a paper (*Phys. Rev.* xxiii. pp. 358-369, Nov. 1906) which covers to some extent the same ground. As his results are somewhat at variance with mine it seems of interest to briefly compare and contrast them.

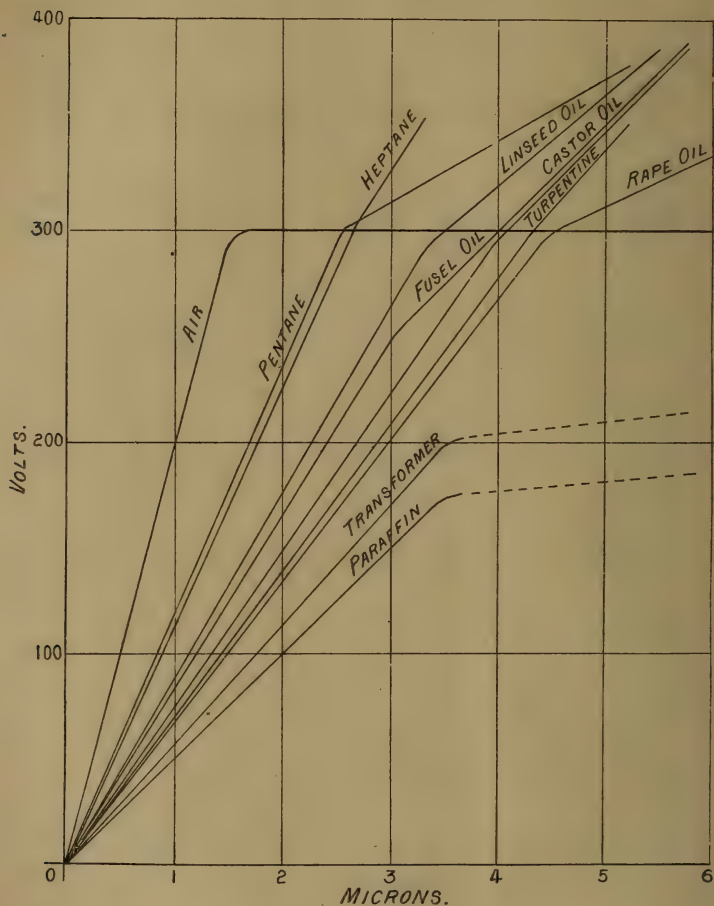
The conditions are different in several ways:—(1) He uses a 110-volt 60-cycle circuit stepped up by a transformer to 2200 volts; I used a 400-volt steady current, a town-lighting circuit. (2) He works up to 1000 volts; I used up to only 400 volts. (3) He uses a plane surface and a spherical one of 25.4 mm. diam., whereas I used a plane and a small sphere of 2 mm. radius. (4) His surfaces are steel, mine of iridio-platinum. (5) He immerses the whole of the contact surfaces in the oil used; I used a drop between the two surfaces, renewed for each experiment. (6) The smallest discharge distance in his work is 3 microns, whereas I used distances as small as 0.25 micron. (7) His work is confined to four oils, whereas I worked on fifteen oils and other liquids.

Seeing how the conditions differ in his apparatus and mine, it is scarcely strange that the results also should differ. To contrast the results:—(1) He finds the potential gradient for small distances greater than for large ones, there being a “knee” for potential 330. In my curves, half of the liquids show such variation in strength, as potential varies, that for them it seems impossible to state a potential gradient. (2) The potential gradient found by him is in all cases much less than in my experiments, and the curves for liquids do not appear to cross the air-line, as is the case in all my experiments. (3) He finds, as I did, that for small distances air insulates better than liquids. (4) The most striking result of his research is the indication of the “knee” in the curves. He finds this to be at 330 volts, which is the same as for sparks in air.

In view of this last result I have examined my results to see if there is any indication of a “knee” in my curves also. As the accompanying figure shows, there seems to be a “knee” in some cases. It is pronounced for pentane, linseed oil, fusel oil, and rape oil. It is slight for heptane, castor oil, and turpentine; whereas for olive oil, codliver oil, hexane, and octane there is no clear indication of a change of the kind.

* Communicated by the Author.

Again, as in Mr. Earhart's results, the "knee" occurs at the same voltage as the "knee" for air, which in my apparatus is 300 volts. But for Mr. Earhart's paper probably I should not have noticed this peculiarity in these curves.



The "knee" is not distinct as for air; and since all sparking distances are very uncertain it is natural to draw a line roughly through the plotted points and to disregard variations from the straight.

Though this "knee" seems to be established for some cases I am not convinced that it is true always, *e. g.* for olive oil, codliver oil, hexane, and octane, under the conditions of my experiments.

Mr. Earhart tested four liquids—olive oil, paraffin, transformer oil, and kerosene. For the first three of these direct comparison can be made with my results.

For paraffin and transformer oil I found a breaking-down of insulation (*i. e.* the equivalent of a “knee”) at 175 volts and 200 volts respectively, whereas for these as for the other liquids Earhart obtained the change at 330 volts. These considerable differences may be explained by the very different conditions of the two methods.

XVIII. *On the Introduction of Doppler's Factor into the Solution of the Equations of the Electron Theory.* By L. DE LA RIVE*.

THE transformation of Maxwell's equations takes place as follows. The equations are †:—

$$\text{curl } \mathbf{D} = -\frac{1}{4\pi v^2} \cdot \frac{d\mathbf{H}}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{curl } \mathbf{H} = 4\pi \frac{d\mathbf{D}}{dt} + 4\pi \rho \mathbf{u}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For regions of space not containing any electric charges,
 $\text{div } \mathbf{D} = 0.$

For a volume-distribution of electricity of density ρ measured in electromagnetic units,
 $\text{div } \mathbf{D} = \rho.$

The distribution of $\mathbf{H}$ being solenoidal, we put

$$\mathbf{H} = \text{curl } \mathbf{A}; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

whence $\text{curl } \mathbf{H} = \nabla \text{div } \mathbf{A} - \nabla^2 \mathbf{A}. \quad . \quad . \quad . \quad . \quad . \quad (4)$

Putting curl $\mathbf{A}$ for $\mathbf{H}$ in (1), we have

$$\text{curl } \frac{d\mathbf{A}}{dt} = -4\pi v^2 \text{curl } \mathbf{D}; \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and, on integrating,

$$4\pi v^2 \mathbf{D} = -\frac{d\mathbf{A}}{dt} - \nabla \phi, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$\nabla \phi$ being a constant of integration whose curl is zero. Let the auxiliary vector $\mathbf{A}$ be arbitrarily connected with the function ϕ by the relation

$$-v^2 \nabla \mathbf{A} + \frac{d\phi}{dt} = 0. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

* Communicated by Prof. J. H. Poynting, F.R.S.

† *Mathematische Einführung in die Elektronentheorie*, von A. H. Bucherer, p. 76.

Differentiating (6) with respect to t , we obtain

$$4\pi \frac{d\mathbf{D}}{dt} = -\frac{1}{v^2} \frac{d^2\mathbf{A}}{dt^2} - \frac{1}{v^2} \nabla \frac{d\phi}{dt}; \quad (8)$$

hence, using (2) and (8),

$$\text{curl } \mathbf{H} = -\frac{1}{v^2} \cdot \frac{d^2\mathbf{A}}{dt^2} - \frac{1}{v^2} \nabla \frac{d\phi}{dt} + 4\pi\rho u, \quad . . . (9)$$

and by (7)

$$\frac{1}{v^2} \frac{d^2\mathbf{A}}{dt^2} - \nabla^2\mathbf{A} = 4\pi\rho u.$$

A similar equation may be obtained for ϕ ; and so we arrive at the system of equations:—

$$\frac{1}{v^2} \frac{d^2\mathbf{A}}{dt^2} - \nabla^2\mathbf{A} = 4\pi\rho u, \quad (I.)$$

$$\frac{1}{v^2} \frac{d^2\phi}{dt^2} - \nabla^2\phi = 4\pi\rho v^2, \quad (II.)$$

$$\mathbf{H} = \text{curl } \mathbf{A}, \quad (III.)$$

$$\mathbf{D} = \frac{1}{4\pi v^2} \frac{d\mathbf{A}}{dt} - \frac{1}{4\pi v^2} \nabla \phi. \quad . . . (IV.)$$

Consider the point P_0 of the field at the instant t_0 , and suppose that the values of ϕ and $\mathbf{A}$ have been propagated to it from the electron with the velocity of light v . They have therefore been emitted by the electron at an instant t such that the radius vector at that instant has the value

$$r = v(t_0 - t). \quad (6)$$

Let now t_0 be regarded as constant, and let ϕ and $\mathbf{A}$ be expressed as functions of r by means of equation (6). Then

$$\frac{d^2\phi}{dt^2} = v^2 \frac{d^2\phi}{dr^2}, \quad \frac{d^2\mathbf{A}}{dt^2} = v^2 \frac{d^2\mathbf{A}}{dr^2};$$

and equations (I.) and (II.) become

$$\frac{d^2\mathbf{A}}{dr^2} - \nabla^2\mathbf{A} = 4\pi\rho u, \quad (I. a)$$

$$\frac{d^2\phi}{dr^2} - \nabla^2\phi = 4\pi\rho v^2. \quad (II. a)$$

A theorem of Beltrami's gives for a function ϕ of r at a point P_0 ,

$$4\pi\phi_0 = \frac{1}{r} \left(\frac{d^2\phi}{dr^2} - \nabla^2\phi \right) d\tau,$$

the volume integral being extended throughout infinite space. Hence, by (I. a) and (II. a),

$$\phi_0 = \int_{t=t_0-\frac{r}{v}} \left(\frac{\rho v^2}{r} \right) d\tau,$$

and a similar expression may be obtained for **A**. In accordance with (6), each volume element of the integral should be considered at the instant t defined by the relation

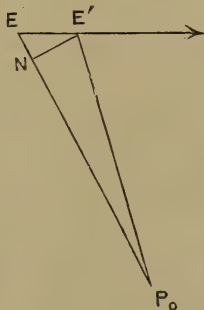
$$t = t_0 - \frac{r}{v}.$$

Let us consider the nature of the variable t defined by (6), t_0 being regarded as constant. Let the whole of space be divided up by spherical surfaces of radius r , r varying from zero to infinity, corresponding to values of t varying from t_0 to minus infinity. I call t the *time of emission*; and it is evident that to a positive increment dr there corresponds a negative increment vdt , since all the emissions are considered with respect to the same instant t_0 at P_0 .

On the other hand, Maxwell's equations and their transformations are directly applicable to points such as P_0 of the field, which may be termed the transmission field, where the function ϕ and its derivatives, whether with respect to time or with respect to the space variables, are the result of the propagation of these quantities from the emitting source. Since our equations apply to the point P_0 , it is necessary to consider an elementary variation of t_0 and of the position of P_0 , the differentials concerned being denoted by d_0 , dx_0 , dy_0 , dz_0 .

Regarding both t and t as variable in (6), we have

$$dr = v(dt_0 - dt). \quad . \quad . \quad . \quad . \quad . \quad (7)$$



Let E and E' be the two positions of the electron at the instants t and $t + dt$ of emission. The path EE' described by

the electron is udt ; and on projecting it on the radius vector EP_0 , we have

$$EN = -dr = udt \cos(ur),$$

reckoning r in a direction from E towards P_0 . On inserting for dr its value given by (7), we find

$$dt_0 = dt \left(1 - \frac{u}{v} \cos(ur) \right). \quad . \quad . \quad . \quad (8)$$

I call t_0 the *time of transmission*; and the relation (8) shows that for a point P_0 the ratio of the elementary variations of t and t_0 is constant. The ratio $\frac{dt_0}{dt}$, which may be termed

Doppler's Factor, is the ratio of the duration of a phenomenon of emission to the duration of the same phenomenon when transmitted, such duration being supposed sufficiently short in comparison with the time required by light to travel from E to P_0 . Its value becomes unity when the velocity of propagation is at right angles to the radius vector, since in that case the source of emission neither recedes nor approaches; and it would become zero if u were equal to v and the angle between the two directions were to vanish, since the source itself would then, with all its consecutive emissions, arrive at the same instant at P_0 .

Now that the ratio $\frac{dt_0}{dt}$, which we shall denote by K , is known and has been shown to be constant whatever the value of r , and that equation (II.) as well as the other equations of the system relate to the transmission space, we see that, replacing, for the sake of greater clearness, the vectorial by the cartesian expressions,

$$\frac{1}{v^2} \frac{d^2\phi}{dt_0^2} - \left[\frac{d^2\phi}{dx_0^2} + \frac{d^2\phi}{dy_0^2} + \frac{d^2\phi}{dz_0^2} \right] = 4\pi\rho v^2,$$

or, putting

$$dt_0 = Kdt, \quad dx_0 = Kdx, \quad dy_0 = Kdy, \quad dz_0 = Kdz,$$

$$\frac{1}{v^2} \frac{d^2\phi}{dt^2} - \left[\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} \right] = K^2 \pm \pi\rho v^2.$$

The first member of the equation again assumes the form which lends itself to the change of variable and the application of Beltrami's theorem, but it becomes necessary to examine whether this theorem is not modified by the change of variables.

The volume-integral

$$\int \frac{d^2\phi}{dx^2} d\tau,$$

in which dx, dy, dz are respectively equal to dx_0, dy_0, dz_0 divided by K , is identically equal to

$$\frac{1}{K} \int \frac{d^2\phi}{dx_0^2} d\tau_0;$$

and this holds good even on replacing $\frac{d^2\phi}{dx^2}$ by $\frac{d^2\phi}{dr^2}$, since $dr_0 = Kdr$.

On the other hand, in the demonstration of Beltrami's theorem, as in that of an analogous theorem of Green's, the point P_0 is surrounded by a sphere whose radius tends towards zero, and the surface-integral

$$\int \phi \frac{d\left(\frac{1}{r}\right)}{dr} d\sigma,$$

extended over the surface of the sphere, is considered. On substituting, as above, dx_0, dy_0, dz_0 for dx, dy, dz , the integral becomes

$$\frac{1}{K} \int \phi \frac{d\left(\frac{1}{r}\right)}{dr_0} d\sigma_0 = \frac{1}{K} 4\pi\phi_0,$$

whence

$$4\pi\phi_0 = K \int \left[\frac{d^2\phi}{dr^2} - \nabla^2\phi \right] d\tau,$$

and therefore

$$\phi_0 = K^3 \int \left(\frac{\rho v^2}{r} \right) d\tau.$$

On now replacing $d\tau$ by $d\tau_0/K^3$, the factor K disappears. Thus, taking into account the field of emission and that of transmission, we see that the factor in question is not introduced, although it was necessary to mention this fact.

Since, however, it is necessary to complete the expression for ϕ_0 by the introduction of K into the denominator in order that the analytical results may accord with those obtained directly by Lorentz without the use of the notion of propagation, and with the experimental facts connected with oscillatory and circular motions of the electron, it becomes necessary to have recourse to a mechanical hypothesis involving this modification.

I believe that the hypothesis according to which the

dimension of the electron parallel to the radius vector is modified in the desired ratio is a matter of controversy. In effect, this change of dimension is by no means justified by the subdivision of the electron into its elements of volume, each of which must correspond to an instant of emission, giving a slightly different value to the radius vector. This integration throughout the volume of the electron occurs in determining the mean radius vector, which only differs from the radius vector of the centre by a quantity of the second order.

This may be shown as follows in the case of rectilinear motion. On taking P_0 as origin and the velocity u as parallel to the x -axis, we find, by eliminating $t_0 - t$, the radius vector at the instant of emission expressed as a function of the co-ordinates of the electron at the instant t_0 . This expression, in which s is equal to $(1 - u^2/v^2)$, is as follows * :—

$$r = \frac{1}{s} \left[\frac{u}{v} x + \sqrt{(y^2 + z^2)s + x^2} \right].$$

This corresponds to the centre of the electron; it is required to find the variation of r due to the variation of the co-ordinates, and for this purpose the above expression is differentiated with respect to x, y, z , and then dx, dy, dz are replaced by x', y', z' , which represent the co-ordinates of a point in the interior of the electron relatively to its centre. This gives

$$dr = \frac{1}{s} \left[\left[\frac{u}{v} + \frac{x}{D} \right] x' + \frac{sy}{D} y' + \frac{sz}{D} z' \right].$$

$x/D, y/D, z/D$ are very nearly equal to the cosines of the angles made by the radius vector with the axes, since D is equal to $\sqrt{(y^2 + z^2)s + x^2}$, which is equal to the radius vector at t_0 if we suppose s equal to 1. It is required to find the mean variation of dr for all points of the volume of the electron; and this is done by integrating the above expression over the entire volume and dividing by the volume. Now each term when integrated with respect to the variable which it contains gives a result which is a square and has the same value at the two limits $+l$ and $-l$ which represent the dimensions of the volume; the integral therefore vanishes, and the mean radius differs from the central one only by an infinitesimal of the second order. This is not so if r is multiplied by K , since u/v itself is not even necessarily a very small quantity of the first order.

The hypothesis which I propose as complying with plausible conditions consists in supposing that the action of the electron in motion is due to an oscillatory state of the æther, which is

* Bucherer, *Einführung in die Elektronentheorie*, p. 84.

propagated with the velocity of light. The ratio dt_0/dt is thus found to be introduced as a denominator into the velocity of the oscillating mass, since Doppler's factor modifies the duration of the oscillation propagated. Now the momentum being proportional to the velocity, it is momentum in the electromagnetic field that fulfils the requirements outlined above, and this involves the assumption that field-intensity is of the nature of momentum.

In a memoir published by Bjerknes*, on the analogy between the steady electromagnetic field and the hydrodynamic field, the author shows that field-intensity in the electromagnetic field is analogous to momentum in the hydrodynamic one. There is between his conclusions, to which I would draw attention, and the necessity of introducing Doppler's factor, a concordance which is worthy of notice.

XIX. On the Thermodynamical Theory of Radiation.

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

MR. TUCKERMAN's criticisms† appear to be based on a misunderstanding of my position. He brings as an objection to my argument, that the temperature T is "a quantity which is and can be defined only in terms of the properties of matter."

My argument was directed against the thermodynamical proof of the laws of radiation. It is therefore sufficient for my *reductio ad absurdum* that according to the thermodynamical theory underlying this proof it is possible to attribute temperature to the æther. (In the thermodynamical proof in question we are concerned with an imaginary heat-engine, using æther as working substance, and working between temperatures T_1 and T_2 .)

Apart from this, however, I wish to take this opportunity of remarking that it is quite easy to eliminate the idea of temperature completely from my original paper‡. Let us agree to measure the warmth of the radiator by a quantity E , which is to be equal to the average energy of a principal degree of freedom in the radiator, and therefore in the ordinary notation, to $\frac{1}{2}RT$.

The radiation-function ϕ must be a function of λ , E , V , e , m , and K . There are now six independent quantities and four independent physical units, namely those of length, mass, time, and inductive capacity. As before, we can combine the six quantities in two independent ways so as to

* *Archives des Sciences phys. et nat.* 1905, t. xx. p. 325.

† *Phil. Mag.* Nov. 1906, p. 498.

‡ *Proc. Roy. Soc. A.* lxxvi. p. 545.

obtain a pure number, say

$$c_1 = Em^{-1}V^{-2}, \quad c_2 = \lambda EK e^{-2};$$

and, in order that ϕ may be of the dimensions of energy per unit volume per unit wave-length, we find that ϕ must be of the form

$$\phi = \lambda^{-4} E f(c_1, c_2);$$

whence, on passing to the limit as before, we obtain

$$\phi = \lambda^{-4} E f(\lambda E).$$

Stefan's formula follows on integration with respect to λ in the form SE^4 , the value S being of course identical with the former $16 R^{-4}\sigma$.

Hence it appears that the argument can be modified without trouble, so that the conception of temperature disappears altogether, being replaced by the conception of energy. The objection brought against my original argument, namely, that "temperature is and can be defined only in terms of the properties of matter, and in fact in terms of properties which are common to all matter," must, I think, be withdrawn when "energy" is substituted for "temperature."

J. H. JEANS.

Princeton, N.J.

XX. Notices respecting New Books.

The Electron Theory. A Popular Introduction to the new Theory of Electricity and Magnetism. By E. E. FOURNIER D'ALBE, B.Sc., A.R.C.Sc. With a Preface by G. JOHNSTONE STONEY, M.A., Sc.D., F.R.S. London: Longmans, Green, & Co. 1906. Pp. xxiv + 312.

THE great interest which is being evinced by all educated people in the new views regarding the nature of matter that have been forced upon the scientific world by the brilliant researches of the last decade should secure a wide circulation for this book. In simple and easy language, free from unnecessary technicalities, the author presents to his readers an account of the electron theory, and shows how the theory may be made to interpret in a satisfactory manner all the old familiar facts, as well as discoveries of more recent origin. A brief sketch of the development of the electron theory is first given, and is followed by an account of the elementary facts of electricity and magnetism, couched in terms suitable to the new conceptions introduced by the electron theory. This part of the book is, in a sense, an elementary text-book written on entirely up-to-date lines. A commendable feature is the frequent use of arithmetical calculation for the purpose of arriving at results which give the reader a correct notion of the *magnitude* of the various effects discussed. The later chapters are mainly devoted to recent advances, and include an account of the electromagnetic theory of light, of magneto-optic phenomena, of radio-activity, and a discussion of the constitution of the

electron and the dimensions of electrical quantities. There are a few obvious slips, such as the one on p. 69, line 6 from the bottom, where "dynes" should read "ergs."

To all who wish for a simple and easily-digested account of the electron theory we can heartily recommend this work, to which Dr. Johnstone Stoney contributes an interesting preface.

Aether: a Theory of the Nature of Aether and its Place in the Universe. By HUGH WOODS, M.D., M.A.O., B.Ch., B.A., D.P.H. London: "The Electrician" Printing and Publishing Co., Ltd. 1906. Pp. xii+100. ✓

ACCORDING to the author of this book, aether consists of very minute particles in complete contact, without any intervening spaces between. The solar system is immersed in "an enormous volume of swiftly-flowing aether." The resistance offered to the free flow of aether by the partially impervious bodies floating in it gives rise to "whirls," which, "by their continual action, make the bodies more or less spherical, and set them rotating," while the whirl surrounding a body tends to draw other bodies in its neighbourhood "centripetally towards the centre of the solid spherical body." Hence gravitation may be accounted for "in a perfectly rational manner." All other physical occurrences, such as the phenomena of heat, light, magnetism, electricity, &c., are made to follow in a similar "rational manner" from "the author's theory." And lest there should be some captious critics ready with objections, we have the following defiance flung at them on one of the concluding pages: "If our theory is scouted and cast aside, what other theory with a shadow of probability can be proposed in its stead?"

XXI. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. xii. p. 532.]

November 7th, 1906.—Sir Archibald Geikie, D.C.L., Sc.D.,
Sec.R.S., President, in the Chair.

THE following communications were read:—

1. 'The Upper Carboniferous Rocks of West Devon and North Cornwall.' By E. A. Newell Arber, M.A., F.L.S., F.G.S.

After a reference to the previous work in the area, the author gives a description of the coast-sections, which display a highly-disturbed sequence of Upper Carboniferous rocks. Special attention is paid to two lithological types:—The Carbonaceous Rocks which contain inconstant and impersistent beds of the impure smutty coal, known locally as 'culm': these beds have yielded plant-remains; and the Calcareous Rocks, partly of marine and partly of freshwater origin, consisting of well-marked, impersistent bands of impure limestone, and conglomeratic beds of calcareous nodules embedded in shales. One of the limestone-bands, the Mouthmill Limestone, is marine, and contains an abundant fauna; while in others the only fossils are *Calamites Suckowi* and *Alethopteris*

lonchitica. The calcareous nodules, which are not ferruginous, occur in thin shale-beds, 2 to 3 feet thick; they commonly contain a marine fauna, with goniatites, lamellibranchs, and fish-remains. The fossil flora and the horizon of the beds are next discussed, lists of plants being given from various localities; and the conclusion is reached that the rocks belong to the Middle Coal-Measures, and that there is no evidence of Upper Coal-Measures. Conglomerates of waterworn pebbles are also present, the pebbles being derived from plant-petrifications, possibly of earlier age than the sandstones and shales in which they occur. Two distinct and unmingled faunas are present in the rocks: one consists of freshwater lamellibranchs, and the other of marine fishes, cephalopods, and lamellibranchs; and the evidence as to horizon obtained from them agrees with that yielded by the plant-remains. The paper concludes with a bibliography.

2. 'The Titaniferous Basalts of the Western Mediterranean.' By Dr. Henry S. Washington, For. Corr. Geol. Soc.

In 1905 the author visited the volcanic districts of Catalonia, Sardinia, Pantelleria, and Linosa. He recognizes the existence in this region of a hitherto unrecognized petrographic province, in which the basalts contain a remarkably-high percentage of titanium. The rocks are of Tertiary age. Labradorite, augite, and olivine are the essential minerals, with titaniferous magnetite and apatite, and in some cases subordinate nepheline. The rocks vary from compact to highly vesicular; they are not conspicuously porphyritic, though small phenocrysts of augite and olivine, and occasionally feldspar, are sometimes present. Olivine-nodules occur in some cases. The textures are those common in basalts, and ophitic relations are almost wanting. A series of analyses is given: the silica varies from 44 to 52 per cent., the alumina from 12 to 19, and the titanium-oxide from 2 to 5. The last oxide appears to be chiefly contained in titaniferous magnetite. Phosphoric pentoxide is distinctly abundant and appears to vary, as the titanium does, with the amount of iron-oxides. Soda is dominant over potash in every instance. The augites are almost, if not quite, colourless. The classification of the rocks according to the quantitative system is discussed, and the majority of the basalts fall into the 'Salfemane' class, the exceptions being some of the Sardinian basalts and that of Graham's Island.

The extent of the region is as yet problematical, and the author points out that along the southern coast of France there are several 'basaltic' volcanoes, and it is possible that these may eventually turn out to be connecting-links between the rocks of Sardinia and those of Catalonia, or possibly extrusion southward is indicated by the occurrence of phonolite at Maid Gharian near Tripoli.

The Impulsive Motion of Electrified Systems.

By G. F. C. SEARLE.

Correction:—

On page 121 of the Philosophical Magazine for January 1907:
 for "Since $\mathbf{e}$ is at right angles to the plane of $\mathbf{r}$ and $\mathbf{u}$ "
 read "Since $\mathbf{ie}$ is in the plane of $\mathbf{r}$ and $\mathbf{u}$ "

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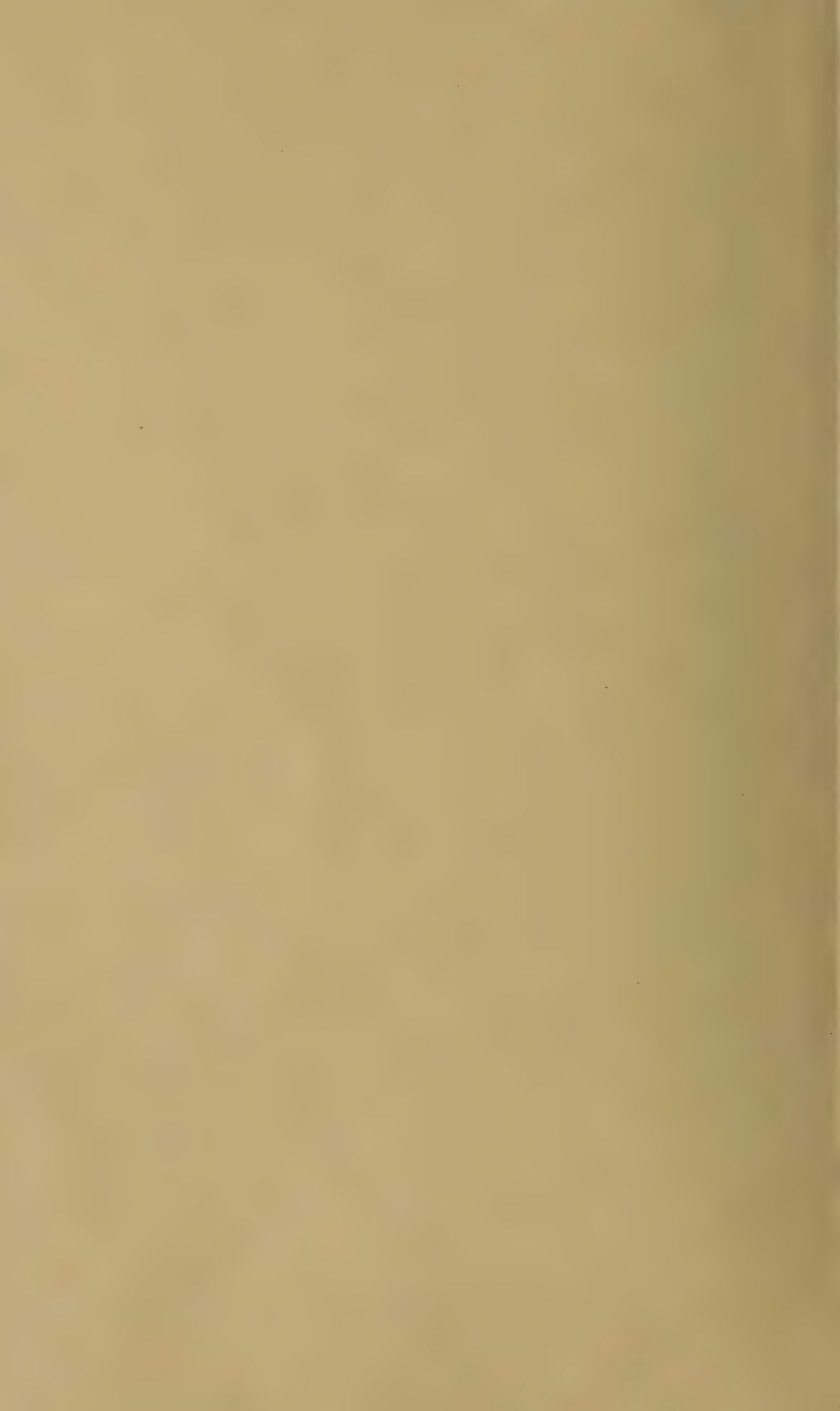


FIG. 1.

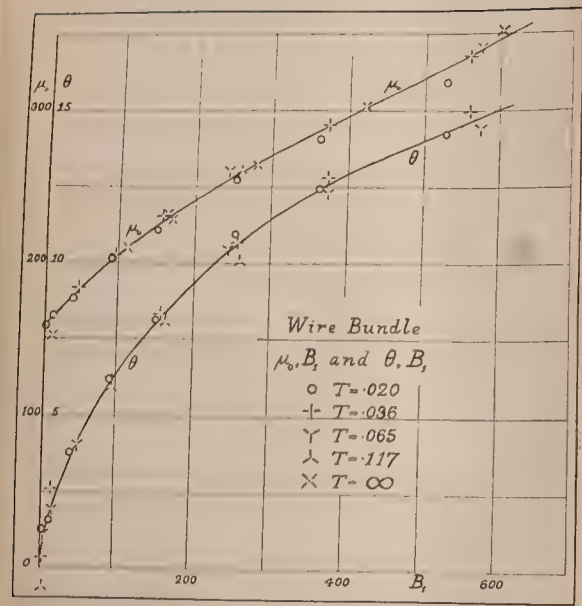


FIG. 3.

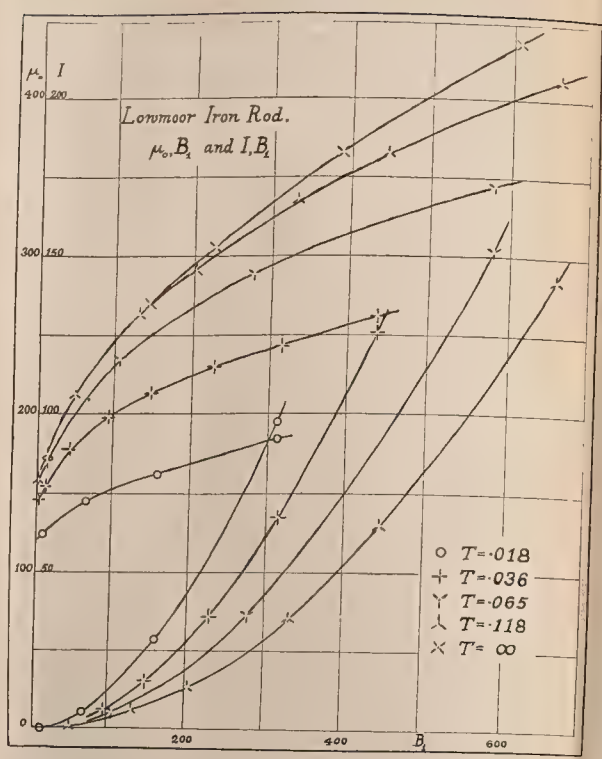


FIG. 5.

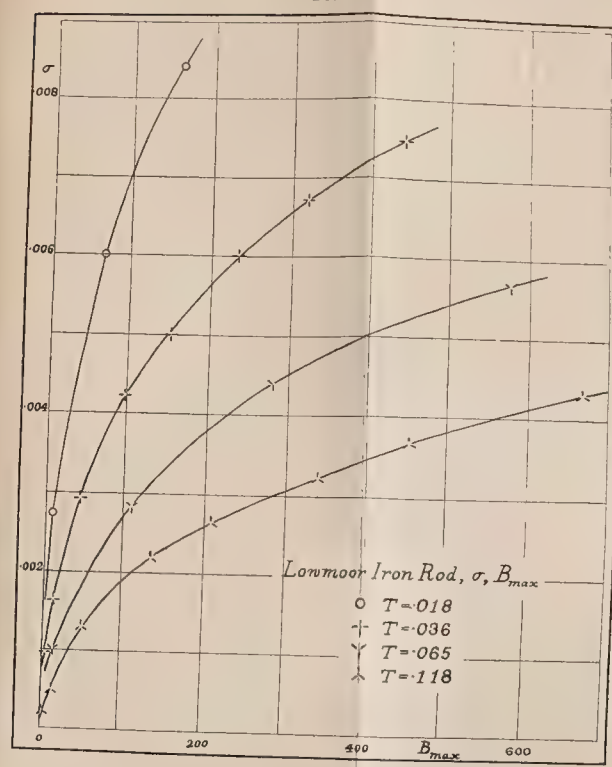


FIG. 2.

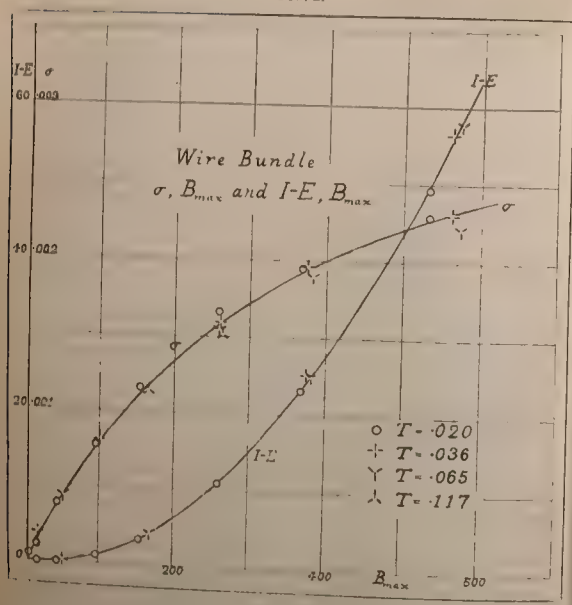


FIG. 4.

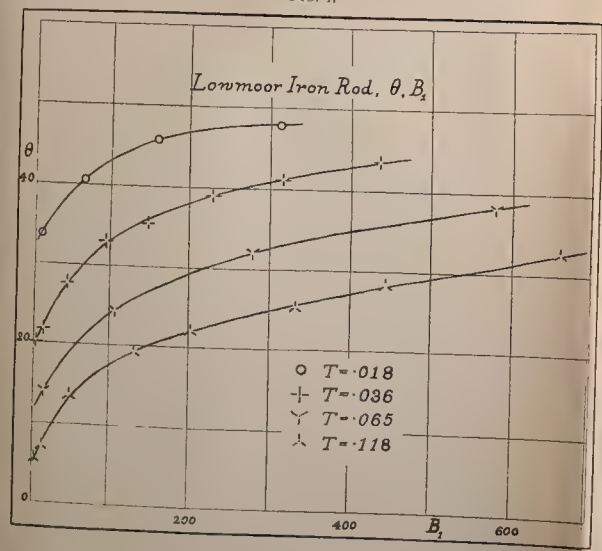
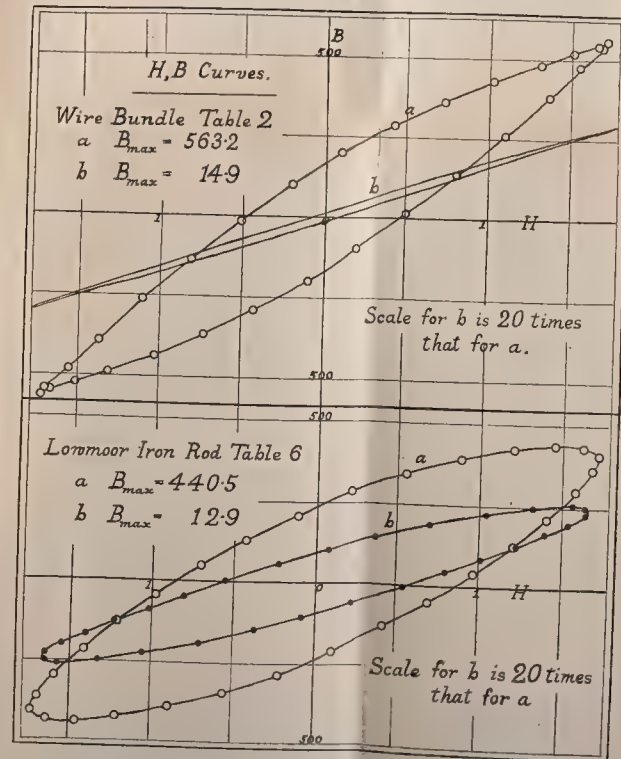


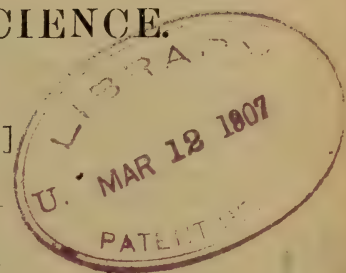
FIG. 6.



THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

MARCH 1907.



XXII. *Chemical Effects of the Electric Discharge in rarefied Hydrogen and Oxygen.* By Rev. P. J. KIRKBY, *Fellow of New College, Oxford*.*

IN the Philosophical Magazine for January 1905 I described the chemical effect of passing electricity through a mixture of hydrogen and oxygen at low pressures and between electrodes whose distance apart varied between $\cdot 25$ cm. and $1\cdot 5$ cm. I explained the results by the hypothesis that the gaseous ions as they pass from their origin to the electrodes, break up into uncharged atoms those molecules of the gas which they strike under favourable conditions, and thus set the atoms free to group themselves afresh into molecules of water-vapour. The experiments showed that when the circumstances of the discharge were constant, the number of molecules of water was proportional to the quantity of electricity that passed and so to the number of ions which reached either electrode, and was independent of the size and material of the electrodes. The formula which approximately symbolized the results seemed to indicate that the chemical combination was most intense within the region of the cathode fall of potential, and that it went on to a smaller extent elsewhere, though the discharge was too short to throw this latter effect into relief.

These conclusions were partly confirmed by some experiments made with an apparatus which will be described at the

* Communicated by the Author.

Phil. Mag. S. 6. Vol. 13. No. 75. March 1907.

Y

end of this paper. These seemed to show that when the electrodes are widely separated the chemical combination within the positive column may become of equal importance with that within the region of the cathode fall of potential.

To examine completely the part played by the gaseous ions in causing chemical action, one ought to determine the chemical effect of their passing through a centimetre of hydrogen and oxygen at various pressures and under various fields of uniform force. But unfortunately the number of molecules of water formed by the motion of a pair of ions is so small* that no results can be obtained by using the small steady currents that can be generated by the help of ultra-violet light in a uniform field of force, even though the force is increased—the current increases with the force owing to the multiplication of ions by collision—to its sparking value.

Recourse must therefore be had to the large steady currents between parallel plate electrodes that are attended by a glow. And though with such currents† the electric force varies greatly in different parts of the discharge, yet, if the distance between the electrodes is great enough, it nevertheless becomes practically uniform towards the anode, the region of uniformity being known as “the positive column.”

It was therefore intended by these experiments to determine the amount of chemical action produced by the passage of a coulomb in a centimetre of the positive column when an electric discharge was passed through a mixture of hydrogen and oxygen in equivalent proportions, and also to determine the electric force in the positive column. These determinations would disclose the chemical effect of ions moving through a centimetre in hydrogen and oxygen at a known pressure and under a known uniform force.

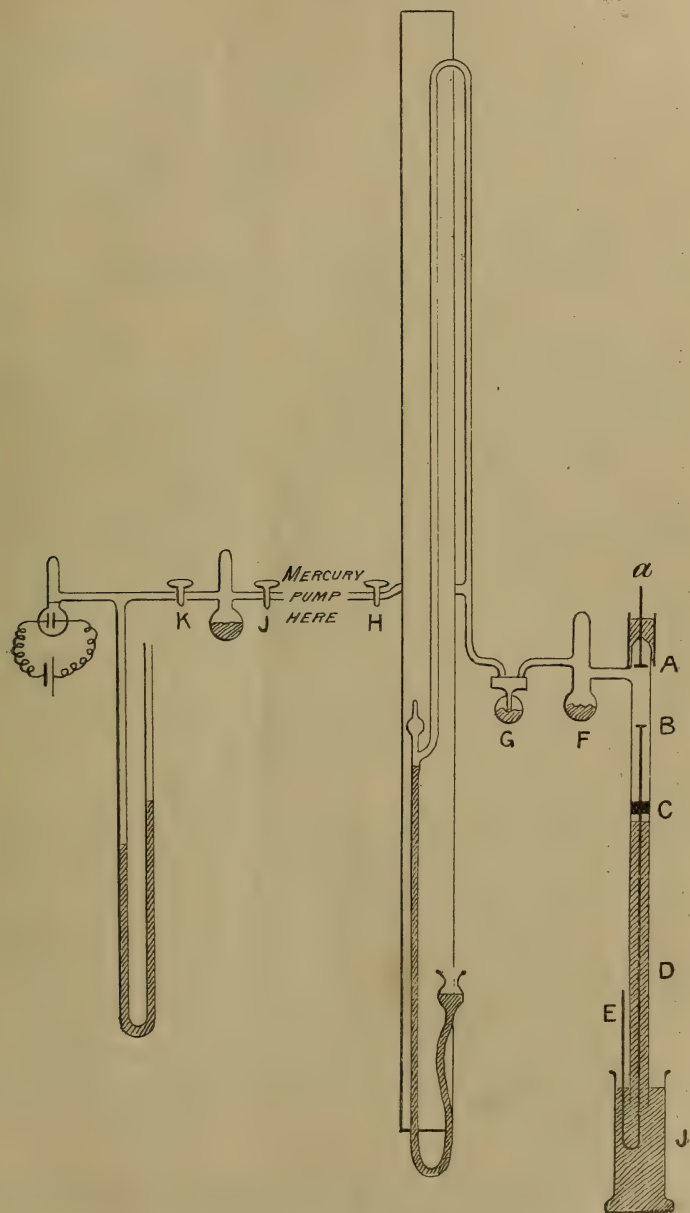
Description of the Apparatus.

The apparatus designed for this purpose is illustrated diagrammatically in fig. 1. ABCD is a long vertical piece of glass tubing about 115 cms. in length and 2.2 cms. in diameter dipping into a jar of mercury J. The end A

* When a luminous discharge passed between electrodes whose distance apart did not exceed 1.5 cm. this number never exceeded 10.

† In the case of very small currents attended by a glow Professor Townsend has shown that the force is practically uniform. These, however, would be too small for our purpose and far too unsteady. For the complete theoretical solution of the distribution of force between two parallel plate electrodes when a steady self-maintained discharge is passing, see Townsend, *Phil. Mag.* March 1905.

Fig. 1.



was fused together round a piece of iron wire *a* which carried the zinc cathode whose diameter was about 2 cms. Iron was used instead of platinum to avoid the chemical effects which might arise if the cathode became hot and heated the platinum sufficiently (P. J. Kirkby, *Phil. Mag.* Oct. 1905). The glass did not stick perfectly to the iron, but the joint was made thoroughly air-tight by placing some mercury above it in a piece of rubber tubing as shown in fig. 1. The anode, near the point B of the figure, was also of zinc*, its diameter was 2 cms., and it was supported by one end of an iron rod which passed up through the glass tubing and was bent at the bottom so that its other end E protruded above the mercury in the jar. At the point C, inside the glass tube, was a steel guide, fixed by springs to the inner wall of the glass tube. By raising or lowering the end E the anode B could be adjusted at any distance from A not exceeding 16 cms. A piece of glass tubing of large bore, fixed horizontally, led from a place in the vertical tube AD near the cathode A into the vessel F containing pentoxide of phosphorus, and permitted the water-vapour, as it was formed, to disappear rapidly. Glass tubing connected F to G, another drying vessel, which was similarly connected to two McLeod gauges (only one is shown in the fig.) and an ordinary manometer (not shown), which enabled any pressure to be measured. All these, which were in permanent connexion, were connected by a single tube to a large mercury pump; but the tube leading to the pump could be closed by a stopcock H, which thus greatly reduced the space where the falls of pressure were measured and so increased the accuracy of the measurements.

Hydrogen and oxygen were generated together by the electrolysis of pure barium hydrate contained in a small sealed glass vessel which was provided with a U-shaped manometer; were passed on through the stopcock K to a drying chamber, and then were admitted by the stopcock J to the mercury pump and by H to the rest of the apparatus. Every joint was of sealed glass, and the stopcocks and the vessel G were protected by mercury so that every part of the apparatus was thoroughly air-tight. By working the pump with the stopcock H open and so evacuating the apparatus, and then by replenishing it with pure mixed gases, the air or gas vitiated by use could be eliminated to any desired extent.

* Zinc electrodes were used because they are most easily worked; and I have shown that the effects under investigation are independent of the material of the electrodes.

The space occupied by mercury and the part of the drying vessels occupied by pentoxide of phosphorus are indicated by shading.

The electrodes A, B were connected by means of the rods α , E to the ends of a battery of small lead cells, whose voltage lay between 1800 and 1900, through resistances including when necessary a megohm. And in the few cases when the battery failed to initiate the discharge, which however it always easily maintained, the discharge was started by an induction-coil whose secondary had its ends connected to wires surrounding but not touching the cathode and anode.

Method of Experiment.

Preliminary experiments (made with air) showed that the force in the positive column is independent of the current so long as the pressure is constant, but changes, although less than proportionally, with the pressure, and that the force at other points of the discharge changes with the current as well as with the pressure.

Nevertheless, by keeping the current and pressure constant the potential difference between the electrodes is made a function of the distance between the electrodes. Hence, if with the distances between electrodes as abscissas and potential-differences of electrodes as ordinates a curve is constructed corresponding to a constant pressure and current, the electric force in the positive column corresponding to that pressure and current can be determined by the final slope of the curve, when the curve becomes straight.

The chemical reaction in the positive column should be exhibited by a similar method. For I have already shown in previous papers that the chemical reaction, represented by the fall of pressure Δp , is proportional to the quantity of electricity ΔQ which passed during the discharge, the circumstances of the discharge being constant. Accordingly, $\frac{\Delta p}{\Delta Q}$ should be a function of the distance between the electrodes when the current and pressure are constant. This will certainly be the case if my hypothesis is correct that the chemical action is the outcome of the dissociation into atoms of the molecules of the gas by the impacts of the gaseous ions, for the motion of these ions is wholly determined by the electric force. If, then, a curve is plotted whose abscissas are the distances between the electrodes and whose ordinates are the values of $\frac{\Delta p}{\Delta Q}$ for a given pressure and current, the chemical reaction per centimetre in the positive column would

be determined by the final slope of the curve, if the curve becomes ultimately straight.

To construct such curves it is necessary to find the chemical effects of passing the same current through hydrogen and oxygen at the same pressure but with different distances between the electrodes, and to find the corresponding potential-differences of the electrodes. As the discharge passes the pressure steadily falls and the current changes slightly, generally falling steadily. But if the variations in pressure and current are fairly small, the results will apply with sufficient accuracy to that pressure and that current which are the mean values of the changing pressure and current.

Several series of experiments were accordingly made, each series with approximately the same mean pressure and current. Moreover, the mean current was nearly the same throughout, being in almost every case within 5 per cent. of the value $\cdot 00275$ ampere, which was selected as the norm partly because it was accurately read on the ammeter, partly because it was easily maintained for a sufficient time in the hydrogen and oxygen at the required low pressures, being neither too great nor too small.

The current, as has been said, slowly changed, but it was stopped before it diverged more than about 5 per cent. from its mean value. Thus the currents used in every one of the experiments recorded in the tables below lay always within about 10 per cent. of $\cdot 00275$ ampere. It was an easy matter to adjust the current by means of external resistances except at very low pressures. But it is not so easy to secure the desired mean pressure.

Let p_0 mms. denote the pressure before the current was started, and p_1 the pressure after it was stopped and the water-vapour resulting from the discharge had been wholly absorbed. Then the mean pressure

$$p = \frac{1}{2}(p_0 + p_1).$$

Thus, to carry out an experiment with a given mean pressure, p , p_0 had to be estimated so that after the experiment p should reach the desired value. This is a difficult result to achieve, partly because it is hard to adjust pressures by means of a mercury pump with sufficient delicacy, and partly because the adjusting the current so that its mean value should lie near enough to $\cdot 00275$ ampere frequently involved a considerable fall of pressure. Nevertheless, the mean pressure for almost all the experiments in each series tabulated below was constant within 3 or 4 per cent.

The fall of pressure, Δp , due to the absorption of the

water-vapour ($=p_0 - p_1$), was always large enough to measure with sufficient accuracy and was often as much as 10 per cent of p , and sometimes much more. But it is safe to assume that no errors are involved in the results by taking Δp so large, provided that the current keeps steady or changes but little, for then the distribution of force throughout the discharge cannot be greatly affected. And the consistency of the results given below is sufficient in itself to justify this conclusion.

The difference of potential, X , between the electrodes during the discharge was found thus. Let C denote the current, Z the voltage of the battery, and R the whole ohmic resistance of the circuit exclusive of the gas, then

$$X = Z - CR,$$

and by taking the mean value of C we get the mean value of X .

This method of determining X was found to be far more accurate and satisfactory than the use of an electrostatic voltmeter, because both C and X were thus instantly determined by a single observation, the ammeter being dead-beat.

The quantity ΔQ of electricity which passed through the gas during the discharge was obtained by multiplying the mean value of the current by the time during which the current lasted.

The duration of the discharge varied greatly. It was determined by the rate at which the current changed, for the current was not allowed a variation of more than 5 per cent. on each side of its mean. It ranged from a few seconds for the lowest pressures to 100 for the highest: the lower the pressure, the more rapidly the current changes.

Results of Experiments.

Each of the following tables (p. 297) contains the results of a series of experiments made with practically the same mean pressure; for the mean pressure lay in almost every case within 3 or 4 per cent. of the pressure given at the head of the table.

To recapitulate:—

p = half the sum of the pressures, measured in mms. of mercury, before and after the passage of the electricity, the water-vapour being completely absorbed, $= \frac{1}{2}(p_0 + p_1)$.

Δp = the fall of pressure in mms. caused by the passage of the electricity $= p_0 - p_1$.

ΔQ = the quantity of electricity measured in coulombs which passed, namely, the product of the mean current and its duration.

X = the mean potential-difference of the electrodes during the discharge.

D = the distance in centimetres between the parallel plate electrodes.

The mean current used throughout was approximately constant and 0.0275 ampere.

The capacity * of the apparatus where Δp was observed was 840 c.c.

These tables show that when the distance D between the electrodes is large, the weight of water formed per coulomb passed through the gas is greater for the lower pressures than for the higher. Thus the amount of the chemical reaction is by no means proportional to the concentration of the gases. On the contrary, the amount of the reaction per coulomb is absolutely increased, and very substantially increased, by diminishing the pressure down to about 1.4 mm. provided that the distance between the electrodes exceeds a certain value. When the pressure falls below 1.4 mm., the reaction per coulomb becomes less. The greatest effects were observed at 1.4 mms. pressure.

The tables are illustrated by the curves now given (p. 298).

The abscissas in these curves are the distances between the electrodes measured in centimetres. The ordinates are the values of $\frac{\Delta p}{\Delta Q}$, and therefore measure the chemical reaction per coulomb passed through the mixed gases. Each curve is determined by one of the tables and so corresponds to a definite mean pressure, which is marked upon it. But, to avoid confusion, only four of the tables are represented. With a battery whose voltage was less than 2000 it was not found possible to extend the curve for the pressure 9.3 mms., even with the help of an induction-coil in starting the discharge, because the current did not at once assume its steady value, and so accurate measures both of ΔQ and of X were out of the question. It will be seen that $\frac{\Delta p}{\Delta Q}$ is always roughly the same when D is very small. For a given pressure $\frac{\Delta p}{\Delta Q}$ rises in a well-marked way up to

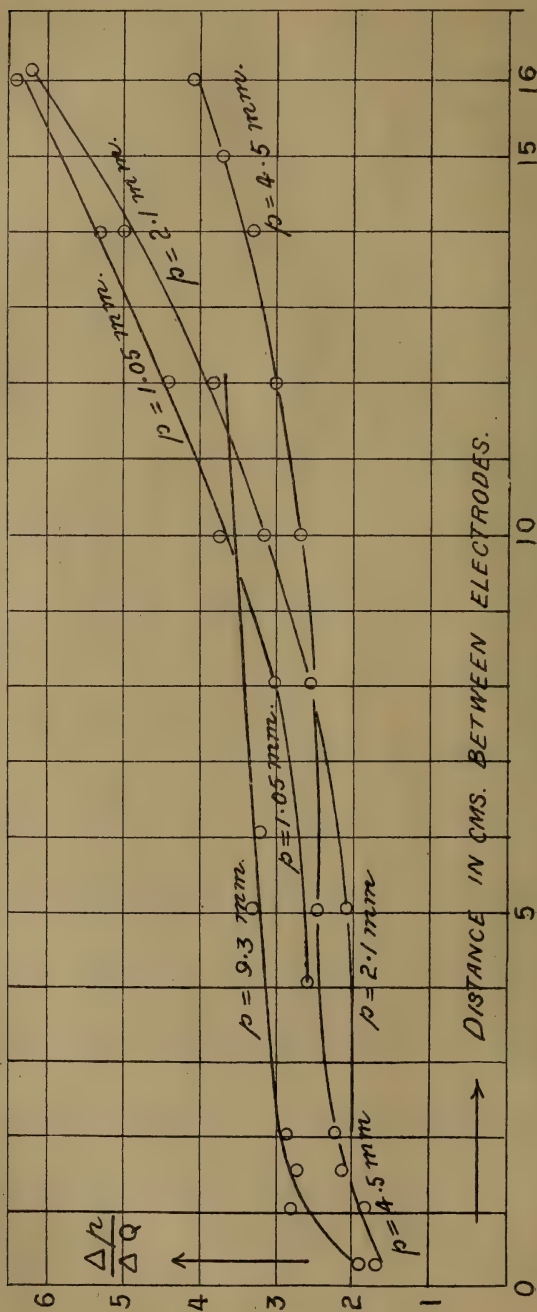
a certain point, while the distance D between the electrodes increases to about 2 cms. After that, the rise in $\frac{\Delta p}{\Delta Q}$ is very

* This was measured by comparing it with the capacity of a glass bulb through an obvious application of Boyle's Law.

gradual and almost imperceptible during a large increase in D. But in the case of the lower pressures, $\frac{\Delta p}{\Delta Q}$ eventually increases once more in a well-defined manner with increase of distance.

Table I., $p=9\cdot3$.			Table V., $p=2\cdot6$.		
D.	$\Delta p/\Delta Q$.	X.	D.	$\Delta p/\Delta Q$.	X.
·25	1·91	340	1·5	2·07	370
1	2·82	390	8	2·2	540
1·5	2·74	410	12	3·12	630
2	2·88	440	13	3·5	640
5	3·3	670	14	4·24	680
6	3·2	690	16	5·3	800
10	3·4	1000			
12	3·66	1260			
Table II., $p=5\cdot5$.			Table VI., $p=2\cdot1$.		
D.	$\Delta p/\Delta Q$.	X.	D.	$\Delta p/\Delta Q$.	X.
4	2·8	480	2	2·05	350
8	2·9	710	5	2·1	390
12	3·2	840	8	2·57	490
14	3·3	980	10	3·18	560
			12	3·82	600
			14	5·0	700
			16	6·3	800
Table III., $p=4\cdot5$.			Table VII., $p=1\cdot4$.		
D.	$\Delta p/\Delta Q$.	X.	D.	$\Delta p/\Delta Q$.	X.
·25	1·7	335	4	2·2	410
1	1·86	360	8	2·54	460
1·5	2·11	380	12	4·3	610
2	2·24	360	14	5·5	700
4	2·5	410	16	6·45	730
5	2·5	455			
8	2·6	570			
10	2·7	700			
12	3·0	710			
14	3·3	810			
15	3·7	870			
16	4·1	900			
Table IV., $p=3\cdot1$.			Table VIII., $p=1\cdot05$.		
D.	$\Delta p/\Delta Q$.	X.	D.	$\Delta p/\Delta Q$.	X.
4	2·27	470	4	2·56	440
8	2·4	590	8	3·0	500
14	3·57	710	10	3·72	550
16	5·24	810	12	4·4	610
			14	5·3	660
			16	6·4	700
Table IX., $p=\cdot65$.					
D.	$\Delta p/\Delta Q$.	X.			
4	3·2	590			
8	3·0	610			
10	3·0	590			
12	3·5	630			
14	4·3	660			
16	4·5	690			

Fig. 2.



As an illustration, it may be seen from the curves that when the distance between the electrodes is 12 cms. the quantity of water formed per coulomb is perceptibly greater at the low pressure of 1 mm. than at the pressure 4 mms., and even than at the pressure 9 mms.

The points corresponding to a given pressure fall upon a curve, or at least do not deviate further from it than errors of observation (about 5 per cent.) should allow; and thus the remark made above is justified, that if the current and pressure are kept constant, the chemical action per coulomb that passes is a function of the distance between the electrodes.

It should be noticed that when D exceeds 10 cms. the curve corresponding to the pressure 1.05 mm. lies above that corresponding to 2.1 mms. which lies above that corresponding to 4.5 mms.; but that the curve corresponding to .65 mm. (not plotted in fig. 2) lies *below* those corresponding to 1.05 mm., 2.1 mms.

Explanation of the observed effects.

I shall endeavour to show that the phenomena presented by the forms and relative positions of these curves can be completely explained by the same hypothesis which I made in my previous paper and mentioned above, namely, that the ions moving swiftly under the electric force dissociate into uncharged atoms those molecules which they strike under favourable conditions and with sufficiently high velocity. But first it is necessary to recall the theory of the electric discharge in any gas between two parallel plate electrodes as established by Professor J. S. Townsend.

When the current exceeds a certain small value there is an intense field of force near the cathode, namely, the region of the cathode fall of potential where ions are generated by the impacts of both the positive and negative ions with the neutral molecules of the gas. From this region of intense force—due to the accumulation near the cathode of the more slowly moving positive ions—the potential rises more gradually up towards the anode: and the electric force tends to become uniform if the distance between the electrodes is great enough. There seems to be no reason to suppose that a sudden rise of potential occurs at the anode (Townsend, *Phil. Mag.* June 1906), as has been supposed. The region of approximately uniform* force terminating at the anode and called “the

* Although the number of positive ions which emerge from the positive column during a discharge is a very small fraction of the number of negative ions which traverse it, yet the numbers of ions of

positive column" is not sufficiently intense to cause the negative ions that pass through it to ionize the gas except to a small extent.

Thus in the case of a gas where no chemical action takes place far fewer positive ions than negative pass through the positive column per second. Hence any chemical action that is observed in the positive column when a discharge passes through hydrogen and oxygen should on our hypothesis be attributed to the motion of negative ions, unless the chemical union of these gases is attended by the generation of ions. Now it is true that the explosive union into water of hydrogen and oxygen is attended by the formation of ions; but the number of ions formed is incomparably smaller than the number of molecules of water (C. E. Haselfoot and P. J. Kirkby, *Phil. Mag.* Oct. 1904). We conclude then that any chemical action that occurs in the positive column is exclusively due to the impacts against the gaseous molecules of the negative ions. But before estimating the chemical action per centimetre of the positive column, it is convenient to investigate the force there upon which depends the motion of the ions.

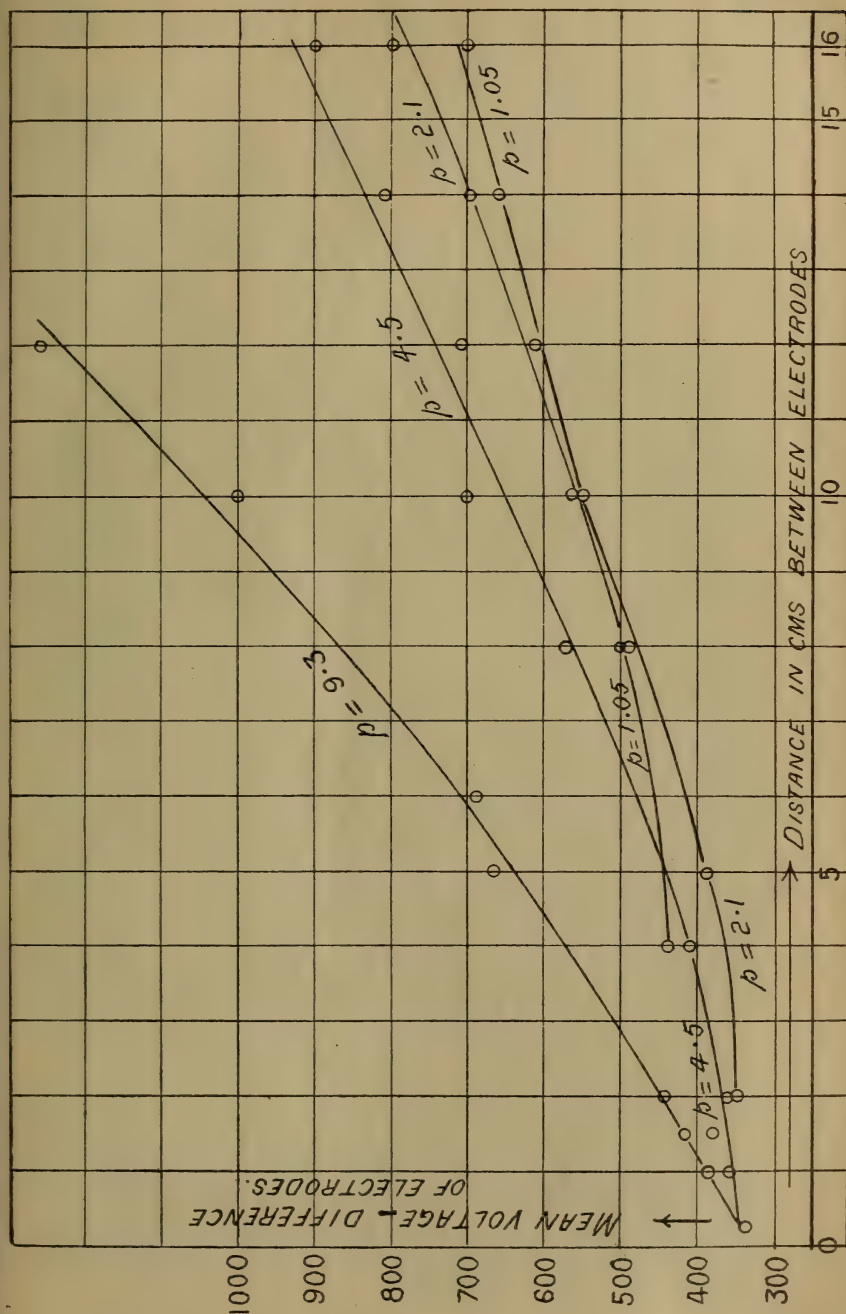
Estimation of Electric Force in the Positive Column.

To determine approximately the electric force in the positive column during experiments made at the same mean pressure, curves have been drawn having as abscissas the distances between the electrodes and as ordinates the mean potential difference in volts of the electrodes. Each of these curves applies to one pressure and was constructed from the numbers in one of the tables. But to avoid confusion only four of them are shown in fig. 3, and these four are each derived from the same table which contributed one of the

each kind within it are always almost exactly equal. For the constancy of the force in the positive column requires that the stream of negative ions has the same density as the stream of positive ions. The former probably moves with a much greater velocity than the latter, since the pressure is so low; hence the number of negative ions which traverse the positive column per second is, in the same proportion, greater than the number of positive. Thus if u , v are the velocities of the streams of negative and positive ions, C_1 , C_2 the currents in amperes carried by these streams, so that $C_1 + C_2$ is the whole current, and if A is the cross-

section of each stream, the equation $\frac{dX}{dx} = 4\pi\rho$ becomes in the positive

column $\frac{dX}{dx} = \frac{4\pi \times 9 \times 10^{11}}{A} \left(\frac{C_1}{u} - \frac{C_2}{v} \right)$, X being in volts per cm. Hence if X is nearly constant, $C_1 : C_2 = u : v$ almost exactly. I assume in what follows that no positive ions are generated near the anode, the explanation does not essentially depend on that assumption at all (see below).



curves in fig. 2. Thus the curves in fig. 2 correspond to those in fig. 3, and apply to the same pressures.

The curves tend to become straight when the distance between the electrodes exceeds a certain value dependent upon the pressure and current. The explanation of this is that the electric force in the positive column (defined as the region terminating at the anode where the electric force is approximately uniform) is independent of the distance between the electrodes. But that distance must be great enough to allow the positive column to be established. The region where ions are generated by collision lies near the cathode; and though the highest activity of reproduction within that region is limited to a narrow strip close to the cathode, yet an appreciable number of ions must be produced outside the domain of highest activity. This will be especially the case at the lowest pressures when the mean free paths of the negative ions are more considerable. In fact the influence of the cathode fall is propagated to a distance of some centimetres from it when the pressure is so low as 2 mms. (See the curves p. 437, J. J. Thomson, 'Conduction of Electricity through Gases'; p. 533 in 2nd ed. For a discussion of the distribution of force see c. xvi. *ib.*) But when the distance D between the electrodes is so large (5 or 6 cms. in the curves just referred to) that the region near the anode is well removed from the region where ions are generated, so that a positive column (as defined above) exists, any further increase in D merely adds to the length of the positive column.

Let the electric force in the positive column be denoted by Y , then, if the distance between the electrodes is increased by δ and their potential difference by $Y\delta$, the pressure being unchanged, the same current will pass as before and nothing will be altered within the discharge except the length of the positive column, which is increased by δ : the positive column remains unchanged in character. (See J. S. Townsend's paper, already referred to, *Phil. Mag.* March 1905.)

Thus the electric force in the positive column for the pressures of these experiments can be inferred from the slope of the curves of fig. 3, and others plotted similarly, when they became straight.

The electric force Y in the positive column at various pressures, determined in this way, are given in the following table. It is obvious, however, that the errors of estimation may be considerable, so that the table cannot claim to be more than roughly true except perhaps for the lowest pressures, for the curves corresponding to these are accurately determined by the observations.

Table X.

<i>p.</i>	<i>Y.</i>
9.3	80
4.5	0
2.1	40
1.05	25
.65	15

It must be observed that the theory that, with a given pressure and current, the force in the positive column is independent of its length, necessitates the conclusion that extremely few positive ions are generated within it in comparison with the number of negative ions that traverse it. To make this obvious, let us suppose that a negative ion by collision breaks up one molecule into ions in going through the positive column which is 5 cms. long, so that half the current is formed within the positive column. Then, if the positive column were doubled in length, the force within it remaining the same, the passage of a negative ion through it would give birth to 3 other negative ions, so that $\frac{3}{4}$ of the current would originate within the positive column. In that case, if the circumstances of the discharge outside the positive column were unchanged, the current would be doubled, instead of remaining the same. Similar reasoning shows that it greatly to extend the positive column does not sensibly affect the force in it, a sensible proportion of the ions which traverse it cannot have been generated in it by collision. If then there is no anode fall of potential where positive ions might be generated, we conclude that comparatively few positive ions pass through the positive column; and otherwise, that all but a small percentage of the positive ions within it have to pass through its whole length.

Estimation of the Chemical Action per centimetre of the Positive Column.

I have discussed the theory of the positive column with reference to the curves of fig. 3, before estimating the chemical action per centimetre of the positive column by means of the curves of fig. 2, because the curves of fig. 3 enable us to interpret those of fig. 2. Their relation is well illustrated

by the curves in each figure corresponding to the pressures 2.1, 1.05 mms.: the pair of curves corresponding to each of these pressures begin their approximately uniform upward slope when the distance between the electrodes exceeds about the same value. If the chemical combination is the result of, and is proportional to, the number of molecules dissociated by the collision of the ions, the proportion contributed to the whole observed chemical reaction by each strip of the discharge must depend upon the electric force within that strip. That being so, we could only account completely for the forms of the curves of fig. 2, just as we could only account for the forms of the curves of fig. 3, by a complete knowledge of the distribution of force in the neighbourhood of the cathode. But the portions of the curves of both figures, where their slope is practically constant, are explained on the same lines. When the distance between the electrodes is great enough, any further separation of them, the current and pressure constant, merely extends the positive column without altering its character. That being so, the final and nearly uniform slope of the curves of fig. 2 gives the chemical action per coulomb that takes place in a centimetre of the positive column. Now this chemical action cannot be due to the positive ions which are generated by collision within the positive column: if it were, the slope of the curves (fig. 2) would not become constant, but would become proportional to the length of the positive column, and the chemical action to the square of its length. Hence unless positive ions are generated near the anode in large numbers, the chemical action within the positive column must be attributed exclusively to the negative ions; and then it is easy to calculate the number of molecules of water which result from the motion of a negative ion through a centimetre of the positive column.

In the first place, let W denote, as in my previous papers, the number of molecules of water, whose formation is the effect of the motion to the electrodes of each pair of ions; then it is very easy to show that, with the present apparatus,

$$W = 3 \frac{\Delta p}{\Delta Q} \text{ almost exactly.}$$

For let N be the number of molecules in a c.c. of gas under normal temperature and pressure, then in a vessel of capacity S a fall of pressure Δp means the disappearance of $\frac{N \times S \times \Delta p}{760}$ molecules, while ΔQ coulombs is carried by

$\Delta Q \times \frac{3 \times 10^9}{e}$ ions, e being the electrostatic charge on each.

If then we divide the number of molecules of hydrogen and oxygen which disappeared by the number of ions which reached each electrode and observe that $Ne = 1.22 \times 10^{10}$ (Townsend, *Phil. Trans.* 1899), while in the present research

$S = 840$ c.c., we get the number $4.5 \frac{\Delta p}{\Delta Q}$. But 3 molecules of hydrogen and oxygen go to form 2 molecules of water. Therefore

$$W = 3 \frac{\Delta p}{\Delta Q}.$$

That being so, if we find the slope (increase of ordinate per centimetre-increase of abscissa) of one of the curves of fig. 2, and multiply it by 3, we obtain the number of molecules of water which result from the motion of a negative ion through a centimetre of the positive column corresponding to the curve in question. Let w denote this number of molecules; then w depends both on the pressure p of the gas (namely that marked on the curve) and on the electric force Y within the positive column. In fact w is the chemical effect of the motion of a negative ion under the force Y through hydrogen and oxygen at the pressure p .

The fact that the chemical action per coulomb is for large distances between the electrodes so much greater when the pressure is very low than when the pressure is much higher, is now easily explained by the hypothesis here adopted, namely, that the ions dissociate into uncharged atoms those molecules which they strike under favourable conditions and with sufficient velocity, thus setting the atoms free to recombine into water. At the higher pressures the free paths of the ions are smaller than at the lower pressures. Now an ion acquires its velocity of impact by moving freely under the electric force between successive impacts. Hence at the lower pressures an ion has greater velocities of impact, unless the electric force is reduced in the same proportion with the pressure. But in the positive column diminution of force does not keep pace with diminution of pressure. Accordingly, though at the lower pressures the number of the impacts made by an ion in moving through a centimetre is smaller than at a higher, yet the velocities of impact are greater. Hence it follows that more molecules may be dissociated per centimetre in the positive column at a lower than at a higher pressure, the fewness of the impacts at the lower pressure

being more than balanced by the higher velocities of impact.

But to be precise, it follows from our collision-hypothesis that w , p , Y are connected by the functional relation

$$\frac{w}{p} = f\left(\frac{Y}{p}\right).$$

A simple proof of this is the following, which is practically the same as one given by Professor Townsend in connexion with his theory of the genesis of ions by collision.

Let us suppose that a mass of hydrogen and oxygen is contained in a gas-tight vessel formed by closing a straight glass tube with two metallic electrodes at right angles to the axis of the tube and fitting the tube exactly. Let one of the electrodes be movable like a piston so that, while the mass of gas remains constant, the distance x between the electrodes can be varied. Let the electrodes be kept at a constant potential-difference, and let an ion cross from one to the other in the field of uniform force Y . Then when x

is changed $\frac{Y}{p}$ remains constant, for both Y and p vary inversely as x . But for all values of x the number of collisions made by the ion in crossing the gas is the same, and the circumstances, including the velocity, of each impact are invariable, because the diminution of force is exactly balanced by increase of free path or *vice versa*. Hence the whole number of dissociated atoms and therefore of molecules of water that result is constant. Therefore the number of molecules of water formed by the passage of the ion through a centimetre is inversely proportional to x . But p is also inversely as x .

Therefore $\frac{w}{p}$ is constant while $\frac{Y}{p}$ is constant.

Hence

$$\frac{w}{p} = f\left(\frac{Y}{p}\right).$$

The following table gives the values of w and Y for four pressures at which they can be determined from the curves with greatest precision, although the errors of determination may be larger than (*e.g.* double) the errors of observation, which may be taken to be about 5 per cent. In the last two columns

the values of $\frac{w}{p}$, $\frac{Y}{p}$ are given.

Table XI.

p .	w .	Y .	w/p .	Y/p .
4.5	.75	50	.17	11
2.1	1.5	40	.7	19
1.05	1.3	25	1.24	24
.65	.85	15	1.3	23

The last two rows of the table show an interesting agreement of the observations with the theory advanced to explain them. The fact that under certain conditions the chemical action per coulomb can be increased by lowering the pressure down to a certain point, after which the chemical action is diminished, is thus explained. In fact the theory enables one completely to account for the complicated and paradoxical effects observed and to interpret the forms and explain the relative positions of the curves (such as those of fig. 2) which give the values of $\Delta p/\Delta Q$ for the various pressures. It is unfortunate that the method here pursued does not lead to a greater variety of results. A more elaborate apparatus and a battery of far more than 2000 volts would be necessary to extend the range of observations. And even then the range of force in the positive column, however accurately that force might be determined, would be far too limited for the purpose in view.

In thus accounting for the observed phenomena I have assumed that only negative ions pass through the whole length of the positive column. But the explanation given does not *essentially* involve that assumption. It will still hold good in a slightly modified form if we suppose that positive ions originating near the anode (see J. J. Thomson, 'Conduction &c.' p. 489) traverse the positive column with chemical results as well as negative ions, provided that the proportion of one kind of ions to the other does not sensibly change with change of pressure and electric force in the positive column. In that case the equation

$$\frac{w}{p} = f\left(\frac{Y}{p}\right)$$

is still true, only w is now the number of molecules of water resulting from the dissociation effected by both kinds of ions in a centimetre of the positive column. In fact these

experiments throw no light on the question whether positive ions do or do not form a sensible proportion of the current within the positive column : whatever view is taken these chemical effects are equally well explained by the collision of the ions present in the positive column with the molecules of gas.

*Estimate of work required to dissociate
a Molecule of Oxygen.*

If one is right in attributing the chemical action within the positive column to the motion of negative ions, the numbers given above can be applied to estimate the work done in breaking up a molecule of oxygen into its atoms. The mean free path of a negative ion in hydrogen at 1 mm. pressure is $\frac{1}{3}$ cm. (Townsend, Phil. Mag. June 1902) and in air is about $\frac{1}{15}$ cm. Hence we shall not be far wrong in taking the mean free path of a negative ion in hydrogen and oxygen to be $\frac{1}{8}$ cm. At the pressure 4.5 mms. a negative ion will therefore make 36 collisions in passing through a centimetre. During this passage it dissociates enough molecules of the gas to produce .75 molecule of water. If we assume that to be the result of dissociating .37 molecule of oxygen, it follows that one dissociating collision occurs out of nearly 100 collisions made by a negative ion in moving through the mixed gases at pressure 4.5 mms. and under a force of 50 volts per cm. Now one free path out of 100 is very nearly 4.6 times the mean free path, which, for a negative ion in the case under consideration, is $\frac{1}{8}$ cm. It follows that to dissociate a molecule of oxygen cannot require more energy than that due to a fall of a negative ion through a potential difference of $\left(\frac{50}{36} \times 4.6 =\right)$ 6.4 volts. And since the charge

on a negative ion is nearly 3×10^{-10} in electrostatic units, the work required to dissociate a molecule of oxygen cannot exceed 6.4×10^{-12} ergs. Probably, however, this limit is rather high. For even if the molecules of hydrogen do not have to be dissociated as well as the molecules of oxygen before water can be formed, it is highly probable that of the dissociated atoms of oxygen some will recombine into molecules of oxygen. Hence the above limit may be a small multiple of the work which it is necessary to expend in order to separate the atoms of a molecule of oxygen.

Again, Professor Townsend has shown (Phil. Mag. Feb. 1901, p. 220) that the energy required to break up a molecule into ions is not greater than that gained by an ion in moving

down a potential fall of 5 volts. Now, the experiments recorded in this paper and previous papers prove that the breaking up of a molecule into atoms needs the expenditure of less work than the breaking of it up into ions, for the ions are generated by the collisions of the moving ions with the molecules of the gas, and at least 5 times as many molecules of water were always formed as pairs of ions. Hence, on these grounds we can fix 5×10^{-12} ergs, which is practically the same figure as before, as a superior limit to the work involved in separating the atoms of a molecule of oxygen.

Very little Ozone present.

Nothing has so far been said about the formation of ozone during the discharge, although it is known that that gas is always present. Some of the atoms dissociated by the colliding ions group themselves into molecules of ozone, but the number of these molecules can be safely neglected beside the number of molecules of water which are formed. For it has been repeatedly verified in the course of these experiments that the same results are obtained whether one uses freshly prepared hydrogen and oxygen or those gases after they have been already used till more than half has been made to combine into water. Now, if any appreciable quantity of ozone were formed, the error involved in the calculation by neglecting it could not in such circumstances escape notice.

Possible influence of the form of Apparatus.

A few of the observations recorded in this paper were made under conditions similar, in respect of pressure and distance between electrodes, to observations contained in my last paper on this subject. The latter were approximately covered by the formula

$$W = \frac{pD}{10} + \frac{3X}{800} \text{ (same notation),}$$

and with the present apparatus $W = 3\Delta p/\Delta Q$. It is therefore possible to make some comparisons between the two sets of observations. In the case of the smallest distance .25 cm. the agreement is good enough, but in the other cases discrepancies occur which cannot be accounted for by errors of observation, since they reach in two cases about 30 per cent. Now I have shown that these chemical effects are not dependent upon the nature of the electrodes, so that one is forced to conclude that the form of the apparatus plays a

considerable part in these reactions. This would be the case if the rate at which water was formed depended to an appreciable extent upon the amount of water-vapour present, for then the observed reaction would depend to some extent upon the rate at which the apparatus permitted the water-vapour to diffuse away from the region of the discharge. Now on our hypothesis the chemical reaction resulting from the motion to the electrodes of each pair of ions is very likely to be affected by the amount of water-vapour present, for water-vapour condenses on ions of both kinds (J. J. Thomson, 'Conduction, &c.,' pp. 153, 154), retarding their motion and increasing their effective mass. The result of this condensation may be to increase the chemical reaction : an impinging ion may be more likely to dissociate a molecule if the mass of the ion is thus increased up to a certain point even though its velocity of impact is diminished. Perhaps this is the reason why the final slope of the curve ($p=4.5$ mm.) of fig. 2 is not so uniform as that of the curves corresponding to the pressures 2.1, 1.05 mms. : the slight increase in the slope may be due to this, that as the positive column is extended the ions which traverse it have a greater chance of attaching water-vapour to them, for all but a small percentage of the ions in the positive column pass through its whole length. This effect, however, would tend to disappear at the lower pressures owing to the greater speed of the ions.

In view of these considerations and of the discrepancies alluded to above, the observations recorded in this paper cannot claim to be independent of the form of the apparatus except very roughly. Nevertheless, they agree among themselves with great precision, and therefore furnish an accurate record of the amount of chemical action which took place in the present apparatus in the various circumstances of the discharge.

Influence of the Current.

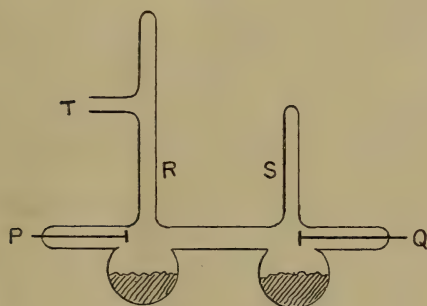
It was shown in the preceding paper that $\Delta p/\Delta Q$ is independent of the current for small values of the distance between the electrodes. This, however, should cease to be true when that distance is large, for then the distribution of electric force may be so modified by a large change in the current that the chemical reaction is appreciably changed. This conclusion was confirmed by observation. Thus, when the distance between the electrodes was 16 cms. and the current .0015 ampere was used, $\Delta p/\Delta Q$ proved to be 5.5

(X being 750) for the pressure 1.85 mm. ; whereas with the current $\cdot 00275 \Delta p / \Delta Q$ would have been 6.4 (X about 770).

Another Experiment.

At the beginning of this paper I referred to another apparatus. This was designed with the view of confirming my conclusion that the greatest intensity of chemical activity existed in the cathode fall. A section of it is shown in fig. 4 which is drawn fairly to scale.

Fig. 4.



It was made of glass tubing from the side of which two bulbs, roughly spherical in form, were blown. Pieces of platinum carrying aluminium electrodes were fused into the ends P, Q of the tubing. The vertical tubes R, S permitted the introduction of phosphorus pentoxide (shown by the shaded areas) into the bulbs without soiling the electrodes, after which R and S were closed at the top. The distance between the electrodes was 11.6 cms. and the diameter of each was 1.4 cm. The diameter of the bulbs was 4 cms. and their orifice roughly a circle of 3 cms. diameter. Hydrogen and oxygen entered the apparatus when evacuated of air by the side tube T.

The terminals P, Q of the electrodes were connected to a battery of 1200 volts and the pressure was sufficiently reduced to allow a current of about $\cdot 0026$ ampere to pass between them. The current was started by means of an induction-coil and was maintained by the battery for about half an hour. As the gas disappeared through chemical action it was replaced by fresh gas, the pressure varying between about 3 mms. and 7 mms., in such a way as to keep the current about the same as its initial value and continuous. The

current never exceeded $\cdot 0052$ ampere. The mean value of the pressure may be taken to be 3 or 4 mms.

The phosphorus pentoxide in the bulbs was carefully watched for the first symptom of absorbing water, and it was observed that the phosphorus pentoxide under the cathode was distinctly soiled by water-vapour before that which was under the anode appeared to be soiled at all. And later, when the latter showed also the yellow effect of absorbing water-vapour, the former continued to maintain a deeper colour.

The experiment proved that the chemical activity was greater in the neighbourhood of the cathode than elsewhere in the discharge.

[After obtaining this result some quantitative experiments were made. The values of $\Delta p/\Delta Q$ for several pressures were determined and compared with values obtained with the previous apparatus when the distance between electrodes did not exceed 1.5 cm., due allowance being made for change of capacity.]

This comparison showed a large increase in the chemical combination per coulomb, presumably due to dissociation within the positive column, and led to the present investigation.]

We may apply the curves of fig. 2 to obtain a rough estimate of the relative amounts of water-vapour contributed by the positive column and by the region of the cathode fall of potential during the experiment just described.

Taking the mean pressure to be 3.3 and the mean current $\cdot 00275$, our curves give, for $D=11.6$, $\Delta p/\Delta Q=3.5$. Hence $W=10.5$. Again, taking D about 1 cm., so as to eliminate the effect of the positive column, we obtain $\Delta p/\Delta Q=2$ about, and $W=6$, as the contribution of the cathode fall. The quantities of water-vapour formed in the positive column and in the cathode fall were therefore in the ratio 4.5 : 6. The former would diffuse about equally into both bulbs ; of the latter by far the greater part would go to the bulb beneath the cathode which thus would receive about four times as much as its companion.

I must express my thanks to Professor J. S. Townsend, to whom I am greatly indebted for some most valuable suggestions.

XXIII. *An Attempt to explain the Radioactivity of Radium.*
By Lord KELVIN*.

§ 1. **O**NE chief action concerned in radioactivity is the shooting out of electrions from a non-electrified solid or liquid body. In the equilibrium of kinetic averages in any solid or liquid, every individual electrion must occasionally have so high a velocity that it is shot out of the body. Hence every solid or liquid body has something of radioactivity.

§ 2. The Radium atom must, so far as we can at present judge, be assumed to have a special property of being adapted to store enormously more energy, by an electrion within it, than the atom of any other substance hitherto known to us. In a short article, published in the *Phil. Mag.* of Dec. 1905, I explained the plan of an atom by the purely Boscovichian assumption of mutual force in the line between the centre of the atom and an electrion anywhere within it, according to which there is for the electrion one position of stable equilibrium, near the boundary of the atom, with very small potential energy; and another position of stable equilibrium at the centre of the atom, with very great potential energy. For brevity I shall call the atom "loaded," when there is an electrion at its centre, or anywhere within the range of the stability of the central position: and I shall call the atom "unloaded," when there is no electrion within the central range of stability.

§ 3. In a solid crystal of Bromide or Chloride of Radium we may suppose the Bromine or Chlorine, and the Helium which Ramsay and Soddy produced from it, to be not directly concerned in the marvellous radioactivity, which the crystal presents. For brevity at present I shall assume that the radioactivity depends primarily on the Radium atoms in the compound. Suppose now the crystal to be given with every Radium atom in it unloaded. In a very short time of progress towards equilibrium of kinetic averages, perhaps the millionth of a second, or the millionth of the millionth of a second, some of the atoms will become loaded. As time advances a greater and greater proportion of the Radium atoms will become loaded until, perhaps in the course of a few months, a permanent average of loadings and unloadings will be reached. The energy of the work done in loading the Radium atoms is taken from the energy of thermometric heat in the crystal. A cooling is thus experienced by the

* Communicated by the Author.

crystal, until heat conducted *and radiated* in from the surrounding matter compensates the cooling effect of the loadings, and a permanent equilibrium of temperature is reached.

§ 4. A certain definite proportion of all the loaded atoms, probably a very small proportion, will, according to the equilibrium of kinetic averages, become unloaded every second of time. Electrions will be projected out of these atoms with enormous velocities ; sufficient no doubt to shoot them through the substance of the crystal into space outside. It will be convenient for us to call this action a discharge, or explosion ; just as the immediate result of igniting the cordite in a loaded great gun is called a discharge, or explosion. The exciting cause of our supposed atomic explosion is a shaking of the electron from its stable equilibrium in the centre, far enough out to get beyond the range of the stability, and to be expelled away by repulsion. This repulsion increases to a very high maximum, and thence diminishes to zero, and changes continuously to the relatively small amount of the attraction between the resinous electron and the vitreous atom, experienced when the electron passes out of the atom.

§ 5. Those of the discharges of loaded atoms which send the electron inwards, relatively to the crystal, would, by the recoil, force the unloaded atom outwards : and, if it is near enough to the surface of the crystal, would send it out into the surrounding space, with comparatively small velocity and energy. The unloaded atoms, being vitreously electric, constitute the " α radiations," when they are sent outside the crystal by the force of the explosion. They would generate comparatively little heat in being brought to rest by the resistance of the matter outside the crystal. The generation of heat by the electrions which are shot out, would be much greater because of their much greater kinetic energy.

§ 6. It seems to me that it must be chiefly the electrions (the " β particles") which do the work of producing heat at the rate of one hundred gram water centigrade calories, per gram of Radium, per hour : but the " α particles" must also contribute to it.

§ 7. This process can go on forever, without violating the law of conservation of energy, and without any *monstrous*, or *infinite*, store of potential energy in the loaded Radium atom. The shooting out of electrions with prodigious velocities generates heat locally in the material around the Radium crystal ; while heat is taken into the crystal by conduction, *and radiation*, to supply the energy for the continued loadings of unloaded Radium atoms.

§ 8. Go back to the "equilibrium of temperature" reached at the end of § 3. Suppose the matter around the Radium to be a globe of lead of 50 cms. radius, surrounded by a spherical sheath so arranged as to let a stream of water flow steadily along meridional semicircles of the globe. Every " α particle," and " β particle," shot from the Radium into the lead, will generate the heat-equivalent of its kinetic energy in the lead. A certain proportion of the heat will pass inwards into the Radium to *cooperate with radiation in* supplying energy for the continued loadings. The remainder will be conducted outwards and will be carried away by the water. The arrangement suggested will form a calorimeter to measure the steady, permanent, thermal, effect of Radium.



§ 9. If the outer surface of the lead were coated with an ideal impermeable varnish, the whole heat generated in the lead by the electrions and atoms shot out from the crystal, would return into the crystal by conduction and would supply the energy for the loadings.

§ 10. It has occurred to many radioactive minds, that by making the lead globe large enough the heating effect might possibly be all lost. This is no doubt true, according to the explanation of radioactivity now suggested: because, with thicker and thicker lead, a greater and greater proportion of the heat generated in the lead will be conducted inwards to supply the energy for the loadings in the crystal.

§ 11. If the Radium is sealed in a glass tube or globe, and if its heating effect is tested by a stream of water, flowing around it through a glass tube of small bore, a large proportion of

the heat generated by the " β particles" would not get into the calorimetric water, but would be lost in heating the outer air around it. If, in the arrangement of § 8, the globe were only half a centimetre instead of fifty centimetres radius, a considerable proportion of the " β particles" would get away without leaving their energy in the lead or water, but practically all the " α particles" that get out of the Radium would be stopped by the lead, and would give up their energy to it.

§ 12. The dimensions stated in § 8 are chosen merely for the sake of illustration, and are not suggested as suitable for any practical experiment.

§ 13. The words "*radiated*" and "*radiation*," printed in italics in §§ 3, 7, and 8, above, indicate what seems to me the only possible way of escaping the conclusion that Radium contains a quasi-infinite supply of energy, which can be drawn upon for hundreds of years, without any compensating extraneous source. It seems to me not absolutely impossible that Radium may be, as it were, an exceedingly black body, relatively to waves of ether so short that lead and other solid and liquid substances are transparent for them.

XXIV. *Acoustical Notes.*—VII.

By Lord RAYLEIGH, O.M., Pres. R.S.

[Plate VIII.]

Sensations of Right and Left from a revolving Magnet and Telephones.—Multiple Harmonic Resonator.—Tuning-Forks with slight Mutual Influence.—Mutual Reaction of Singing Flames.—Longitudinal Balance of Tuning-Forks.—A Tuning-Fork Siren and its Maintenance.—Stroboscopic Speed Regulation.—Phonic Wheel and Commutator.

Sensations of Right and Left from a revolving Magnet and Telephones.

AMONG the methods available for the production of a pure tone in a telephone circuit is that where the electromotive force has its origin in the revolution of a small magnet about an axis perpendicular to its length, the magnet acting inductively upon a neighbouring coil which forms part of the telephone circuit. It was by experiments made partly in this manner that I formerly† determined the minimum of current necessary for audibility in the telephone. In connexion with recent work upon the origin of the lateral sensation in binaural audition‡ I have again employed this

* Communicated by the Author.

† Phil. Mag. xxxviii. p. 285 (1894); Scientific Papers, iv. p. 109.

‡ Phil. Mag. [6] xiii. p. 214 (1907).

method, and I now propose to give a brief account of the results, which were not available in time for incorporation in the paper just cited.

The object of the experimental arrangements is the separate presentation to the two ears of pure tones, practically in unison, in such a manner as to allow the effect of a variation in the phase-relationship to be appreciated. When the sounds proceed from tuning-forks vibrating independently, the phase-difference passes cyclically through all degrees, and if the beat be slow enough, there is good opportunity for observation. But it is not possible to stop anywhere, nor in some uses of the method to bring into juxtaposition phase-relationships which differ finitely. I thought that it would be of interest to observe under conditions which would allow any particular phase-relation to be maintained at pleasure, and to this the revolving magnet method naturally lends itself.

The propulsion is by means of wind (under about two inches water pressure) from a well regulated bellows. The blade forming the magnet may be bent wind-mill fashion and receive the wind directly, but in the present experiments it was combined with a diminutive turbine, the whole revolving about a vertical axis. The speed was about 190 per second, giving in the telephones a note of pitch *g*. Two inductor-coils* were used, the circuit of each being completed through a telephone. The planes of the coils were vertical, their centres being at the same level as the magnet. One was fixed and the other was so mounted that it could revolve about an axis coincident with that of the magnet and turbine. The angle between the planes represents of course the phase-difference of the periodic electromotive forces, subject it may be to an ambiguity of half a period, dependent on the way the connexions are made. If the circuits are similar, as is believed, the phase-difference of the currents and of the electromotive forces is the same. The telephone-discs were loaded, but not so heavily as to bring them into tune with the sounds employed. The circuit of one telephone included a commutator by which the current through the instrument could be reversed, corresponding to a phase-change of 180° .

In commencing observations the first step is to adjust to equality the sounds heard from the two telephones. This can be effected by varying the distances between the magnet and the inductor-coils. The telephones are then brought into simultaneous action at the two ears, and the effect is observed.

* They were also employed in the 1894 experiments and are there spoken of as "wooden coils" from the fact that the wire is wound upon wood.

A rotation of the movable coil may then be made, or the current in one telephone may be reversed by means of the commutator. The results were for the most part in harmony with what had been expected from the experiments with forks. But one anomaly must be noted, relating to the neutral condition where no pronouncement can be made in favour of either right or left. This should occur when the phases of vibration at the ears are either the same or precisely opposed; and it had been expected that the condition would be realized when the planes of the inductor-coils were strictly parallel. There is no difficulty in determining the neutral position, where neither right nor left has the advantage in either state of the commutator; but I was surprised to find that according to my own judgement the neutral position deviated very appreciably, perhaps 10° and one occasion even more, from that of parallelism. At first I supposed that the explanation of the anomaly was to be sought in the behaviour of the telephone plates, whose vibrations may not have the same relation in the two cases to the electric currents actuating them. It is possible that this cause of disturbance may have been operative to some extent; but that it was not a complete account of the matter became evident later when it was found that in Mr. Enock's judgement the neutral position did coincide sensibly with parallelism of the coils. There is no doubt at all but that the judgements of the two observers really differed; each repudiated the setting of the coil satisfactory to the other.

In the judgement of the individual observer, the neutral position can be determined with considerable precision by the observed absence of lateral effect, in conformity with the results of the tuning-fork experiments. In using the commutator in order to ascertain that no change in respect of lateral effect accompanies reversal, a complication arises from the fact that during reversal one circuit is momentarily broken. The telephones may be removed from the ears during commutation. If this be not done, care must be taken that the judgement made relates to the permanent effect, or errors may ensue due to the momentary action of the sound on one ear only.

When the neutral position of the coil is departed from, a lateral sensation—say to the right—is experienced, and this increases until the displacement reaches 90° . A reversal at the commutator changes the *right* into a *left* sensation, having the same effect as a rotation of the coil through 180° .

When the adjustment is such that the combined lateral sensation (to the right) is a maximum, it is interesting to observe the effect of applying the telephones to the ears

consecutively. If the right telephone be the first applied, the sensation of course is of a sound to the right. When the left telephone follows, the sound remains on the right and appears louder. If on the other hand the left telephone be the first applied, the sound appearing originally to be on the left transfers itself to the right as the second telephone comes into action. Under the best conditions there seems to be nothing remaining over on the left.

The results are thus confirmatory of those obtained from tuning-forks. Unquestionably we are able to take account of the phase-difference at the two ears, and this in the case of low pitch is the foundation of the secure judgement as to direction that we are able to form when a single sound is heard from the right or from the left. With respect to the convenience of the two methods of experimenting, much of course depends upon what appliances are available. In most laboratories, I suppose, the tuning-forks would be preferred.

Multiple Harmonic Resonator.

The use of Helmholtz resonators to demonstrate the compound character of a musical note is now familiar. The harmonic component tone which has the pitch of the resonator is specially reinforced and so rendered conspicuous even to untrained ears. By changing the resonator, the fundamental tone or any of the harmonics may be intensified in succession.

Such effects are rendered far more striking if the necessary changes of pitch can be brought about in a single resonator, which then remains continuously connected with the ear. We may do a little in this direction with a resonator of the usual König pattern. Choosing one of somewhat high pitch and listening to a harmonium note two or three octaves down, we find that various harmonics swell out in turn as we pass the finger over the aperture, thereby gradually lowering the pitch to which the resonator responds.

The idea is carried out more completely if the resonator is provided with a number of separate apertures, any or all of which can be completely closed with the fingers. According to the simple approximate theory* the natural frequency (N) of the resonator is given by

$$N = \frac{a}{2\pi} \sqrt{\left(\frac{c}{S}\right)}, \dots \dots \dots (1)$$

where a is the velocity of sound in air, S the volume included in the resonator, and c the electrical conductivity between the interior and exterior calculated upon the supposition that air is a conductor of unit specific conducting power and that

* Theory of Sound, §§ 304, 305, 306.

the walls behave as insulators. For a circular aperture in a thin wall c is equal to the diameter of the aperture. If there are several apertures, well separated from one another, c is equal to the *sum* of the diameters, or as we may write it,

$$c = d_1 + d_2 + d_3 + \dots \quad (2)$$

Hence, if the first aperture acting alone give the fundamental tone, the first and second together the octave, the first three together the twelfth and so on, we have so far as relative magnitudes are concerned,

$$d_1 = 1, \quad d_1 + d_2 = 4, \quad d_1 + d_2 + d_3 = 9, \text{ \&c.};$$

or

$$d_1 : d_2 : d_3 : \dots = 1 : 3 : 5 : \dots \quad (3)$$

The ratios (3) may give some idea of the proportions, but for many reasons—among them the neglect of the thickness of the walls—they are only roughly applicable. There is no reason for insisting on a circular form of aperture; indeed, when the aperture is large, an elongated form lends itself better to closure by a finger. Extreme cases excluded, the effectiveness of an aperture depends mainly upon its *area*.

When, as would be especially likely to happen in the case of the fundamental tone, a simple aperture would be very small, it may be well to replace it by a channel of finite length. If R be the radius of a tube of circular section and L its length,

$$c = \frac{\pi R^2}{L + \frac{1}{2} \pi R}, \quad \dots \quad (4)$$

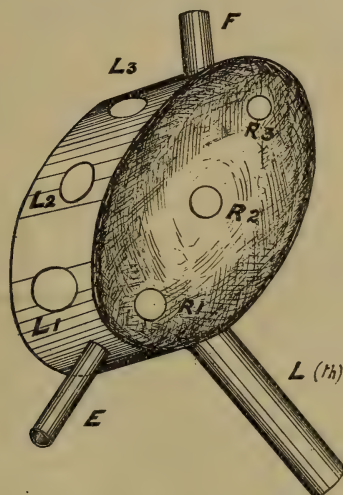
from which it will be seen that the area of aperture may be much increased as compared with what would be admissible if $L=0$.

Two compound resonators on this principle have been constructed. The first was made from the upper part of a glass bottle which had been cut off square near the neck. Over the wide opening a rather stout zinc plate was cemented through which the various apertures were bored. Through a cork, fitted into the neck, passed a short piece of brass tubing, by means of which and a suitable india-rubber prolongation connexion between the ear and the interior of the resonator was established. In tuning the instrument it is necessary to begin with the lowest tone and care must be taken to complete the adjustment of each aperture, or channel, before the next is attempted. Further details are hardly required. If the resonance is improved by shading an aperture with the finger, it is a sign that the aperture is already too large. If on the other hand the resonance

improves up to complete withdrawal of the finger, the aperture may still be too small. Before finally enlarging the aperture it may be well to ascertain that the resonance is improved by a partial use of the one next in order.

The second resonator was constructed entirely of metal. It consists of an elliptical box of sheet zinc, with top and bottom also of zinc and slightly dished for the sake of enhanced rigidity (fig. 1). The capacity (S) is about 140 c.c.

Fig. 1.



The apertures for the fundamental tone (F) and for the octave L(th) are formed of brass tubing soldered into position. The other passages are simple perforations in the wall and in the top of the box. E represents a short length of brass tubing over which is slipped an india-rubber attachment passing to the ear. To sound the fundamental tone of about 128 vibrations per second (B of my harmonium), F, about 8 mm. in diameter and 20 mm. long, is alone open. For the octave tone L(th), 11 mm. in diameter and 53 mm. long, is opened *in addition*. R₃ then gives the twelfth, R₂ the double octave, R₁ the higher third, L₃ the sixth component (octave + twelfth), L₂ the harmonic seventh, and finally L₁ the triple octave. The diameter of R₃ is about 5 mm. and that of L₁ (the largest aperture) say 13 mm. The letters are intended to indicate the fingering. Thus L₁, L₂, L₃ denote the first, second, and third fingers of the left hand; R₁, R₂, R₃ the corresponding fingers of the right hand. The octave tube L(th) is closed with the thumb of the left hand.

The performance of this instrument is very satisfactory. The seventh and eighth components are a little weak, but the others, and especially the twelfth and higher third, are loudly heard. The experimenter should bear in mind that when working in-doors much depends upon the precise position. The room is intersected with nodes and loops of approximately stationary vibrations, whose position varies from one tone to another. If a particular harmonic is ill heard, it may only be that the situation of the resonator is unfavourable. A motion of a few inches will often make a great difference. Usually the effects are best when the resonator is held pretty close to the reed in action.

In general a harmonium note is the most convenient for experiment and demonstration, but other instruments are of course available. A man's voice singing the proper note (B as above) gives excellent results.

Tuning-Forks with slight Mutual Influence.

Two forks giving 128 vibrations per second are independently maintained, each making and breaking its own contacts at a mercury cup. If mutual influence be altogether excluded, the "beat" may be made as slow as we please. But although the electric circuits may be entirely distinct, if the forks stand on the same table there may be enough mutual influence to bring about absolute unison. The best method of observation is by Lissajous' figures. The permanence of the ellipse is a sign that there is mutual control and that absolute unison is established; otherwise the ellipse undergoes more or less slowly the usual transformations.

A series of observations on this subject were made in 1901. Mutual influence may arise from both forks being connected with the same battery. If the electric circuits are in a series which includes two Grove cells, the forks keep together indefinitely; but this arrangement is rather akin to the familiar one in which a single interrupter-fork drives another, the second having no break of its own. Even when the fork circuits are in parallel and are fed from two Grove cells*, there is, or may be, sufficient reaction to maintain absolute unison. The feebler the reaction, the more nearly must the natural frequencies approach to identity. When the reaction is just insufficient for control, it is interesting to watch the cycle of the beat, as revealed by Lissajous' figure. At one part of the cycle the changes are very slow and at the opposite part relatively very quick.

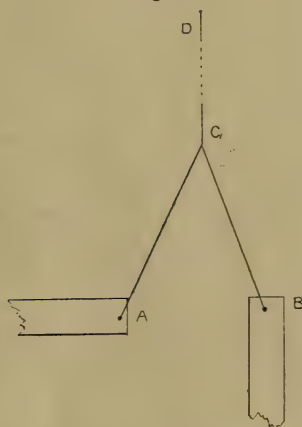
In another set of experiments the electric circuits were quite distinct, each fork being driven by a separate Grove

* The internal resistance of the cells comes into play here.

cell. A sufficient mutual reaction could be obtained through the air. To this end a large resonator was constructed by cementing a wooden plate over the opening of a bell-glass. In the plate were two similar apertures, to which the free ends of the forks were presented, the pitch of the resonator being equal of that of the forks (128). In this way an adequate control was secured, but the margin was narrow.

A more powerful controlling reaction accompanies a connexion between the two forks by means of slender cotton threads. The arrangement employed is indicated in the figure. A and B are the free ends of the upper prongs seen

Fig. 2.



from above. To them is attached a Y-shaped thread ABCD, the tension of which can be adjusted at D. When the control is established, the Lissajous' ellipse is stationary and usually open. An ellipse closed in upon its major axis would indicate that the natural frequencies were identical, independently of the control. By touching a fork judiciously with the rubber tip of the exciting hammer, the phase may be disturbed without stopping the electric maintenance. If one fork be touched, the ellipse closes in, while a similar operation upon the other fork opens it out. In a short time the ellipse settles back, showing that the original phase-relationship is recovered.

Mutual Reaction of Singing Flames.

In a former paper * I discussed the mutual influence of organ-pipes nearly in unison, showing that the disturbances depend upon the approximation of the open ends, and not sensibly upon the circumstance that they may take their wind

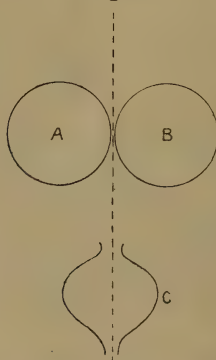
* Phil. Mag. vii. p. 149 (1879); Scientific Papers, i. p. 409.

from the same source. When the reaction suffices, only one note is sounded, and that is usually higher in pitch than the notes of either pipe separately. It is proposed to record the results of some observations of a similar character recently made upon so-called singing flames, *i. e.* tubes caused to speak by means of hydrogen flames.

The tubes were of glass from the same length, each 30 cm. long and 16 mm. internal diameter. The hydrogen bottles were also similar and were provided with burners formed from 14 cm. lengths of glass drawn down at the upper ends. Very small flames suffice. The tubes were held vertically and so that their upper (and lower) ends were at the same level.

When the distance between the tubes is considerable, say 30 cm., and draughts are avoided, fairly slow beats may be obtained by suitable tuning, as by approach of the finger to one end of that tube which vibrates the quicker. But when the distance is reduced to perhaps 8 or 10 cm., a difficulty begins to be experienced in producing slow beats. Either they are rather quick or else, when the tuning does not allow of that, they disappear altogether, the vibrations as it were engaging. On the margin where beats still occur, their character is peculiar. They appear unsymmetrical, the swell being protracted and the fall hurried. The phenomenon is the same as that observed optically in the case of forks (p. 323). When the tubes are as close as possible—they may conveniently be tied together with string,—even moderately slow beats are excluded. In this situation the sound is much

Fig. 3.



attenuated, indicating that the phases of vibration are opposite, at any rate in the ideal case. The ideal case is, however, rather difficult to attain. There should be complete cessation of the principal tone in a resonator (C) whose mouth is held

near the upper ends (A, B) symmetrically in the median plane (fig. 3). Frequently a better silence may be reached by moving round a little, and even then it is not absolute. By use of the finger to give a finishing touch to the tuning, the most silent position may be driven to the median plane, but even so the residual tone may not be quite extinguished. It is evident that the ideal condition is easily disturbed a little by slight failures of symmetry, probably connected with the flames.

As so far described, the disappearance of the principal tone, sometimes very nearly realized, leaves a considerable amount of octave outstanding. A remedy may be applied by the insertion, at the middles of the tubes, of rectangular blocks of wood, about two diameters long and forming a loose fit. They are held in their places by springs. In this way the outstanding octave may be very much reduced.

Longitudinal Balance of Tuning-Forks.

The vibrations of a well-constructed tuning-fork are approximately isolated and are conveyed to the stalk in only a limited degree. When, as in the ordinary use of small forks, the stalk is pressed against a sounding-board, the principal tone is attended by a considerable accompaniment of octave, especially at first when the vibrations are vigorous. The substitution of a suitably tuned resonance-box for the sounding-board may easily render the octave sound preponderant*. The experiments now to be recorded were an attempt to ascertain how far it was possible to carry the isolation of the principal tone. It should be remembered that however complete may be the isolation as regards the stalk, there is necessarily a certain small amount of direct communication from the vibrating prongs to the surrounding air. For our present purpose this is to be disregarded.

At first sight it may appear that the desired state of things must be very approximately attained in the usual construction where the prongs are parallel. Something will depend upon the manner in which the transition takes place between prongs and stalk. In what follows I have more particularly in view a construction in which the prongs form a U of tolerably uniform thickness, to which a cylindrical stalk is attached without much excess of metal at the junction. As a rough approximation we may suppose that the inertia of the fork is

* Phil. Mag. iii. p. 456 (1877); Scientific Papers, i. p. 318. Even when the box is tuned to the fundamental note, the octave and twelfth are often easily audible. I have observed this effect with three different 256-forks when mounted upon a particular resonance-box. There was no suggestion of looseness or chatter.

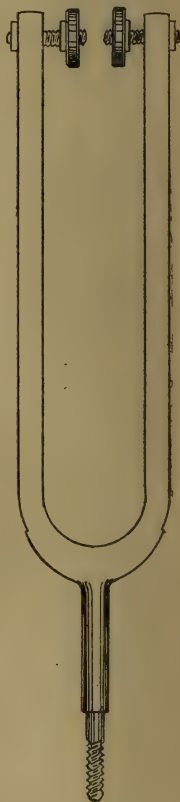
concentrated at the ends of the prongs. Then if the fork be free in space, these ends can move only backwards and forwards along the line joining them. The question we have to ask is—does the stalk remain at rest? A little consideration makes it fairly clear that in the case of parallel prongs the answer is in the negative. As the prongs approach one another the curvature of the bend is increased and the stalk moves along its length *outwards*, *i. e.* away from the prongs*. Similarly half a period later the opening of the prongs is accompanied by an approach of the stalk. Under these conditions if the stalk be brought into contact with a sounding-board, a motion of the first order is communicated and the principal tone is heard.

It is evident that the effect to be expected when the prongs are parallel may be compensated by a suitable permanent bending of the prongs *inwards*, or what comes to the same by a suitable *loading* on the inner sides. The motion of the stalk during the vibration is then composed of two parts which have opposite signs—the one already considered depending on the variable curvature at the bend, the other on the obliquity of the prongs to the line of motion at the ends. It would appear then that by this adjustment it should be possible to secure that the stalk remains at rest, so far as motion of the first order is concerned. It is assumed that everything is symmetrical, so that the stalk, if it moves at all, does so along its length and (in view of its dimensions relatively to the wave-length of vibration in steel) practically as a rigid body.

For the purposes of the experiments a large fork was constructed by Mr. Enock. The U was from a single length of steel 60 cm. long and of section 1.275 cm. square. The prongs were parallel, 5.35 cm. apart (inside measurement), and the stalk was attached by brazing (fig. 4).

In its unloaded condition it gave 128 vibrations per second and could be screwed to a resonance-box appertaining to a large fork by König of the same pitch. When

Fig. 4.



* Somewhat as if by a violent local bending at the middle of the U the prongs were brought into contact throughout their whole length.

excited by bowing it emitted a very powerful sound and, largely in consequence, came somewhat rapidly to rest. The isolation of the vibrations was thus far from complete.

The principal loads, of 40 gms. each, were in the form of nuts and travelled along screws passing through the prongs near their ends and parallel to the direction of vibration. Suitable lock-nuts kept all tight. In consequence of the loading, the pitch fell about a major third, and the tuning of the resonance-box had to be readjusted by a piece of board obstructing the mouth. It soon appeared that, as had been expected, when the loads were outside the prongs the sound diminished as they were moved inwards as far as possible. To obtain a minimum, the loads must be *inside* the prongs; and a great falling off was readily achieved by adjustment of their position.

At this stage considerable difficulty was experienced in appreciating the quality of the residual sound, but it was suspected that most of it was octave, in spite of the fact that the resonance-box was tuned to the fundamental tone. The device appropriate to stop tones of a particular pitch from gaining access to the ear is a Quincke tube. A straight length of composition metal tubing, open at both ends, was provided with a lateral connexion at a distance from the outer end amounting to $\frac{1}{4}\lambda$ of the *octave* tone. The whole length was nearly the double of this, and the other end was inserted in the resonance-box. At the same time the ear was connected with the lateral branch with the aid of an india-rubber prolongation. When the outer end of the straight tube is closed with the thumb, that end becomes a node of the stationary vibrations of octave pitch executed therein, and as the junction with the lateral tube is distant $\frac{1}{4}\lambda$ from the end that place is a loop, and consequently no (octave) vibration is propagated to the ear. The application of the thumb accordingly has the effect of freeing the possibly compound sound from its octave component, while it leaves the fundamental tone in full vigour.

The application of this test proved at once that by far the greater part of the residual sound heard when the loads were in approximate adjustment was in fact octave. Immediately after bowing, when the vibrations of the fork are vigorous, a loud sound is heard when the outer end of the Quincke tube is open, but comparative silence ensues when the thumb is applied. As the vibration dies down, closing the end has less effect.

In this way it appeared that in reality a great measure of

success had been already attained in isolating the fundamental tone, only obscured by the accompaniment of octave in unexpected amount. A sensible revival of the fundamental tone ensued when the loads were rotated from their best adjustment though a quarter turn each, corresponding to a lateral shift inwards or outwards of $\frac{1}{80}$ inch. Since the test is of such delicacy, we may perhaps consider the isolation of the fundamental tone to be practically complete.

The fact remains, and must not be slurred over, that it was not possible by any adjustment of the loads to eliminate the fundamental tone *entirely*. The residual sound did not come directly through the air from the prongs, but was propagated through the stalk to the resonance-box. It is a little difficult to trace the nature of this residue. Upon the supposition that the vibrations of the various parts of the fork are all in one phase, and of complete geometrical and mechanical symmetry in the construction of the fork, it would appear that some adjustment of the loads *must* eliminate the fundamental tone. There was, indeed, evidence of actual lack of symmetry, which could not in any case be mathematically perfect. When with the aid of a handle the fork was held horizontally so that its stalk rested upon a wooden edge supported in its turn upon the top of the resonance-box, sound was heard from the box, which varied as the fork rotated round its stalk as axis and in fact nearly vanished in two asymmetrical positions. It would seem that the residual fundamental tone heard in the more normal use may be connected with a *lateral* movement of the stalk, dependent upon some failure of symmetry.

As a variation upon the above arrangement the prongs were now bent inwards so that at the outer ends the distance from metal to metal was reduced from 53.5 mm. to 38 mm., a bending intended nearly to represent the effect of the loads. With the 40 gm. loads it was no longer possible to reduce the fundamental tone to silence and, as soon appeared, for this reason that the proper position for the loads was unattainable, being that occupied by the metal of the prongs themselves. When smaller (20 gm.) loads were substituted, interior positions could be found allowing the elimination of the fundamental tone to about the same degree of perfection as before. In listening with the Quincke tube to the dying sound with alternate application and removal of the thumb at the outer end, it was recognized that the low tone was practically gone (thumb on) while the octave (thumb off) was still fairly audible. About the same displacement of the

loads as before ($\frac{21}{80}$ inch) was sufficient to cause a perceptible augmentation of the residual fundamental tone.

As to whether these results can be turned to practical account in the construction of forks, we must remember that if a fork is to be used in conjunction with a sounding-board or resonance-box a too complete isolation of the fundamental tone would defeat the intention. On the other hand if, as in Helmholtz's vowel experiments, a fork is to be employed to excite an air resonator placed near the ends of its prongs, a suitable turning inwards of these prongs and consequent quiescence of the stalk would be of advantage.

In conclusion attention may be drawn to the circumstance that a symmetrical *bell* with stalk attached would not need any particular adjustment in order to ensure the isolation of the vibrations of the first order. If the stalk tend to move outwards when contraction occurs along one particular diameter of the circumference, the same tendency must repeat itself half a period later when the contraction is transferred to the diameter at right angles to the first. A similar remark would apply to a symmetrical compound fork, such as we may imagine to be produced by cutting away all the material of the bell, except in the neighbourhood of two perpendicular meridians.

A Tuning-Fork Siren and its Maintenance.

When in 1901 I was experimenting upon the work absorbed in various cases of the production of sound* I had at my disposal a Trinity House "Manual" Fog-horn. In this instrument the wind is generated by cylinders and pistons, and a much higher pressure is available than is usual in laboratory apparatus. Among the experiments then tried was the substitution of what I called a tuning-fork siren for the natural reed and conical horn of the instrument. Fitted to a wind-chest was a metal plate, carefully faced internally and carrying a rectangular aperture about 10 mm. broad. This aperture could be nearly closed by a plate 3 mm. wider, which vibrated laterally in front of it. The vibrating plate was mounted upon the side of one prong of a fork making 128 vibrations per second. When the fork was at rest, the aperture was obstructed and the fit of the moving and fixed plates was so good that the leakage of wind was not serious.

* Phil. Mag. vi. p. 292 (1903).

But as the fork vibrated the aperture was in part uncovered, and that twice during each complete vibration of the fork, so that the pitch of the instrument, considered as a siren, was 256. The fork was driven electrically from another interrupter-fork of the same pitch situated outside. Fitted to the aperture externally was a resonating tube whose length could be adjusted to give the maximum effect. One of the objects was to be able to vary the resonance without disturbing the maintenance or the pitch of the siren. The pressure employed was sometimes as low as 2.5 cm. of mercury, and the consumption of wind about $3\frac{1}{2}$ litres per second, corresponding to .015 horse-power. At this rate of working the pumps could be kept going by hand, or rather by legs, for a moderate length of time.

One unexpected effect presented itself, which seems worthy of record. As has been mentioned, the intention had been to keep the fork in motion electrically. But it was found that, at any rate after being once started, it remained in vigorous vibration under the action of the *wind alone*, although the electric connexion was cut off. It will be observed that this case is altogether different from that of a *reed*, where the tongue approaches and recedes from the aperture normally. It is more analogous to the æolian harp, where, as I have formerly shown *, the vibration is executed in a plane perpendicular to the direction of the wind. So far as I am aware, no adequate mechanical explanation of this singular behaviour has been given.

Stroboscopic Speed Regulation.

The stroboscopic method † has often been employed for testing and regulating the speed of revolving shafts. I used it extensively in my determinations of absolute electrical units ‡, referring the speed of revolving coils to the frequency of vibration of an electro-magnetically maintained fork. But I doubt whether even now the convenience of this method for general purposes is appreciated. A few years ago the late Mr. Gordon drew for me upon card rows of alternate black and white "teeth" from 20 to 40. Photographs from this upon flexible paper could be mounted upon a revolving shaft so as to form reentrant circles of teeth for

* Phil. Mag. vii. p. 149 (1879); Scientific Papers, i. p. 413.

† Plateau (1836); Töpler, Phil. Mag. Jan. 1867.

‡ See for example Proc. Roy. Soc. xxxii. p. 104 (1881); Scientific Papers, ii. p. 8.

observation by intermittent view. In my use of it an electrically maintained fork of large dimensions and of home construction was employed. The fork was provided with solid (platinum) contacts and made 64 vibrations per second. At the ends of the prongs were blackened plates of thin metal perforated with slits, so disposed as to be opposite to one another in the equilibrium position. When the vibrations were excited by one or two cells, there were 128 views per second through the slits.

Viewed past the fork, some of the circles of revolving teeth appear nearly stationary. Usually two neighbouring circles can be picked out, which appear to revolve slowly in opposite directions. From these data the necessary information is obtained in a way that need not be further explained. It is thought that it may be of service to give a reproduction of Gordon's drawing (fig. 5, Plate VIII.). Photographic copies can easily be made upon any desired scale adapted to the shafts round which it is intended to mount them. Care should be taken to effect the junction properly, so that the circles of teeth are continued through it without irregularity.

Phonic Wheel and Commutator.

By the use of the phonic wheel, invented independently by La Cour and myself *, the speed of revolving shafts may be not merely compared with a fork but automatically governed thereby. I have used this method for driving a commutator of the kind required for passing a regular succession of condenser charges through a galvanometer, as for example in determining the ratio of the electrical units (velocity of light). The contacts required are such that a piece A in connexion with the insulated pole of the condenser shall make contacts alternately with a piece B representing the insulated pole of the battery and another C connected to earth and to the other pole of the battery and condenser. Of course the contacts must be good, and it is essential that both be not made at the same time.

As in Thomson and Searle's work †, the commutator was of the usual type adapted to a revolving motion, except that the cycle of contacts was repeated so as to occur twice in each revolution. The developed form is shown in the

* Phil. Trans. 1883, p. 295. Scientific Papers, i. p. 355; ii. p. 179.

† Phil. Trans. 1890, p. 607.

accompanying diagram (fig. 6) where the shaded parts represent brass pieces separated by ebonite insulation. Springs lightly bearing against the exterior continuous portions of metal correspond to B and C, while A corresponds to a brush bearing near the centre and making contact alternately with the two metal pieces. Provision was made for varying the pressures at these contacts during the running and without disturbing the insulation. The problem is to secure a uniform rotation of this commutator, whose diameter was 28 mm.

Fig. 6.



The phonic wheel, mounted on the same shaft as the commutator, takes its time from a vibrating fork (44 per second *) acting as interrupter of an electric current. The current (about 4 amperes) is from 3 secondary cells and excites not only the electro-magnet by which the vibrations of the fork are maintained but also the electro-magnet of the phonic wheel. Four soft iron armatures are mounted round the circumference of the drum and in their passage complete approximately the magnetic circuit. The holes through which the fork is viewed are also four in number.

The most advantageous action of the regulating current occurs when one armature passes for each complete vibration of the fork. Under these circumstances the prong, or rather a projecting wire attached for the purpose, is seen stationary and *single*. There are then 11 ($=\frac{1}{4} \times 44$) revolutions of the wheel per second and 22 charges and discharges of the condenser. But the wheel may also be run at double or triple this speed, and then the projecting wire is in general seen doubled or tripled. The regulating current from the fork is of itself capable of maintaining the rotation at single or double speed when once the necessary *engagement* has been secured. For this purpose the speed must be raised to the required point by means of a string passed round the shaft and worked with the fingers, and even then it may be only after many trials that engagement ensues. For the triple speed the power of

* For the design of steel vibrators and for rough determinations of frequency, especially when below the limit of hearing, the theoretical formula is often convenient. We may take

$$\text{frequency} = 84600t/l^2,$$

where l is the total length of a prong and t the thickness in the plane of vibration, both being reckoned in centimetres. ('Theory of Sound,' § 177.) At any rate the *octave* is never uncertain.

the fork current is insufficient, and recourse must be had to independent driving by an electric motor or water-power engine. If the driving power be in excess, the fork currents are equally capable of holding the wheel back. The whole behaviour is evident on observation of the fork through the revolving apertures, and so long as the engagement lasts the wheel can never gain nor lose a complete cycle relatively to the fork.

I have used the commutator thus driven to observe the charges of a condenser in their passage through a galvanometer. The only inconvenience was the necessity of a considerable separation between the galvanometer and the rest of the apparatus to obviate magnetic disturbance. The galvanometer deflexion was steady and apparently independent of the force exercised at the springs of the commutator. I believe that the arrangement might be used with advantage in such work as determining the ratio of the electrical units.

XXV. *On the Ionization of Various Gases by the α Particles of Radium.*—No. 2. By W. H. BRAGG, M.A., Elder Professor of Mathematics and Physics in the University of Adelaide*.

[Plate IX.]

Introduction.

IN a paper with a similar title (Proc. Roy. Soc. of South Australia, vol. xxx. p. 1, and Phil. Mag. May 1906, p. 617) I have given a preliminary account of an attempt to determine the relative amounts of ionization produced in various gases and vapours by the α particle of RaC. The present paper contains an account of the further progress of this work.

In the first place, I have here discussed the validity and the experimental details of the method used, and have brought forward evidence in favour of the hypothesis that $\delta\iota$, the ionization produced in consequence of the expenditure of a small quantity of energy $\delta\epsilon$ by the α particle, is related to the latter quantity by the equation $\delta\iota = kf(v)\delta\epsilon$, where $f(v)$ is a function of the velocity of the particle only, and k a constant for each gas.

Secondly, I have given the result of the attempts to determine for several gases the constant k , which may be called the specific ionization of a gas for α radiation, air being taken as the standard.

* Communicated by the Physical Society of London: read February 22, 1907. Part of the paper was also read before the Royal Society of South Australia, Oct. 2, 1906.

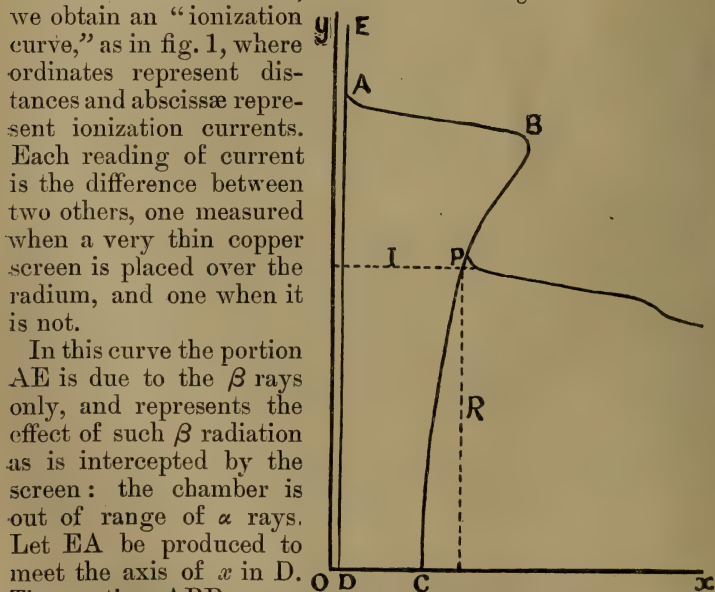
In conclusion, I have discussed very briefly the form of the function $f(v)$, and such conclusions as it seems legitimate to draw from the results so far obtained. Amongst these is the following:—The ionization per molecule (ks , where s is the stopping power) is closely connected to the molecular volume.

§ 1.

The method of this research has already been described briefly in the preliminary paper (*loc. cit.*). For the sake of clearness, however, and in order to facilitate a discussion of the validity and the experimental details of the method, it will be well to insert a short description here also.

A small platinum plate is coated with a very thin layer of radium bromide, and placed below a horizontal ionization-chamber of 3 mm. width, at a distance which can be altered at will. (See Plate IX.) A set of narrow vertical tubes is placed over the radium, and stops all α particles which move in any direction which is not almost vertical. Thus the particles cross the narrow chamber at right angles to its greater dimensions, and all spend 3 mm. of their paths in the air within it. The resulting ionization being plotted against the distance from the radium to the middle of the chamber,

Fig. 1.



In this curve the portion AE is due to the β rays only, and represents the effect of such β radiation as is intercepted by the screen: the chamber is out of range of α rays. Let EA be produced to meet the axis of x in D. The portion ABP represents part of the effect of the α particles from RaC.

If no other radioactive substances were present, the curve would show a continuation of the portion BP down to the axis of x , in some such manner as PC.

If the ionization curve were completed in this way, the area ABPCD would represent the total ionization due to the α radiation from RaC. If now the air were removed, another gas substituted for it, and the air re-measured, a comparison of the values obtained would give the result which this research aims at. We may call it the specific ionization of the gas. But the complete determination of the boundaries of the area is so long and complicated a process as to render this procedure impracticable. It can, however, be shown that the product of the coordinates of a certain point on the curve may be taken as a measure of the area of the curve, provided certain assumptions are made. The point is at the intersection of the top portion of the curve representing the effects of RaA with the side of that showing the effects of RaC. The coordinates of this point are comparatively easy to obtain.

Now, it might appear that it would be better to measure of one time the whole of the ionization produced by the particle, rather than to determine the ionization point by point along its path; since, if this were done, it would no longer be necessary either to find the exact form of the ionization curve, or to depend upon the validity of assumptions. We might spread a layer of radioactive material on the floor of an ionization-chamber, and so arrange the temperature and pressure of the gas in relation to the dimensions of the chamber, that all the α particles completed their paths within the gas. But the potential gradient required to separate and collect the ions made by the α particle is generally very great. For example, in ethyl chloride at 30 cm. pressure and ordinary temperatures, about 1000 volts per cm. is desirable, if saturation is to be certain. With such gas it would be necessary to make the height of the chamber about 4 cm., in order to allow all the α particles to complete their ranges; even if the radioactive material were uranium or polonium. Thus, a potential of 4000 volts would be required, and such large electromotive forces are out of the question. If the pressure of the gas were lowered, less electric force would be sufficient; but the paths of the α particles would be longer, the chamber would need to be higher, and the total potential as great as ever.

It is absolutely necessary to use a narrow ionization-chamber if sufficient electric force is to be obtained without the use of enormous battery-power. Clearly it would be no gain to use such a chamber if the radioactive material were

spread on one of its walls. For in this case some of the particles would complete their full ranges within it, others only part, and an estimate of the ionization to be expected would render it necessary to take into account the amount of the range completed by each particle as determined by the nature and physical conditions of the gas and the dimensions of the chamber, the reckoning being further complicated by the fact that the ionization produced by the particle is not constant along its path. It is possible that an experiment might be arranged in which a thin sheet of α radiation entered the chamber through a slit at the side, and spent itself within the chamber without touching the walls. It would be necessary to make sure that the same portion of the range was completed within the chamber by the particle, no matter with what gas the chamber was filled. I have not yet tried this plan.

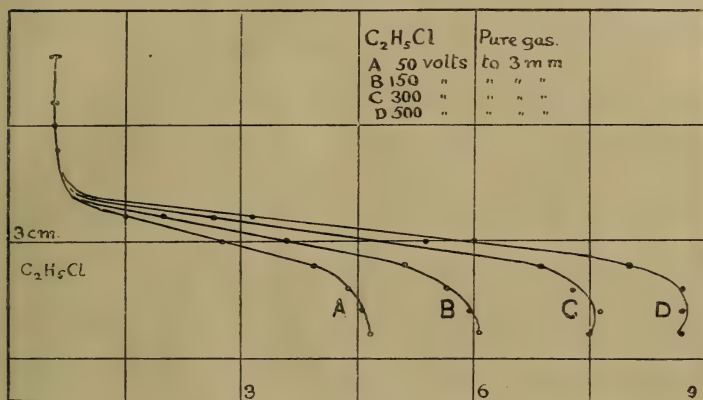
It will now be clear, I think, that the method actually used is not without its advantages. It avoids the use of very high potentials, and does not require lengthy and uncertain calculations. It has also this in its favour, that it gives the range of the particle in the gas, so that it is possible to make a sufficiently accurate estimate of the amount of any air that may be present. The presence of this air can then be allowed for.

Let us, therefore, proceed to consider the assumptions and approximations which the method requires.

In the first place it is necessary to consider whether any disadvantages are likely to arise from the use of a sheet of gauze as the lower wall of the ionization-chamber. The electric field must be distorted in the neighbourhood of the gauze; some very small portions of the chamber which are just over the openings in the gauze must be under feeble forces, and the ions made there be separated only when the potential is high. It is easy, however, to show that this effect is negligible by a consideration of the ionizations due to β rays. This ionization does not show initial recombination, as in the case of the α rays; a fact first demonstrated by R. Kleeman, formerly of this University. In fig. 2 are drawn the upper parts of the ionization curves of ethyl chloride under different potentials. It will be seen that in the portion which represents the effects of β rays only, saturation is complete when 50 volts are applied to the 3 mm. chamber; but the ionization due to α rays is far from being collected completely by ten times that potential. Now, if the field distortion due to the gauze were appreciably effective, we should find the β rays also producing an ionization which appeared to increase at higher potentials; and there is no trace of any such effect.

The same result shows that, although ions are very apt to be drawn through a gauze by a strong field on one side, yet in this case nothing of the sort takes place. To prevent it, a second gauze has been placed 3 mm. below the first, and earthed, so that there are strong, equally-balanced fields on both sides of the latter.

Fig. 2.



A thin uniform metal sheet might replace the gauze, but unless it were very thin it would cut off more of the range than can generally be spared; and if it were thin it would be liable to flexure by the powerful electric forces, so that the depth of the ionization-chamber might become indeterminate.

We must now consider the assumption that the area of the ionization curve may be represented by the product of the coordinates RI as already defined. This is really equivalent to the supposition that the ionization resulting from the expenditure of a quantity of energy $\delta\epsilon$ by the α particle is equal to $kf(v)\delta\epsilon$, where $f(v)$ is a function of the velocity of the particle and k is a constant, depending on the nature of the gas molecule. It implies in the first place that the area of the ionization curve in any gas is not dependent on pressure and temperature, and that, if the form of the curve is altered by a variation of these conditions, it is only in so far that all the ordinates are multiplied by some factor, and all the abscissæ divided by the same factor. It implies, in the second place, that the ionization curve of one gas can be made to coincide with the curve of any other gas, by multiplying all the ordinates by some factor, and all the abscissæ by some other factor. Let us examine the evidence in favour of these statements.

If the hypothesis is true, RI must be independent of pressure
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and temperature. As regards pressure, some results were quoted in a paper "On the Recombination of Ions in Air and other Gases"*, which showed this to be correct in the cases of air and ethyl chloride; and further evidence will be found in the results given at the end of this paper. For, without having made any exhaustive comparison of the values of RI at different pressures in each gas, I have often used various pressures in the determination of the specific ionization of a gas; and the general agreement between the results obtained is good evidence that pressure is without effect.

In the same way, since many determinations in the case of the same gas have been made at different temperatures, the close agreement shows also that temperature has no influence on the ionization. More direct confirmation can be obtained from the following results. During a number of the determinations of RI, the ionization-chamber was connected in parallel with a second chamber containing a uranium layer. The ionization currents acted against each other. Thus the values of the currents in the radium apparatus could be determined by balancing against the uranium: the latter was always at the temperature of the room, and therefore formed a fixed standard. The extent of the surface of the uranium could be varied by means of a semaphore, having a graduated circle on the same axis. It was then found that although the RI in air appeared to decrease as the temperature of the radium apparatus was raised, yet when the readings were expressed in terms of the uranium scale, the value of RI was constant. The decline was merely apparent, and due to leakage through the heated glass insulators. The actual values of RI were:—

Five determinations, 20° to 60° C.: 320, 326, 318, 314, 314; mean, 320.

Five determinations, 60° to 80° C.: 296, 314, 311, 334, 327; mean, 316.

The experiments were made at various times, and some of the irregularities are probably due to slight alterations in the amount of the RaC present.

Furthermore, it has already been shown with respect to ionization in general that pressure and temperature have no effect (Patterson, *Proc. Roy. Soc.* lxi. p. 277, 1901, and *Phil. Mag.* Aug. 1903). I have thought it well, however, to reconsider the point with special reference to the circumstances of this experiment.

It is convenient at this stage to state that temperature does not seem to have much effect on initial recombination.

* *Trans. Roy. Soc. S. A.* 1905, p. 187; *Phil. Mag.* April, 1906, p. 466.

The latter decreases rapidly as pressure is lowered. This has been shown by Kleeman and myself ("On the Recombination of Ions, &c."). But when the alteration in density occasioned by a rise of temperature has been allowed for, there appears to remain only a slight diminution in initial recombination, which can be ascribed to the direct result of the increase in temperature. This is shown with some clearness in some experiments which I have made with CO_2 . They may be tabulated as follows, the ionization at an electric force of 1000 volts per cm. being taken as 100:—

CO_2 .	Ionization at 1000 volts per cm.	Ionization at 333 volts per cm.	Ionization at 166 volts per cm.
(a) Pressure, 651 mm., 20° C. ..	100.0	95.0	90.2
(b) Pressure, 760 mm., 72° C. ..	100.0	96.8	94.0

A repetition of the experiment gave practically the same result. The pressures and temperatures were so arranged that the density was the same in each experiment.

I also tried the experiment with ethyl chloride, but the results were not so definite; that is to say, change of temperature produced no very obvious effect.

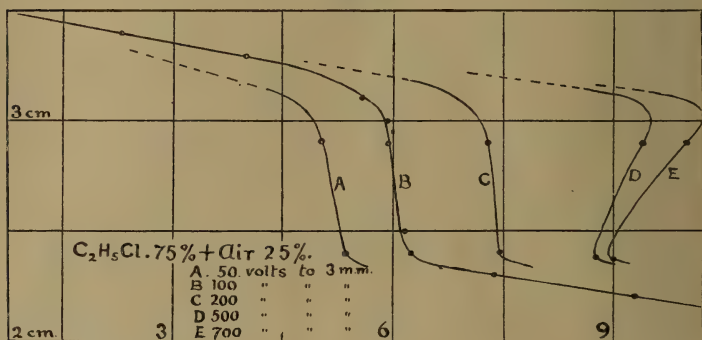
It is further assumed that the curves for different gases are of the same form; in other words, that the function $f(r)$ is the same for all gases.

A complete test of this hypothesis would require an accurate delineation of the ionization curve in the case of each gas. As has already been said, this would be a difficult task, inclusive, indeed, of our present purpose. But a comparison of the curves in different gases, so far as they have been obtained, shows that the principle is at least approximately true. For example, the ratio of the range of RaC to that of RaA is the same in all gases within errors of experiment, and again the ratio of the maximum abscissa of the RaC curve to the abscissa I is also constant, so far as I have measured it. As examples of the constancy of the first of these two ratios, I have at different times found it to be 1.46 in air, 1.47 in pentane, 1.47 in ethyl chloride, 1.44 in carbon dioxide, 1.48 in ethyl alcohol, and 1.49 in ethylene. The differences here are probably experimental only. As regards the second ratio, I have found it to be 1.36 in air, 1.37 in ether, 1.44 in ethylene, 1.35 and again 1.41 in ethyl chloride. This ratio is much more liable to error than the former; for all ionizations are harder to measure correctly than ranges, and the peak of the ionization curve is an especially uncertain point. Also there is a special

difficulty due to the existence of a peculiar phenomenon, which must now be considered.

It is to be observed that the ionization curves in different gases will not correspond unless the potential employed is enough to saturate at all points of the path of the α particle. More electric force is required as the particle slows down. This may be deduced from figures given in the paper "On the Recombination of Ions, &c." p. 196. It is there stated that the ratio of the saturated ionization current to that at 25 volts per cm. in the case of the ions made by the α particle of RaC at a distance of 6.25 cm. from its origin was found to be 1.29; whereas, when the distance was reduced to 5.05 cm., it was found to be 1.19. Each of these ratios is the mean of four determinations. (By an arithmetical error, one of the latter is incorrectly given in the paper quoted: 1.23 should be 1.20.)

Fig. 3.



Again, the effect is clearly shown by the curves of fig. 3, which represent the results of experiments on a mixture of ethyl chloride and air. It will be seen that the curve does not show the characteristic increase of ionization with distance when the electric force is small, the reason being that it is so much more difficult to collect the ions made by the α particle at the end of its path.

It is necessary to refer to one more assumption which is made in calculating the results, viz., that the RI of a mixture can be determined from a knowledge of the RI of each component. For example, it is supposed that the RI of air being 100 and of ethyl chloride 132, then the RI of a mixture in such proportions that the α particle spends half its energy in each is 116.

For I have not been able to prevent the leakage of air into the apparatus when raised above ordinary temperatures, and it is necessary to measure and allow for the air present in each experiment. The apparatus holds very well when not heated; but it is sometimes necessary to raise the temperature to 60° or 70° C. in order to obtain a sufficient density of the vapour under treatment. Fortunately, however, the air present may be a considerable fraction of the gas when measured by pressure, and yet be of little importance when measured in terms of the energy spent in it. Thus the correction for air present is usually quite small, as will be seen from a consideration of the numerical results in § 2.

The assumption is by no means an obvious one. If any part of the ionization in a gas is secondary, and is due to radiation originating in one molecule and acting on a neighbouring molecule, it might well be that complications would arise in a mixture of gases. I have made several direct attempts to find whether any such effects existed: the results of some of them are shown in the following tables. Each table refers to a set of experiments carried out consecutively. The percentage of gas in each mixture is determined from the stopping power, and the percentage of energy spent in the gas is then calculated. The value of RI for the gas is calculated by multiplying the observed value for air by the specific ionization of the gas, as taken from the final tables given at the end of the paper. For example, in the first set RI for air is 198.5, and RI for ethyl chloride is taken to be $189.5 \times 1.32 = 262$. The RI for each mixture is then calculated. In the table the calculated and observed values are put side by side, and it will be seen that there is a good agreement:—

	Percent. of Gas.	Percent. of Air.	Percent. of Energy spent in Gas.	Pressure in mm.	Temp.	RI observed.	RI calculated.
C_2H_5Cl:							
1.	0	100	0	760	37	198.5	—
2.	88.5	11.5	94.5	421	37.5	253	259
3.	39.7	60.3	61	437	38	235	237.5
4.	17.3	82.7	33	433	38	220	220
5.	8.5	91.5	18	441	38.2	212	210.5
6.	0	100	0	760	39	198.5	—
C_2H_3Cl:							
1.	0	100	0	760	32	200.5	—
2.	91.5	8.5	96	294	34.5	258	260
3.	35.3	64.7	56.5	310	36.5	232	235
4.	0	100	0	760	37	198.5	—
C_3H_{12}:							
1.	0	100	0	760	41	202	—
2.	83.5	16.5	95	321	43	262	264
3.	19.2	80.8	45.5	351	45	232	229.5
4.	0	100	0	760	45.6	195	—

Nevertheless, in a number of cases in which I have attempted to calculate the value of RI of one gas from a knowledge of the values of RI for air, and for a mixture in known proportions, I have obtained an unexpectedly high result, and when I began some direct experiments on the question I was quite prepared to find that the ionization of a mixture of air and gas was more than the sum of the ionization of air and gas separately. Further experiment will, no doubt, make everything quite clear. In the meantime it is sufficiently evident that the principle is at least nearly true. For the purpose of this investigation it may be taken as quite true, since the correction to be made for the presence of air is, at the most, only small.

The quantity I, as measured, includes a small proportion of β ray ionization. It must be shown that this does not harm the result.

In the form of apparatus which I use the ionization in the portion AE of the curve (see fig. 1) is nearly 6 per cent. of the ionization at P, and I have not found enough variation from gas to gas to justify an attempt at correcting for it. Of course, the quantity is only small.

The curve shows only this β ionization above A; that below is hidden. But I have found by experiment that it varies very little throughout the whole distance from the axis of α . This I did by placing over the radium just enough tinfoil to cut off all the α rays.

In the foregoing will be found, I think, sufficient justification for the choice of the method of this paper, and for the assumptions made during the calculation of the results.

In the previous paper I have already given a brief description of the process of an experiment. Some points, however, deserve reconsideration in the light of further experience, and some changes have been found convenient. These are best discussed in relation to an actual experiment; I will take a determination of RI in carbon bisulphide.

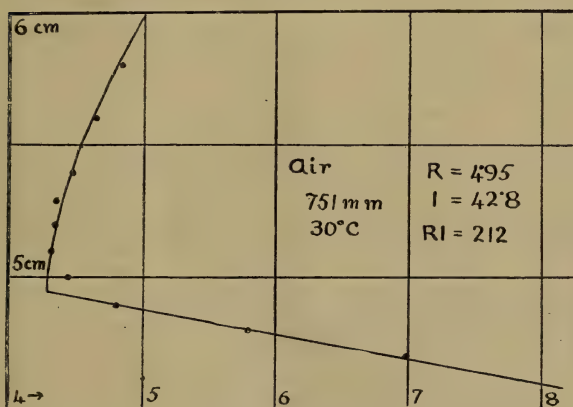
I have found it best to separate experiments whose object is to determine RI from those whose object is to find the stopping power of the gas. In the former the chief difficulty lies in overcoming initial recombination. This requires the pressure of the gas to be low, and the applied potential to be high. A little leakage of air into the apparatus, which can hardly be avoided under these circumstances, is no serious disadvantage, since the proportion of air can be found from a knowledge of R, and of the pressure and temperature at the

time when R is measured ; and these data are easily obtained. In the latter, any moderate voltage will do, since the range does not depend on potential ; but it is desirable to have as much gas as possible, and no leakage of air during the experiment, so that when the bulb containing a sample of the gas is taken away and weighed in order to find the proportion of the mixture, it may truly represent the condition of things during the earlier part of the experiment. It is best to work at a high temperature, if such is required to fill the chamber with gas which is nearly at atmospheric pressure.

Carbon bisulphide vapour is well superheated at a temperature of 30° and a pressure of 25 cm. The apparatus is, therefore, heated to that temperature ; and RI for air is first measured.

Fig. 4 shows the readings obtained, and the curve which is drawn to find R and I . These are determined to be 4.95

Fig. 4.

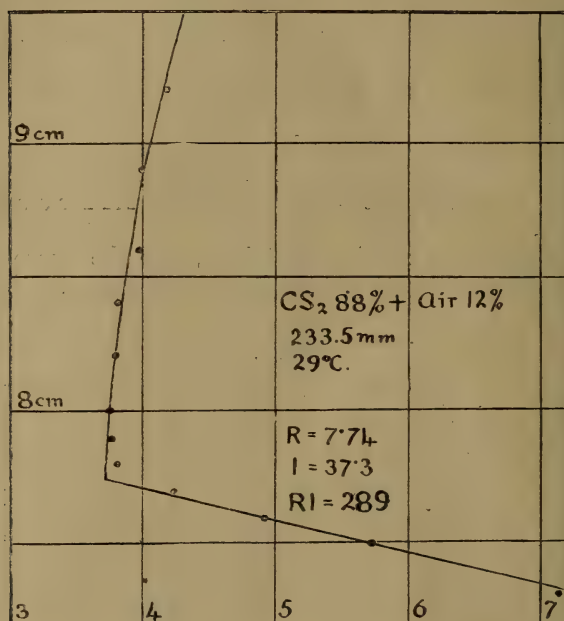


and 42.8 respectively, so that $RI=212$, the temperature being $30^\circ C$., and the pressure being 75.1 cm. The apparatus is then exhausted and filled with CS_2 vapour to a pressure of about 24 cms. It is known from a separate experiment that 1000 volts per cm. is a saturating potential gradient, and a battery of 300 volts is therefore put on to the 3 mm. chamber.

The readings then taken, and the curve drawn are shown in fig. 5. It appears from these that $R=7.74$, $I=37.3$, so

that $RI=289$. The pressure has altered about 1 cm. during the determination of the curve, but was found to be 23.35 at the moment when the corner (R, I) was passed. The temperature at the same time was 29° . Now the stopping-power of

Fig. 5.



CS₂ is 2.20, and the stopping-power of the mixture is (comparing with the previous experiment)

$$\frac{495}{774} \cdot \frac{751}{233.5} = 2.06.$$

Hence :

If x be the percentage of gas, we must have

$$x \times 2.2 + 1 - x = 2.06, \quad \therefore x = 88.5.$$

The vapour is then cleared out of the apparatus, and dry air admitted. The value of RI for air is again determined, as shown in fig. 6, and found to be 214.

In the second experiment the proportion of gas to air, by pressure, is as 88.5 to 11.5; but, according to the energy spent by the α particle, as 88.5 to $11.5/2.2$, i. e., as 94.5 to 5.5. Hence

the true value of RI is found from the equation

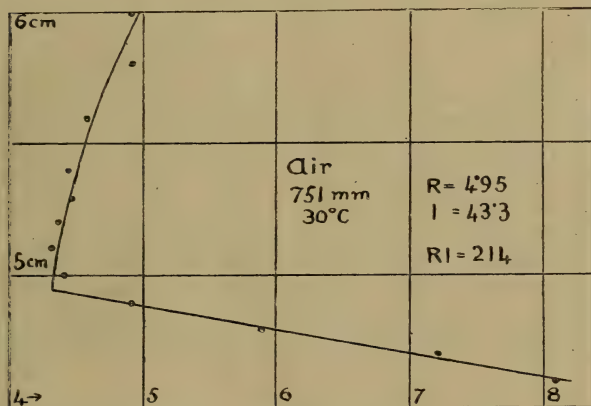
$$\cdot 945 \text{ RI} + \cdot 055 \times 213 = 289;$$

whence $\text{RI} = 294.$

Hence specific ionization of $\text{CS}_2 = 294/213 = 1.38.$

The results of this particular experiment are recorded in the second line of the results for carbon bisulphide in § 2.

Fig. 6.



It seems probable that the determination of the ionization in various cases due to the α rays may be of considerable importance, and I therefore attach a drawing of the apparatus which I have used (Plate IX.) in the hope that it may save the time of any other workers in this direction.

In the figure, P is one of the three glass pillars which support the high potential plate. I have also used a glass plate, as shown by the dotted surface, to insulate the upper plate of the ionization-chamber. Sulphur and ebonite do not stand the heat. The upper gauze, *gg*, is the lower wall of the chamber, *g'g'* is the lower gauze and is earthed; it is supported by three brass pillars, only one of which is shown. The vertical tubes are shown at *TT*, and the radium plate at *RR*. The semaphore, *ss*, is made of thin sheet-copper, and can be turned round so as to uncover the radium. It may be worth while mentioning that I have found it better to keep the plate, *QQ*, "out of sight" of any insulating material connected with the high-potential plate; and if this is not done then the creep of electricity over the insulators, which is apt to occur when the potential is changed, exerts a troublesome electrostatic effect. *DEFG* is the outline of the electric oven.

Volts. per cm.	Pressure in cm.	Tempe- rature.	Percent. of Gas by Pressure.	Percent. of Gas by Energy.	RI.	RI (corr'd.).	RI (air).	Ratio.
<i>Ethyl Chloride.</i>								
1670	23.3	14	89	95	244	247	191	1.29
3000	38	16	93	97	270	272	204	1.33
"	58	26	91.5	96.5	266	268	203	1.32
1670	26.2	72	81	91	202	211	166	1.27
"	31.8	60	85	93	238	242	186	1.30
"	32.8	60	85	93	239	243	186	1.30
"	29.4	34.5	91.5	96	258	261	199.5	1.31
"	42.1	37.5	88.5	94.5	253	256	198.5	1.29
								<i>Mean 1.30</i>

<i>Carbon Tetrachloride.</i>								
1670	25.0	60.7	89	97	257	259	197	1.315
"	22.8	61	72	91	259	265	199	1.33
"	27.9	61	83	95	253	256	199	1.285
"	25.7	54	81	94.5	264	267	204	1.31
"	26.4	53.5	90	97.5	264	266	202	1.32
								<i>Mean 1.31</i>

<i>Ethyl Iodide.</i>								
1000	29.2	65	75.5	91	223	228	177	1.29
1700	25.2	68.5	90.5	97	261	263	207	1.27
								<i>Mean 1.28</i>

<i>Ethyl Alcohol.</i>								
1000	34.8	72	85	98	213	216	174	1.24
"	33.0	67	73	85	217	224	184	1.22
								<i>Mean 1.23</i>

<i>Methyl Alcohol.</i>								
1000	35.9	65	92	94	217	219	179	1.22

<i>Methyl Iodide.</i>								
1000	35.35	47	88	95	210	213	160	1.33

<i>Benzene.</i>								
1670	27.4	62	71	89.5	226	232	181	1.28
"	27.2	61.5	77.5	92.2	235	240	185	1.30
1000	25.6	67	89	96.5	251	253	194	1.30
1670	29.3	67	88	96	248	250	198	1.26
"	27.0	67	83	94	252	256	198	1.29
								<i>Mean 1.29</i>

Volts per cm.	Pressure in cm.	Temperature.	RI.	RI (air).	Ratio.
<i>Acetylene.</i>					
1000	Atmo.	54	274	223	1.23
"	"	37.5	298	232	1.28
"	"	30.5	294	229	1.28
"	"	70	252	201	1.25
					<i>Mean 1.26</i>

<i>Ethylene.</i>					
1000	Atmo.	34.5	290	227	1.28

Volts per cm.	Pressure in cm.	Temperature.	RI.	RI (air).	Ratio.
<i>Carbon Dioxide.</i>					
1000	Atmo.	20	235	215	1.09
"	"	72	192	176	1.09
"	"	31	237	225	1.05
					Mean 1.08
<i>Nitrous Oxide.</i>					
1000	Atmo.	29	240	229	1.05
<i>Oxygen.</i>					
1000	Atmo.	20	247	226	1.09
<i>Nitrogen.</i>					
1000	Atmo.	19	214	224	.96

Of the measurements recorded in the above tables, those for acetylene, ethylene, carbon dioxide, and nitrous oxide were made some time ago. But they are probably quite correct enough to rank with the rest, which have for the most part been made recently, since they are not affected by temperature and initial recombination difficulties. The measurements most likely to contain error are those of the alcohols and methyl iodide, the latter because I have been unable from lack of material to repeat the one somewhat ancient determination, the former because for some reason the alcohols are very difficult to manage in my apparatus. They are apt to cause—particularly methyl alcohol—very large normal leaks, though other vapours, such as benzene, have no such effect. I believe the cause to be connected with the presence of minute particles of fluff, which bridge across the walls of the ionization chamber, being stretched along the lines of force. Although the apparatus is guarded with plugs of glass-wool, yet things of this sort seem to find their way into the chamber at times, and it is possible that the methyl alcohol sets them free from the sides or base of the apparatus to which they are fastened by traces of grease. I have only once had the apparatus in perfect working order with methyl alcohol: at that time I had gone over the working parts with a magnifying glass to find and remove every foreign particle, and had washed the whole apparatus out with methyl alcohol itself. These good conditions lasted only a short time, and unfortunately a second cleansing process was not equally effective.

I must point out that the results for benzene and acetylene are now close together. In the preliminary paper I believed them to differ considerably, and used them as an illustration of the want of direct connexion between the energy spent and

the ionization produced. It will be seen later that this effect is now clearly shown, but I was unfortunate in using a comparison of benzene with acetylene as an illustration.

§ 3 *.

Though our knowledge of the process of ionization by the α particle is as yet only small and imperfect, it does not seem out of place to draw together what facts we do know, and to endeavour to connect them by some thread of argument, which may be useful for a time.

In the first place, there is the fact that the ionization produced by the α particle increases as its velocity diminishes. Now, Rutherford has recently shown (Phil. Mag., Aug. 1906) that the particle spends energy at a uniform rate along its path. It follows, therefore, that the ionization produced is not proportional to the energy spent. In my preliminary paper I have already given a reason for supposing that the energy spent and the ionization produced are not directly connected, viz., that the former is related to the atomic weight by a simple law and the latter is not.

As a temporary hypothesis let us suppose that there is an intervening link; that the α particle produces a primary effect A, which in turn produces a secondary effect B. The latter consists of ionization, the former may or may not do so. It is in the production of the primary effect that the energy of the particle is spent.

Since the energy spent is related to the atomic weight by a simple law, since it is independent of velocity, and since there is a critical speed at which all ionization ceases, which speed is the same for all atoms, it appears clear that A is a sub-atomic effect. It consists in the performance of some act which always involves the expenditure of the same amount of energy; and the stopping-power of an atom is proportional to the number of times that the act is performed within it. The effect might consist, for example, in some operation upon a common constituent of all atoms, such as an α particle. The critical speed might be that at which the moving α particle failed to penetrate, or, more generally, act upon the α particle of the atom.

In the next place, consider the effect B. The proportion of ionization to energy spent varies from molecule to molecule, and is dependent on the velocity of the α particle. The results described in this paper show that, as already said, $di = kf(v)de$. The nature of the function $f(v)$ is of great

* The greater part of § 3 has been written since the above was read before the Royal Society of South Australia.

interest. In two previous papers I have made attempts to find it. In the first (Phil. Mag. Sept. 1905) I showed that if we assumed the ionization produced to be proportional to the energy spent, and both to v^n , and also assumed all the energy to be spent on ionization, then the form of the curve was most readily explained by taking $n = -\frac{1}{2}$. Later Rutherford showed that the energy of the α particle was not all spent on ionization, but that much still remained when ionization ceased. Using his figures, I then pointed out that with this modification of the hypothesis it seemed probable that $n = -2$ (Phil. Mag. Nov. 1905). But Rutherford's recent work shows that the hypothesis is still fundamentally wrong, because the ionization is not proportional to the energy spent. His results settle the whole question.

If v = the velocity of the particle, r the range yet to be run, d a constant, which Rutherford estimates at 1.25 cm., then his conclusion is that v is proportional to $\sqrt{r+d}$. Now I have shown (Phil. Mag. Nov. 1905) that the ionization produced by the particle during the last r cm. of its path is proportional to $\sqrt{r+d} - \sqrt{d}$ where $d = 1.33$. The two values of d may be taken to be the same. Hence di/dr is proportional to $1/\sqrt{r+d}$, i. e., to $1/v$; which means that $i'(v) = 1/v$, or that the ionization produced at different points of the path in any gas is proportional to the time spent by the α particle in crossing the atom.

The formula which I have used here for the ionization was calculated on the hypothesis that the α particle lost its ionizing power abruptly, and that the slope of the top of the ionization curve was due to the effects of the thickness of the Ra film. Bronson's results (Phil. Mag. June 1906) seem to show that the loss of ionizing power is not quite so sudden as I supposed it to be. But I find that this does not affect the calculation of the form of $i'(v)$. For we may take an extreme view and suppose the whole of the top slope to be due to a gradual decay of the α particle's powers, and none to the thickness of the radium layer. In that case the form of the ionization curve represents the effects of one particle. Now, the ionization at 6.5 cm. (in air) for RaC is nearly $4/3$ of the ionization at 5 cm. At the former distance $r+d = 5 + 1.25 = 75$, and at the latter $2 + 1.25 = 3.25$. But $\sqrt{3.25}/\sqrt{1.75} = 1.36$; which is very nearly $4/3$. Thus the ionization on this hypothesis also is inversely proportional to $\sqrt{r+d}$, and the true explanation of the top slope must lie between the two extremes.

It seems clear, then, that the ionization in the molecule is proportional to the energy spent in it (i. e., to the stopping-power, or the amount of the effect A), to the velocity of the

α particle inversely, and to a quantity k , constant for any one gas, but varying from gas to gas. It is this quantity which is given in the last column of the tables above.

The velocity of the α particle might enter into the formula because A is effective in producing B in proportion to the derangement of the atom or molecule consequent on the presence of the particle within it, and therefore to the time during which the intrusion lasts. There is something odd about this conclusion, which suggests a reconsideration of the position.

At this stage, therefore, it is natural to raise the question whether the effect A really is the cause of the effect B, whether, that is to say, the energy spent by the α particle goes to the production of ions, or the ionization energy comes from some other source and the α particle merely pulls the trigger in its passage through the molecule. The fact that the ionization produced varies as the time of passage is certainly indicative of the truth of the latter hypothesis; whilst the occurrence of the stopping-power in the expression for the ionization is not necessarily evidence against it, because the factor k might be taken in conjunction with s , and ks might be found to represent not some derivative of the energy spent by the particle within the molecule, but some inherent property of the molecule which determined the ionization produced in consequence of the pulling of the trigger.

The quantity ks represents in the first place the specific ionization of the molecule; that is a relative measure of the ionization produced in a molecule when an α particle passes through it at a given speed. Now, it is an extraordinary thing that the values of ks which I have obtained for different molecules prove to be nearly related to already well-known molecular constants, such as the molecular volumes, molecular refraction constants, and so on.

In the following table the values of k , s , and ks of a number of substances are given in the first three columns; the fourth contains the values of the molecular volume v , and the fifth the ratio v/ks . The values of the volumes were for the most part taken from the tables in Ostwald's *Lehrbuch der Allgemeinen Chemie*, 2nd edition, p. 356, &c., but those of C_2H_2 and C_3H_4 were calculated from the general equation for obtaining the molecular volumes of organic compounds, and the values for CO_2 , O_2 , and H_2 were adopted on the assumption that they fell into line with the same equation. This is justifiable since my immediate object is to show a relationship between ks and the atomic volume in combination. As a matter of fact, the molecular volume of O_2 *per se* has been found by Dewar to be 27.4 (Chem. News, June 1898).

This is close to the value in the table, viz., 24.4. But Dewar also finds H_2 to be 28, which is much larger than the value used in the ordinary formula.

	$k \times 10^2$.	$s \times 10^2$.	$ks \times 10^2$.	v .	$v/ks \times 10$.	B .	$ks/B \times 10^2$.
C_6H_6	129	333	430	96.0	223	75.5	5.8
C_5H_{12}	135	359	485	117.0	242	87.5	5.5
C_4H_6	128	135	173	44.0	254	43.0	3.9
C_2H_2	126	111	140	33.0	236	36.0	3.9
$C_4H_{10}O$	132	333	440	106.0	241	83.0	5.3
C_2H_6O	123	200	246	62.0	252	47.0	5.2
CH_4O	122	143	174	42.5	244	...	...
CCl_4	132	400	528	104.0	197	80.0	6.8
$CHCl_3$	129	316	408	85.0	208	72.0	5.7
C_2H_5Cl	132	236	312	71.0	227	55.5	5.6
CH_3I	133	258	343	66.0	193	52.0	6.6
C_2H_5I	128	312	400	86.0	215	68.8	5.8
CS_2	137	218	299	62.0	207	50.0	6.0
CO_2	108	147	159	35.4	222	30.0	5.3
N_2O	105	146	153	...	...	29.0	5.3
O_2	109	105	115	24.4	212	19.0	6.1
N_2	96	96	94	...	...	...	...
H_2	100	24	24	11.0	460	8.6	2.8

The value of k for H_2 is set down as 100. This is only approximate, and is probably too high. Its accurate determination will require the construction of special apparatus.

The agreement between the ratios v/ks in the fifth column is not such as to show that v and ks are directly proportional; but it is good enough to suggest strongly that they both rest immediately on some more fundamental property. The case is even a little stronger than appears at first sight, since it is clear that H_2 contributes an abnormal amount to the molecular volumes; the ratio v/ks is high whenever H preponderates in the molecule. Moreover, the molecular refractions also run closely parallel, as is well known, with the molecular volumes, and in general the connexion between the various physical properties of the molecule and its volume is more obvious than any connexion with its molecular weight. Consequently the quantity ks is closely related to most of the physical properties of the molecule. As a second instance, I have put in the sixth column of the above table the respective values of Sutherland's molecular volume B (Phil. Mag. Jan. 1895), and shown in the last column that this also is closely connected to ks . According to Sutherland, B tends to be proportional to the electric moment of the molecule. In this case also the variations in the ratios (see the last column) seem to be due to abnormalities in B rather than in ks ; *e.g.*, C_2H_2 and C_2H_4 would fall into

line if the values of B for these substances were more in keeping with those for C_6H_6 and C_5H_{12} .

Since k is nearly the same for a number of gases, v/s is also nearly the same. Thus the molecular volume is connected, not very distantly, with the sum of the square roots of the weights of the atoms which make up the molecule.

Each of these physical properties which are so nearly related is partly additive, partly constitutive. For example, the molecular volume of an organic molecule depends in part on the sum of the volumes of the constituent atoms, and in part on the mode of constitution. This suggests that there is some fundamental and purely additive property of the atom itself, on which various semi-additive properties are based. For this reason it appears to be of great interest that the stopping-power of the atom has shown itself to be simply additive, so far as experiment has tried it; and at the same time to be closely connected with the atomic volume, the atomic refraction, and the rest. The additive nature of the constant may be seen from the following table, in which the observed stopping-powers of a number of gases are set alongside those calculated from assumed values for H, C, O, and Cl.

Assume $H_2 = \cdot 24$, $C_2 = \cdot 85$, $O_2 = 1\cdot 03$, $Cl_2 = 1\cdot 78$, the air molecule being taken as standard :—

	C_2H_2 .	C_2H_4 .	C_6H_6 .	C_5H_{12} .	CH_4O .	C_4H_8O .
Calculated	1·09	1·33	3·27	3·56	1·41	2·08
Observed	1·11	1·35	1·33	3·59	1·43	2·00

	$C_4H_{10}O$.	CO_2 .	CCl_4 .	$CHCl_3$.	C_2H_5Cl .
Calculated	3·41	1·47	3·98	3·21	2·34
Observed	3·33	1·48	4·00	3·16	2·36

It is of course too early to say that the stopping-power has been proved to be a perfectly additive property of the atom, yet it is clear enough that it is more so than any other known property, except one. The more nearly experiment shows it to be strictly additive, the greater will be its title to rank with mass itself. I hope to begin soon a fresh and more accurate set of experiments in the endeavour to find to what extent the additive law holds.

The near proportionality of the stopping-power to the atomic square root is an effect which is quite apart from its additive nature. Its existence is a connecting link between the atomic weight on the one hand and the atomic volume, refractive power, &c., on the other. The preliminary paper

on this subject contained a table of stopping-powers as found up to that time. I have made several new measurements of these constants, which are, I believe, an improvement on the old. This is particularly the case with the metals Au, Pt, Sn, Ag, Cu, and Al, since the specimens used were obtained as pure from Messrs. Johnson, Matthey & Co. I find that if the stopping-powers of S, Cl, and I are calculated from those of molecules containing them, on the assumption that the additive law holds, then these fit in very well with the metals. So also does Br fit in very well; it is quite possible that the divergence is due to experimental error, since the only measurement on a molecule containing Br was made at a very early stage of this inquiry. The divergence from the exactness of the square-root law, which I have previously pointed out, seems to occur only in the molecules whose weights are below 30; these have an abnormally low value, as may be seen from the table below, in which the "air-atom" = 1. It is curious that a similar effect should occur in the case of the atomic heats:—

	H.	C.	N.	O.	Al.	S.	Cl.	Fe.	Ni.
s	24	85	94	105	1495	176	178	229	244
$\sqrt{\omega}$	1.00	3.47	3.74	4.00	5.20	5.65	5.96	7.48	7.65
$s\sqrt{\omega} \times 10^3$...	240	246	251	262	287	312	299	307	319

	Cu.	Br.	Ag.	Sn.	I.	Pt.	Au.	Pb.
s	246	260	328	356	344	414	422	427
$\sqrt{\omega}$	7.96	8.93	10.37	10.9	11.2	13.95	14.0	14.35
$s\sqrt{\omega} \times 10^3$...	309	291	316	326	307	297	301	298

[Note added Nov. 8.—The figures in this table will no doubt be somewhat modified by more accurate measurements, and I am revising my methods for this purpose. The values for C and N are certainly, perhaps 4 per cent., too low. As regards C, see also the accurate measurements of Kučera and Mašek (*Phys. Zeit.* 1906, p. 634). The value of H may be in error as this gas really requires a special apparatus. Good sheets of Fe and Ni are hard to obtain. The value for Br requires redetermination. I think the rest are fairly accurate, and I trust the average error in this table will not in the end be found to exceed one or two per cent. The values of s for Sn and Ag certainly show some variation from the square-root law as indicated by the difference of the ratios 316 and 326.

The results of Kučera and Mašek are a very welcome confirmation of the square-root law. Only on one point there seems to be still an obscurity. These authors claim to have

shown that the stopping-power of a metal* depends on the velocity of the α particle, and certainly some effects observed by them and Kleeman and myself would be explained if this were true. On the other hand, this hypothesis is contradicted by Rutherford's direct measurements of the velocity after passing through various sheets of foil. It might be argued that Kučera and Mašek measure the *relative* stopping-powers of metal and air, and that their results really show that the stopping-power of air decreases with increasing velocity of the α particle, not that that of the metal increases. But this is inconsistent with their own measurements in the earlier part of the paper, where they show that two equal Al foils cause twice as much drop as one; and also with the similar and fuller measurements of this effect made by Levin. I may add that in comparing their own results with those of Kleeman and myself, they make a natural mistake in supposing that we placed our foils in contact with the Ra. But as a matter of fact they were about 1.5 cm. away, so that the average speed of the particle in going through a metal equal to 3 cm. of air was about the same as the average speed in a gas.]

One other point invites some consideration. Whilst the saturated ionization curve seems to be the same for all gases, yet the effects of initial recombination vary from gas to gas and from point to point on the curve. This fact can be explained by the consideration that the amount of the ionization produced is an intramolecular effect, and is therefore independent of the physical conditions of the molecule and of the relations of one molecule to another, whilst the amount of initial recombination depends on extramolecular relations, on pressure, perhaps on temperature, and so on. The increase of initial recombination towards the end of the path of the particle may be due in part to the existence of a greater number of molecules that have lost more than one ion, since in such cases recombination would be harder to prevent. This raises the question as to how the ionization is distributed between the molecules which the α particle traverses. There does not appear to be any evidence, as yet, that the chance of an ion being formed from a molecule is dependent on whether the molecule has already lost one or more ions: rather the contrary. If this is the case, occasional molecules must lose several ions. Nor is it yet clear in what mode ionization occurs. Does the α particle simply cause the removal from the molecule of one or more electrons? May there not possibly be a more complete disruption of the molecule, or even the atom? There is one curious parallelism

in numbers which may have a bearing on this question. Ramsay and Soddy (Proc. Roy. Soc. lxxii. p. 204, 1903) found that 50 mmg. of radium bromide in solution evolved gases at the rate of .5 cc. per day—*i. e.*, 2×10^{19} molecules per day. Now, Rutherford has shown that one gram of radium bromide, without its radioactive descendants, produces 3.6×10^{10} α particles per second. Each α particle makes 86,000 pairs of ions. Hence the number of ions made in one day by 50 mmg. is

$$3.6 \times 10^{10} \times .05 \times 60 \times 60 \times 24 \times 172,000 = 2.7 \times 10^{19}.$$

This number is an inferior limit. A superior limit is found by considering all the radioactive products of radium to be present in full, in which case the number will be between five and six times greater. The close agreement of these numbers certainly fits in with the hypothesis that an actual disruption of the water molecule takes place in consequence of the passage of the α particle through it.

I owe my thanks to my assistant, Mr. A. L. Rogers, for the great care and skill with which he has made the apparatus used in this work, and drawn the plate illustrating this paper.

Nov. 14.—The very interesting experiments described by Kučera and Mašek in recent numbers of the *Phys. Zeit.* afford a very welcome confirmation of the square-root law of absorption.

On one point only there is still some obscurity. It was pointed out by Kleeman and myself (Phil. Mag. Sept. 1905) that the α particles from Ra A were less stopped than those from Ra C by a sheet of metal placed at a given distance from the Ra. (The distance was 1.5 cm. in our experiments, not zero as Kučera and Mašek suppose; the point is of some importance in view of their comparison of our results with their own.) This could be explained by supposing that "the slower α particles were a little less affected than the swifter" (Phil. Mag. Sept. 1905, p. 337). Kučera and Mašek come to the conclusion that they have established this fact from their own experiments. The trouble is that this is hardly consistent with Rutherford's experimental determination that successive equal layers of Al foil absorb equal energies from the α particle's motion. Unless indeed it is supposed that metals absorb to an extent which is independent of speed, and that it is the air which is at fault; for the experiments of Kučera and Mašek and those of Kleeman and myself relate

to the relative absorption of metal and air. But this again is opposed to the experiment of Levin (*Phys. Zeit.* Aug. 1906), who found that equal layers of foil dropped the ionization curve by equal amounts; and to a similar experiment described by Kučera and Mašek themselves (*Phys. Zeit.* xviii. p. 630).

The following figures are taken from measurements by Kleeman and myself:—

Drop of curve when sheet of metal is placed 1·5 cm.
above Ra.

	Drop for RaC.	Drop for RaA.	Drop for Emanation.	Drop for Ra.
1. Al	2·31	2·25		
2. Sn	1·92	1·77	1·67	
3. Sn	1·91	1·81		
4. Ag	·86	·81	·50	·73
5. Au	1·63	1·46	1·36	

XXVI. *The Efficiency of the Plug-Cock Fog-Chamber.*

By CARL BARUS*.

1. **INTRODUCTION.**—In the last few years I have had occasion to use the fog-chamber extensively for the estimation of the number of colloidal† nuclei and of ions in dust-free air under a great variety of conditions. These data were computed from the angular diameter of the coronas of cloudy condensation; and it is therefore necessary to reduce all manipulations to the greatest simplicity and to precipitate the fog in a capacious vessel, at least 18 inches long and 6 inches in diameter. To obtain sufficiently rapid exhaustions it is thus advisable to employ a large vacuum-chamber, and the one used was about 5 feet high and 1 foot in diameter. The two vessels were connected by 18 inches of brass piping, the bore of which in successive experiments was increased as far as 4 inches; but 2-inch piping, provided with a 2½-inch plug stop-cock, sufficed to produce all

* Communicated by the Author.

† See Smithsonian Contributions, No. 1309, 1901; No. 1373, 1903; No. 1651, 1906. Carnegie Publications, No. 40, 1906; No. 62, 1907 (forthcoming).

the measurable coronas as far as the *large* green-blue-purple or the bluish type, the largest of the useful coronas producible in a fog-chamber by any means whatever. Moreover, it is merely necessary to open the stop-cock as rapidly as possible by hand, using easily devised annular oil-troughs at top and bottom of the plug, both to eliminate all possible ingress of room-air and to reduce friction. Fog-chambers larger than the one measured were often used, and it is curious to note that the efficiency of such chambers breaks down at a certain high exhaustion abruptly, while up to this point (varying greatly from chamber to chamber) different apparatus behave nearly alike. The vacuum-chamber is put in connexion with an air-pump, the fog-chamber with a well-packed filter by the aid of stop-cocks. Water nuclei are precipitated between exhaustions from the partially exhausted fog-chamber.

2. *The Variables*.—After reading the initial pressures of the fog- and vacuum-chambers, it is expedient to open the stop-cock quickly and thereafter to close it at once before proceeding to the measurement of the coronas. Eventually, *i. e.* when the temperature is the same in both the fog- and vacuum-chambers, they must again be put in communication and the pressures noted, if the details of the experiment are to be computed.

The following series of variables (Table I.), where p denotes pressure, ρ density, τ absolute temperature, π vapour pressure, is to be considered. The ratio of volumes of the fog- and vacuum-chambers was $v/V = .064$.

TABLE I.—Notation.

Drop of pressure $\delta p = p - p_3$, observed; $\delta p = p - p_2$, computed.

States No.	Fog-Chamber.				Vacuum-Chamber.				Remarks.
	p	ρ	τ	π	p'	ρ'	τ	π	
1	p	ρ	τ	π	p'	ρ'	τ	π	Initial states. Chambers separated.
2	p_1	ρ_1	τ_1	π_1	p_1'	ρ_1'	τ_1'	π_1'	Adiabatic states, after exhaustion. Chambers communicating.
3	$\bar{p}_1$	$\bar{\rho}_1$	$\bar{\tau}_1$	$\bar{\pi}_1$	$\bar{p}_1'$	$\bar{\rho}_1'$	$\bar{\tau}_1'$	$\bar{\pi}_1'$	Do., after condensation of water in the fog-chamber.
4	p_2	ρ_2	τ	π	p_2'	ρ_2'	τ	π	Chambers separated <i>before</i> condensation ensued. Original temperature regained.
5	$\bar{p}_2$	$\bar{\rho}_2$	τ	π	$\bar{p}_2'$	$\bar{\rho}_2'$	τ	π	Chambers separated <i>after</i> condensation. Original temperature regained.
6	p_3	ρ_3	τ	π	p_3	ρ_3	τ	π	Chambers communicating after exhaustion. Original temperature restored.

At the beginning (case 1), the fog-chamber is at atmospheric pressure, p (nearly), the vacuum-chamber at the low pressure p' , and both at the absolute temperature τ . On suddenly opening the stop-cock the adiabatic pressures, &c., given under no. 2 appear, supposing that no condensation has yet taken place in the fog-chamber. If the stop-cock could now be suddenly closed and the whole apparatus allowed to regain the original temperature τ , the conditions under no. 4 would obtain. This is virtually the case in Wilson's* piston apparatus, and consequently these variables are comparable with his results, *cf.* § 3. In the present apparatus, however, condensation takes place within the fog-chamber before the stop-cock can be closed, and thus an additional quantity of air is discharged from the fog- into the vacuum-chamber. After condensation and before the stop-cock is closed, the conditions under no. 3 apply; when the stop-cock has been closed and the apparatus thereafter allowed to regain the room temperature τ , the conditions are shown in no. 5, and may be observed (with crude approximation) in the isolated chamber. Finally when the chambers are put in communication, the variables, no. 6, are the same in both.

This account of the phenomena may seem prolix; but it is essential to a just appreciation of the efficiency of the plug-cock fog-chamber. Quantities in the above table referring to a given chamber may be connected at a given time by Boyle's law, as for instance, $p - \pi = R\rho\tau$. This gives eleven equations, some of which may be simplified. Corresponding quantities in groups 1 and 2, as for instance τ/τ_1 , may be connected by the law for adiabatic expansion giving two equations. In addition to this, an equation stating that a given mass of air is distributed in fog- and vacuum-chamber (volumes v and V , respectively) is available; or

$$v\rho + V\rho' = v\rho_1 + V\rho_1' = \&c., = (v + V)\rho_3.$$

All the quantities π are supposed to be given by the corresponding τ , though at high exhaustions the lower limit of known data, $\pi = f(\tau)$, is often exceeded.

3. *Computation of p_1 and p_2 .*—It will first be necessary to compute p_2 , the pressure which would be found in the fog-chamber when it has again reached room-temperature τ , if there were no further transfer of air from fog-chamber to vacuum-chamber, due to the condensation of water-vapour in the former after adiabatic cooling. The pressure p_2 is given

* C. T. R. Wilson, Phil. Trans. London, vol. 192, pp. 405 *et seq.* (1889).

by the gauges of the piston apparatus, since there is but a single chamber. In this respect the plug-cock apparatus differs from it, because the corresponding gauge reading is essentially even less than $\bar{p}_2$, § 5.

The nearly rigorous solution when the air in both chambers is continually saturated, leads to transcendental equations for the adiabatic pressures $p_1 = p_1'$, which can therefore only be obtained approximately. If the vapour pressures π_1 and π_1' correspond to p_1 and p_1' , the results are

$$(p_1 - \pi_1')^{c/k} = \frac{(p_3 - \pi)(1 + v/V)}{(p' - \pi)^{(k-c)/k} + v/V \cdot (1 - (\pi_1 - \pi_1')/(p_1 - \pi_1'))^{c/k} (p - \pi)^{(k-c)/k}}$$

$$(p_1 - \pi_1)^{c/k} = \frac{(p_3 - \pi)(1 + v/V)}{(1 - (\pi_1' - \pi_1)/(p_1 - \pi_1))^{c/k} (p' - \pi)^{(k-c)/k} + v/V \cdot (p - \pi)^{(k-c)/k}}$$

where approximate values must be entered for π_1 , π_1' , p_1 , in the denominator on the right side of the equation.

Similarly,

$$(p_2 - \pi) = (p_1 - \pi_1)^{c/k} (p - \pi)^{(k-c)/k},$$

$$(p_2' - \pi) = (p_1' - \pi_1')^{c/k} (p' - \pi)^{(k-c)/k},$$

$$\tau/\tau_1 = ((p - \pi)/(p_1 - \pi_1))^{(k-c)/k},$$

$$\tau/\tau_1' = ((p' - \pi)/(p_1' - \pi_1'))^{(k-c)/k}.$$

Making use of the values found incidentally elsewhere, the data of Table II. were computed on a single approximation. They are reproduced in the graph.

TABLE II.—Showing successive values of pressure and temperature in the plug-cock fog-chamber. Volume ratio of fog- and vacuum-chambers, $v/V = .064$; $p = 76$; $t = 20^\circ \text{C.}$; $\pi = 1.7 \text{ cm.}$; t refers to $^\circ \text{C.}$, τ to absolute temperature; δp denotes the drop in pressure.

OBSERVED*.				COMPUTED†.											
p_1'	p_3	$\bar{p}_2$	$\bar{p}_2'$	p_1	p_1'	p_2	p_2'	$\bar{p}_2$	π_1	π_1'	$\bar{\pi}_1$	t_1	$\bar{t}_1$	t_1'	
43.5	45.5	47.9	45.6	46.1	46.1	54.7	44.9	49.9	.0	2.2	.7	-17.8	+5.2	+24.1	
51.5	52.5	54.3	...	52.5	52.5	59.6	52.0	55.5	.2	1.9	.9	- 8.3	9.4	21.3	
59.5	59.7	62.2	...	59.3	59.3	64.6	59.4	61.5	.5	1.7	1.1	+ .8	12.7	19.8	
				$\bar{e}p_2 = \bar{e}p_3 =$		$\delta p_2 / \delta p_3$									
				$p - p_2$		$p - p_3$									
				0		0									
				16.3		11.4									
				23.5		16.4									
				30.5		21.3									

* These observations merely illustrate the equations. No attempt was made at accuracy. See Chart.

† The values of $\bar{p}_1/\rho_1 = .91, .93, .95$, respectively.

dry air or not

$$(p_2' - \pi) + v/V \cdot (p_2 - \pi) = (p_3 - \pi) (1 + v/V)$$

is constant for a given absorption,

$$\delta p_2' = -v/V \cdot \delta p_2.$$

Hence if $\delta p_2 = 2$ cm., since $v/V = .064$,

$$-\delta p_2' = .064 \times 2 = .13 \text{ cm. nearly.}$$

This case is illustrated graphically for $p' = 45$ cm. in the notched curves of the figure in a way easily understood. It seems probable that whereas the smaller fog-chamber has lost too much air to even approach the isothermal pressures, p_2 , the large vacuum-chamber is only a millimetre short of them when the cock is again closed. The constancy of the observed difference, $\bar{p}_2 - p_3$, seemed at first to be referable to the systematic method of investigation, though the effect of the precipitated moisture (which has not yet been considered) will largely account for it. See § 5.

Anomalous relations in the data for the fog-chamber, as in the case of $p' = 59.5$ cm., are direct errors of observation. On the other hand, however, since within the ranges of observation and very nearly $p = a$, $p_2 = a_2 + b_2 p'$, $p_3 = a_3 + b_3 p'$, it follows that $(p - p_2)/(p - p_3)$ may approximately be written $A + Bp'$, where a , b , A , B , &c. are constant. Frequently B is negligible, so that $(p_2 - p_3)/(p - p_3)$ is constant, in which case the graph for $p_2 - p_3$ varying with $p - p_3$ passes through the origin.

4. *Computation of $\bar{\tau}_1$.*—To find the temperature of the fog-chamber after the adiabatic temperature τ_1 has been raised by condensation of fog to $\bar{\tau}_1$, it is apparently necessary to compute p_2 first, and then proceed by the method used by Wilson* and Thomson. When the vacuum-chamber is large, however, its pressures vary but slightly, and therefore the pressure observed at the vacuum-chamber after exhaustion, p_3 , when the two chambers are in communication, is very nearly the adiabatic pressure of the fog-chamber, p_1 . This result makes it easier to compute not only $\bar{\tau}_1$, but incidentally the water, m , precipitated per cubic centimetre (without stopping to compute the other pressures), with a degree of accuracy more than sufficient when the other measurements depend on the size of coronas.

To show this, let d , L , and π refer to the density, latent

* C. T. R. Wilson: Phil. Trans. London, vol. 189, p. 298 (1897).

heat of vaporization, and pressure of water (or other) vapour; let ρ , k , c , τ denote density, specific heat at constant pressure, specific heat at constant volume, and absolute temperature of the air, the water-vapour contained being disregarded apart from the occurrence of condensation. As above, let the variables, if primed belong to the vacuum-chamber, otherwise to the fog-chamber. Let the subscripts &c. also be similarly interpreted, so that $\bar{d}$ is the known density of saturated water-vapour at $\bar{\tau}$ degrees absolute.

Assuming the law of adiabatic expansion to hold both for gaseous water-vapour and for wet air in the absence of condensation, it is convenient in a plug-cock apparatus of fog- and vacuum-chamber (where p_1 is nearly given by p_3) to reduce to adiabatic conditions, whence

$$\bar{d} = d \left(\frac{p_1 - \pi_1}{p - \pi} \right)^{c/k} - \frac{\rho c}{L} \left(\frac{p_1 - \pi_1}{p - \pi} \right)^{c/k} (\bar{\tau}_1 - \tau_1),$$

$$\tau_1 = \tau \left(\frac{p_1 - \pi_1}{p - \pi} \right)^{(k-c)/k}, \quad m = \frac{\rho c}{L} \left(\frac{p_1 - \pi_1}{p - \pi} \right)^{c/k} (\bar{\tau}_1 - \tau_1),$$

where m is the quantity of water precipitated per cubic centimetre of the exhausted fog-chamber. Finally $\bar{d}$, the density of saturated water-vapour, must be known as far as $\bar{\tau}$, so that an equation $\bar{d} = f(\bar{\tau})$ is additionally given. Here π_1 , the vapour-pressure at τ_1 , is usually negligible (about .5 cm.) as compared with p_1 , and p_1 may in practice (where great accuracy is not demanded) be replaced by p_3 , which, like p , is read off, while π holds at τ , which is also read off. In conclusion I give a numerical example taken from Table II. for $p' = 43.5$ cm.

If the original equation (isothermal) is taken, $m = 5.36 \times 10^{-6}$ grams per cubic centimetre. If the above equation is taken, $m = 5.35 \times 10^{-6}$. If the above equation is taken and p_1 replaced by p_3 , $m = 5.30 \times 10^{-6}$, the error being 1 per cent. of the true value, near enough in practice to admit of easy correction.

5. *Computation of $\bar{p}_2$.*—Since the plug stop-cock cannot be closed before the water condenses in the fog-chamber after sudden exhaustion, the pressure observed in the fog-chamber when the room temperature reappears is smaller than p_2 . An excess of air has passed to the vacuum-chamber, so that the pressure within the fog-chamber is eventually $\bar{p}_2$, or less. The equations for p_1 and p_1' remain as in § 3.

The new quantities are

$$\bar{p}_2 - \pi = \frac{\bar{\rho}_1}{\rho_1} (p - \pi)^{(k-e)/k} (p_1 - \pi_1)^{e/k},$$

$$\bar{p}_2' - \pi = \frac{\bar{\rho}_1'}{\rho_1} (p' - \pi)^{(k-e)/k} (p_1' - \pi_1')^{e/k},$$

where $\bar{\rho}_1$ is the density of air at $\bar{\tau}_1$. The ratio $\bar{\rho}_1/\rho_1$ may be found when $\bar{\tau}_1$ is known as

$$\frac{\bar{\rho}_1}{\rho_1} = \frac{(\bar{\pi}_1' - \bar{\pi}_1) - (\pi_1' - \pi_1) + (p_1 - \pi_1)(1 + v/V \cdot \tau_1'/\tau_1)}{(p_1 - \pi_1)(\bar{\tau}_1/\tau_1 + v/V \cdot \tau_1'/\tau_1)},$$

where $\bar{\tau}_1'$ and τ_1' , $\bar{\pi}_1'$ and π_1' , p_1 and p_1' are nearly the same.

The last equation may usually be written

$$\bar{\rho}_1/\rho_1 = (1 + v/V \cdot \tau_1'/\tau_1)/(\bar{\tau}_1/\tau_1 + v/V \cdot \tau_1'/\tau_1),$$

and the small quantity involving the vapour-pressures π treated as a correction. It amounts to about 1 per cent of the large quantity. The values of $\bar{p}_2$ are also given in the table and chart. This shows that $\bar{p}_2$ observed is always smaller than $\bar{p}_2$ computed, even when allowance is made for the condensation of water; *i. e.* the fog-chamber begins to appreciably heat itself above the temperature $\bar{\tau}_1$ before the cock can be closed again, so that when isolated the fog-chamber contains less than its proper quantity of air. Only the initial and the final (both chambers communicating) pressures may therefore be taken at the fog-chamber.

6. *Conclusion.*—If the fog-chamber is combined with a large vacuum-chamber, through a sufficiently wide passageway containing an ordinary plug gas-cock to be opened and closed rapidly by the hand, all the measureable coronas of cloudy condensation, due to the presence of colloidal nuclei in wet strictly dust-free air, may be evoked. While such an apparatus admits of capacious fog-chambers and extremely simple manipulation, it has not been shown to be inferior in efficiency to any other apparatus whatever.

The conditions of exhaustion must, however, be computed from the initial pressures of the fog- and vacuum-chambers when separated and their final pressure (after exhaustion) when in communication, all at the same temperature, and the volume ratio of the chambers. The chief pressures and temperatures are shown in the figure for different initial pressures of the vacuum-chamber, the fog-chamber being at atmospheric pressure.

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XXVII. *On the Influence of Past Experience on Future Expectation.* By KARL PEARSON, F.R.S.*

(1) **I**N the whole round of statistical investigation few principles are so much theoretically neglected at present and yet so largely and unconsciously appealed to as that theorem in probability by which our past experience is made a basis for future conduct. It is perfectly true that the logic of this theorem has been criticised and discussed by De Morgan, Boole, Venn, Edgeworth, and many others, but the extent to which it really lies at the basis of most state, municipal and individual actions is too often disregarded. It is not only the insurance companies, but the Chancellor of the Exchequer, the First Lord of the Admiralty, the local officer of health, the merchant, and the general practitioner, who almost daily express their faith in the approximate stability of statistical ratios. One and all, we act on the principle that the statistical ratio determined from our past experience will hold, at any rate approximately, for the near future. This category of the stability of statistical ratios is as all important not only in statistical theory but in practical conduct, as is from a second standpoint in physical theory and also in practical life the principle that the same causes will reproduce the same effects. Neither principle admits of an ultimate logical demonstration; both rest on the foundation of commonsense and the experience of what follows their disregard. Both need considerable care in their application, but what is quite clear is that practical life cannot progress without them. The object of the present paper is to put into a new form the mathematical process of applying the principle of the stability of statistical ratios, and to determine, on the basis of the generally accepted hypothesis, what is the extent of the influence which may be reasonably drawn from past experience.

The problem is of course not novel; it has been treated by a number of the greater writers on probability, but I believe the present investigation is original in character; it extends some of the usual conceptions and shows very serious limitations in the applications frequently made of the old results. Very crude applications of the principle are occasionally made in vital statistics, or even suggested in medico-statistical text-books. Thus past experience has shown that in a certain population a sample of n individuals shows a character in np individuals and an absence of this character in nq individuals ($p+q=1$). These proportions are taken to hold good for future experience, and if a further sample m be

* Communicated by the Author.

expected, the result anticipated will be mp individuals of the given character with a probable error of $\cdot 67449\sqrt{mpq}$. This result is often given without any regard to the relative magnitudes of m and n , or again of p and q . Further, tables of the probability integral are applied to determine the probability of deviations, although many of the users would be much puzzled to cite where the justification for using the normal curve at all is to be found.

It seems therefore desirable to reinvestigate the whole subject. I start as most mathematical writers have done, with "the equal distribution of ignorance," or I assume the truth of Bayes' Theorem. I hold this theorem not as rigidly demonstrated, but I think with Edgeworth that the hypothesis of the equal distribution of ignorance is, within the limits of practical life, justified by our experience of statistical ratios, which *a priori* are unknown, *i. e.* such ratios do not tend to cluster markedly round any particular value. "Chances" lie between 0 and 1, but our experience does not indicate any tendency of actual chances to cluster round any particular value in this range. The ultimate basis of the theory of statistics is thus not mathematical but observational. Those who do not accept the hypothesis of the equal distribution of ignorance and its justification in observation, are compelled to produce definite evidence of the clustering of chances, or to drop all application of past experience to the judgment of probable future statistical ratios. It is perfectly easy to form new statistical algebras with other clustering of chances. We can form other geometries with varied axioms as to parallels. But as the usual axiom of parallels leads to results in close approximation to experience, so the "equal distribution of ignorance" leads us to results which accord reasonably with actual observation. As to the second alternative, even those who criticise Bayes' Theorem apply at every turn in practical life their past experience of statistical ratios to actual conduct.

With this prelude, which appears needful in the present state of opinion as to the basis of probability, I turn to the actual problem before me.

(2) Let the chance of a given event occurring be supposed to lie between x and $x + \delta x$, then if on $n = p + q$ trials an event has been observed to occur p times and fail q times, the probability that the true chance lies between x and $x + \delta x$ is, on the equal distribution of our ignorance,

$$P_x = \frac{x^p (1-x)^q dx}{\int_0^1 x^p (1-x)^q dx} \quad \dots \quad (i.)$$

This is Bayes' Theorem.

Now suppose that a second trial of $m = r + s$ instances be made, then the probability that the given event will occur r times and fail s , is on the *a priori* chance being between x and $x + \delta x$

$$= P_x \frac{m}{r \mid s} x^r (1-x)^s,$$

and accordingly the total chance C_r , whatever x may be, of the event occurring r times in the second series, is

$$C_r = \frac{m}{r \mid s} \frac{\int_0^1 x^{p+r} (1-x)^{q+s} dx}{\int_0^1 x^p (1-x)^q dx} \quad \dots \quad (\text{ii.})$$

This is, with a slight correction, Laplace's extension of Bayes' Theorem.

A further evaluation of C_r is then usually made by expressing the integrals in β -functions, these again in γ -functions, then using Sterling's Theorem, and after a series of approximations, not always closely investigated, showing that C_r can be represented by the ordinates of a normal curve of errors. The present investigation is intended to replace this by a more rigid and non-approximative process.

We have :

$$C_r = \frac{m}{r \mid s} \frac{\Gamma(p+r+1) \Gamma(q+s+1) \Gamma(n+2)}{\Gamma(p+1) \Gamma(q+1) \Gamma(n+m+2)}.$$

Now write :

$$C_0 = \frac{\Gamma(q+m+1) \Gamma(n+2)}{\Gamma(q+1) \Gamma(n+m+2)}, \quad \dots \quad (\text{iii.})$$

then the chances of the successive values of r , 0, 1, 2 m are the successive terms in the series :

$$C_0 \left\{ 1 + \frac{m}{1!} \frac{p+1}{q+m} + \frac{m(m+1)}{2!} \frac{(p+1)(p+2)}{(q+m)(q+m-1)} + \frac{m(m-1)(m-2)}{3!} \frac{(p+1)(p+2)(p+3)}{(q+m)(q+m-1)(q+m-2)} + \&c. \right\} \quad (\text{iv.})$$

This is a simple hypergeometrical series, the momental properties of which have already been studied*.

* Pearson, "On Certain Properties of the Hypergeometrical Series, and on the Fitting of such Series to Observation Polygons on the Theory of Chance," Phil. Mag. Feb. 1899, pp. 236-246.

Now it is clear that the terms of this hypergeometrical series increase as long as

$$\frac{m-r+1}{r} \frac{p+r}{q+m-r+1} > 1.$$

or as long as

$$r < \frac{p(m+1)}{n}. \quad \dots \quad (v.)$$

The nearest whole integer to $p(m+1)/n$ accordingly gives us the *modal* value, or maximum frequency to be expected in second trials. This result is wholly independent of the values of p , q , m , and n .

Let us write $\bar{p} = p/n$, $\bar{q} = q/n$ so that $\bar{p} + \bar{q} = 1$. Then p and q are the chances calculated from the first trial, and the modal or most probable value on the second trial is the nearest whole integer to $(m+1)\bar{p}$.

The value of the mean of the hypergeometrical series is given in Eqn. (8) of the paper cited. Measured from $r=0$ we find :

$$\text{Mean} = \frac{m(p+1)}{n+2} = m\bar{p} + \frac{m}{n+2}(q-\bar{p}), \quad \dots \quad (vi.)$$

whereas from the same origin

$$\text{Mode} = \text{greatest integer in } m\bar{p} + \bar{p}. \quad \dots \quad (vii.)$$

Hence for mode and mean to agree—the first condition for an approach to a Gaussian curve—it is necessary that p should not be very small, and further $m/(n+2)$ should be small. In other words, characters which occur in a very small percentage of the population, or when a relative large second sample is taken, cannot be expected to give on further sampling a *mean* simply determined by the experience of the first sample.

As illustration, let us take the following: 2 p. c. of a certain population, on the basis of a sample of 100, are found to be suffering from malaria. The *mean* value to be expected on the basis of this sample in future samples of 100 is 2.94 or nearly 3 and not 2 p. c. If the first sample were 1000, future samples of 100 might be expected to have an average percentage of 2.10 p. c. For both cases of first sampling the most probable frequency in second samples would be 2 p. c. *But it must be remembered that in dealing with the probable error, we generally take the probable error of the mean and not of the mode.* Hence in rare characteristics considerable caution must clearly be used in tacitly replacing the modes

by the means, from which they may differ, as just shown, by as much as 5 to 50 p. c. of their values.

I now pass to the standard deviation, σ , of the frequency, measuring it as usual about the mean. The value of $\mu_2 = \sigma^2$ is given in the paper referred to (p. 239, eqn. (13)) for a hypergeometrical series. We find, substituting the present values of the symbols, that :

$$\sigma^2 = \mu_2 = m \left(\bar{p} + \frac{\bar{q} - \bar{p}}{n+2} \right) \left(\bar{q} - \frac{q-p}{n+2} \right) \left(1 + \frac{m-1}{n+3} \right). \quad (\text{viii.})$$

It is clear that this can take values differing comparatively widely from the customary $\sigma^2 = m \bar{p} \bar{q}$. This latter carries on the face of it its unfitness to express the desired facts. The variability about the expected average must depend on the extent of the previous experience, but no sign of this is visible in $\sigma = \sqrt{m \bar{p} \bar{q}}$.

Suppose, for example, that the characteristic does not occur in a disproportionately large or small section of the population, then if the samples are considerable as they mostly will be in safe practice $(\bar{q} - \bar{p})/(n+2)$ may be neglected and (viii.) gives :

$$\begin{aligned} \sigma^2 &= m \bar{p} \bar{q} \left(1 + \frac{m-1}{n+3} \right), \\ &= m \bar{p} \bar{q} \left(1 + \frac{m}{n} \right), \end{aligned}$$

if m and n are considerable. Hence a first sample n having given a percentage value of the characteristic p , a second sample of m individuals may be expected to give :

$$100\bar{p} \pm 67.449 \sqrt{\bar{p} \bar{q} \left(\frac{1}{m} + \frac{1}{n} \right)} \text{ per cent.}^* \quad . \quad . \quad (\text{ix.})$$

This is the correct value, and not $100\bar{p} \pm 67.449 \sqrt{\frac{\bar{p} \bar{q}}{m}}$ as is frequently adopted. To justify the latter the first sample must be indefinitely larger than the second, and in many cases this is not true. The first result given shows at once why a small first sample is quite useless in determining the expectancy in the case of a second large sample. The absence of the term $\frac{1}{n}$ in the second formula proves its unfitness.

* This value has been long used in my laboratory and by my statistical students.

Thus far we have seen that a sensible modification in the expected mean may be needful and most probably a still more appreciable deviation in the variability about that mean. It is necessary, however, to consider how far the distribution is really normal round the mean. Otherwise we have no justification for calculating the odds on the basis of tables of the probability integral.

The full values of the second, third and fourth moments about the mean for the hypergeometrical series are given in the paper already referred to (pp. 239-240), and with the changes requisite to suit our formula (iv.) are :

$$\mu_2 = \frac{m(p+1)(q+1)}{(n+2)^2} \left(1 + \frac{m-1}{n+3}\right), \quad \dots \quad (x.)$$

$$\mu_3 = \frac{m(p+1)(q+1)(q-p)}{(n+2)^3} \left(1 + \frac{m-1}{n+3}\right) \left(1 + \frac{2(m-1)}{n+4}\right), \quad \dots \quad (xi.)$$

$$\begin{aligned} \mu_4 = & \frac{3m(m-2)(p+1)^2(q+1)^2}{(n+2)^4} \left(1 + \frac{m-1}{n+3}\right) \left(1 + \frac{m-1}{n+4} \left(1 - \frac{8}{m-2} - \frac{9}{n+5}\right)\right) \\ & + \frac{m(p+1)(q+1)}{(n+2)^2} \left(1 + \frac{m-1}{n+3}\right) \left(1 + \frac{6(m-1)}{n+4} \left(1 + \frac{m-2}{n+5}\right)\right). \quad \dots \quad (xii.) \end{aligned}$$

In any actual case there is no difficulty in calculating μ_2 , μ_3 and μ_4 ; then, if $\beta_1 = \mu_3^2 / \mu_2^3$ and $\beta_2 = \mu_4 / \mu_2^2$ we can find β_1 and β_2 , and these ought to be sensibly zero and 3 respectively, if the system is to have the symmetry and meso-kurtosis, which characterize the Gaussian curve.

The values of β_1 and β_2 can, however, be found directly from the following formulæ :

$$\beta_1 = \frac{(q-p)^2}{m(p+1)(q+1)} \frac{\left(1 + 2\frac{m-1}{n+4}\right)^2}{1 + \frac{m-1}{n+3}}, \quad \dots \quad (xiii.)$$

$$\begin{aligned} \beta_2 = & 3 \left(1 - \frac{2}{m}\right) \frac{1 + \frac{m-1}{n+4} \left(1 - \frac{8}{m-2} - \frac{9}{n+5}\right)}{1 + \frac{m-1}{n+3}} \\ & + \frac{(n+2)^2}{m(p+1)(q+1)} \frac{1 + 6\frac{m-1}{n+4} \left(1 + \frac{m-2}{n+5}\right)}{1 + \frac{m-1}{n+3}}. \quad \dots \quad (xiv.) \end{aligned}$$

Or, if we write $\epsilon = (\bar{p} - \bar{q})/(n+2)$, we can throw β_1 and β_2 into a form which shows their approximation to the values 0 and 3 thus :

$$\beta_1 = \frac{(\bar{q} - \bar{p})^2}{m(\bar{p} + \epsilon)(\bar{q} - \epsilon)} \cdot \frac{\left(1 - \frac{2}{n+2}\right)^2 \left(1 + \frac{2(m-1)}{n+4}\right)^2}{1 + \frac{m-1}{n+3}}, \quad \text{(xiii.)}^{bis}$$

$$\begin{aligned} \beta_2 = 3 \left(1 - \frac{2}{m}\right) & \frac{1 + \frac{m-1}{n+4} \left(1 - \frac{8}{m-2} - \frac{9}{n+5}\right)}{1 + \frac{m-1}{n+3}} \\ & + \frac{1}{m(\bar{p} + \epsilon)(\bar{q} - \epsilon)} \frac{1 + \frac{6(m-1)}{n+4} \left(1 + \frac{m-2}{n+5}\right)}{1 + \frac{m-1}{n+3}}. \quad \text{(xiv.)}^{bis} \end{aligned}$$

Hence if both m and n are large numbers :

$$\left. \begin{aligned} \beta_1 &= \frac{(\bar{q} - \bar{p})^2}{m(\bar{p} + \epsilon)(\bar{q} - \epsilon)} \frac{\left(1 + 2\frac{m}{n}\right)^2}{1 + \frac{m}{n}} \\ \beta_2 &= 3 + \frac{1}{m(\bar{p} + \epsilon)(\bar{q} - \epsilon)} \frac{1 + 6\frac{m}{n} \left(1 + \frac{m}{n}\right)}{1 + \frac{m}{n}} \end{aligned} \right\} \dots \text{(xv.)}$$

If m be small relative to n , they become

$$\beta_1 = \frac{(\bar{q} - \bar{p})^2}{m(\bar{p} + \epsilon)(\bar{q} - \epsilon)}, \text{ and } \beta_2 = 3 + \frac{1}{m(\bar{p} + \epsilon)(\bar{q} - \epsilon)}. \quad \text{(xvi.)}$$

While if n be small relative to m , we have

$$\beta_1 = \frac{4(\bar{q} - \bar{p})^2}{n(\bar{p} + \epsilon)(\bar{q} - \epsilon)}, \text{ and } \beta_2 = 3 + \frac{6}{n(\bar{p} + \epsilon)(\bar{q} - \epsilon)}. \quad \text{(xvii.)}$$

These again reduce to still simpler forms if $\epsilon = (\bar{q} - \bar{p})/(n+2)$ be not comparable with either $\bar{p}$ or $\bar{q}$, *i.e.* if neither $\bar{p}$ or $\bar{q}$ be very small. We then have for :

$$\text{(xvi.), } \beta_1 = (\bar{q} - \bar{p})^2 / (m\bar{p}\bar{q}), \quad \beta_2 = 3 + 1/(m\bar{p}\bar{q}) ;$$

$$\text{(xvii.), } \beta_1 = 4(\bar{q} - \bar{p})^2 / (n\bar{p}\bar{q}), \quad \beta_2 = 3 + 6/(n\bar{p}\bar{q}).$$

Both forms result,—for n or m large and the product of either

with p and q not small,—in $\beta_1=0$ or $\beta_2=3$, *i. e.* in the symmetry and mesokurtosis, which are for practical purposes closely enough represented by the Gaussian curve. But if m and n be commensurable, and either $\bar{p}$ or $\bar{q}$ moderately small, this result by no means follows. Thus suppose we take:—

Illustration I. A sample of 100 of a population shows 10 p. c. affected with a certain disease. What percentage may be reasonably expected in a second sample of 100?

In this case $m=n=100$, $p=10$, $q=90$, and we easily find:

Mode=10. Mean=10.78. $\sigma=4.344$.

Probable error of mean=2.930.

Further: $\beta_1=.2749$, $\beta_2=3.3255$.

There is thus sensible skewness and sensible leptokurtosis.

If we suppose the probable error to still give a sufficiently definite meaning, we should put the answer in the form 10.784 ± 2.930 , or assert that the percentage of the character in the second sample was as likely to fall outside as inside the limits 7.85 to 13.71. If we used the value $m\bar{p} \pm .67449 \sqrt{m\bar{p}\bar{q}}$, often adopted, we find the limits 7.98 to 12.02. The first result gives a very much more considerable chance of percentages arising in the second sample considerably beyond that shown by the first sample. It seems worth while investigating for this case, the actual errors in expectancy introduced by using the normal curve for our skew platykurtic frequency distribution. I have

accordingly calculated out series (iv.). I find $C = \frac{1}{2581}$

nearly, and obtain the distribution of frequency given in Table I. In this table the second column gives in a 1000 second samples of 100 each, the numbers which would show the percentage in the first column; the third column gives the numbers, which would show this percentage or any less percentage. This table proves at once the marked skewness of the distribution which with a modal value at 10 stretches from 0 on the defect to upwards of 30 on the excess side. If we interpolate to find the quartiles we have lower quartile at 7.65 and the upper quartile at 13.48. The median is at 10.40. Thus the lower quartile has 2.75 and the upper 3.08 for range; values in excess of the percentage observed are thus to be expected. Since $\frac{1}{2}(2.75 + 3.08) = 2.92$, we see that the "probable error" thus found is practically identical with the value found from the standard deviation. The range, however, as obtained from placing this probable error either

TABLE I.
Frequency of Percentages in a Second Sample
for Special Case.

(1) Percentage.	(2) Frequency.	(3) Sum of Frequency.	(1) Percentage.	(2) Frequency.	(3) Sum of Frequency.
0	4	4	17	29.5	927.7
1	2.2	2.6	18	22.0	949.7
2	7.1	9.7	19	13.0	965.7
3	15.9	25.6	20	11.4	977.1
4	28.9	54.5	21	7.9	985.0
5	44.8	99.3	22	5.4	990.4
6	61.3	160.6	23	3.6	994.0
7	76.1	236.7	24	2.3	996.3
8	87.0	323.7	25	1.5	997.8
9	92.8	416.5	26	.9	998.8
10	93.3	509.8	27	.6	999.3
11	89.1	598.9	28	.4	999.6
12	81.2	680.1	29	.2	999.8
13	71.0	751.1	30	.1	999.9
14	59.9	811.0	31	.0	1000
15	48.7	859.7			
16	34.5	898.2	Total ...	1000	1000

side of the mean (*i. e.* 7.85 to 13.1) is clearly too high. If we calculate the median by the rule of the third * and place the probable error either side of this theoretical median, we shall approximate closely to the actual range. The distance from mode to mean = $10.78 - 10 = .78$, and one-third of this is .26. Hence $10.78 - .26 = 10.52$ is the theoretical median. This gives 10.52 ± 2.93 , or a range of 7.59 to 13.45 as compared with the actual 7.65 to 13.48 within which half the number of second samples will fall. This is a far better result than the 7.98 to 12.02 of the usual method. The one amounts to saying that half † the second samples will fall between 7.5 and 13.5, which is sensibly in accord with actuality, while the other crowds them into the range between 8 and 12, which result is far in excess, only about 36 p. c. of the samples falling within this range.

This first illustration is, I think, sufficient to demonstrate that it is not possible in judging expectancy from past experience (i.) to neglect the relative sizes of the first and second samples, or (ii.) to neglect, even in characteristics which appear in 10 p. c. of the sample, the sensible deviation from the Gaussian distribution.

The above method of dealing with the problem is merely intended as a ready approximate method. It involves only

* Phil. Trans. vol. clxxxvi. A. p. 375.

† More accurately 51.5 per cent.

the calculation of the mode, the mean and the S.D. from formulæ (vi.), (vii.), and (viii.). For more accurate work we must calculate the frequencies as I have done in Table I., a laborious process, or we can note that the values of β_1 and β_2 correspond to a skew curve of Type I.* We have in fact $\kappa_1 = 6 + 3\beta_1 - 2\beta_2 = 1.737 > 0$. This leads to $r = 70.8325$, $\epsilon = 552.8582$, $m_1 = 7.93$, $m_2 = 60.90$,

and the equation to the curve is

$$y = 93.8 \left(1 + \frac{x}{12.78}\right)^{7.93} \left(1 - \frac{x}{98.13}\right)^{60.90}.$$

The origin is at 9.57 per cent. and the range from -3.2 to 107.7 per cent., extending somewhat beyond the possible limits, with, however, no sensible frequency. Considering that the raw moments have been taken to get the constants, the curve must be considered as giving very good results, and exhibiting in a manner amply sufficient for practical purposes the deviations to be expected in second samples from the Gaussian symmetry and mesokurtosis. The curves are shown in Diagram I.

Illustration II.—I proceed now to a more extreme case. Let us suppose that examination of 1000 members of a population has shown 20 affected with a certain characteristic. What is the expectancy if a second sample of 100 be examined? In this case $p = 20$, $q = 980$, $n = 1000$, and $m = 100$. We find :

$$\text{Mean} = 2.096, \quad \text{Mode} = 2, \quad \sigma = 1.5015,$$

and thus the probable error = 1.013.

There is not much difference between the mean and the mode, and the usual theory would give $\sigma = 1.40$, and state the answer to be expected from the second sample as $2 \pm .94$. The second sample is so much smaller than the first, that the standard deviation of second samples is fairly close to the true value when calculated by the erroneous formula. But the assumption of a Gaussian distribution falls wide of the mark. In Table II. are the calculated frequencies given by the series in (iv.) for a thousand samples †, the data being

* Phil. Trans. vol. clxxxvi. A. p. 369. This is an interesting illustration of the futility of Dr. K. Ranke's criticism of the worth of these curves on the ground that they may give negative and unintelligible values for the constants. In this particular case the n of Phil. Trans. vol. clxxxvi. A. p. 361, has become the *minus* ($n+2$) of this paper, and the pn and qn of that paper are here both negative. Yet we see that the n , p , and q of our present notation are perfectly intelligible, and that the skew curve follows at once from the series in (iv.).

† C_0 of (iv.) = .1328 nearly.

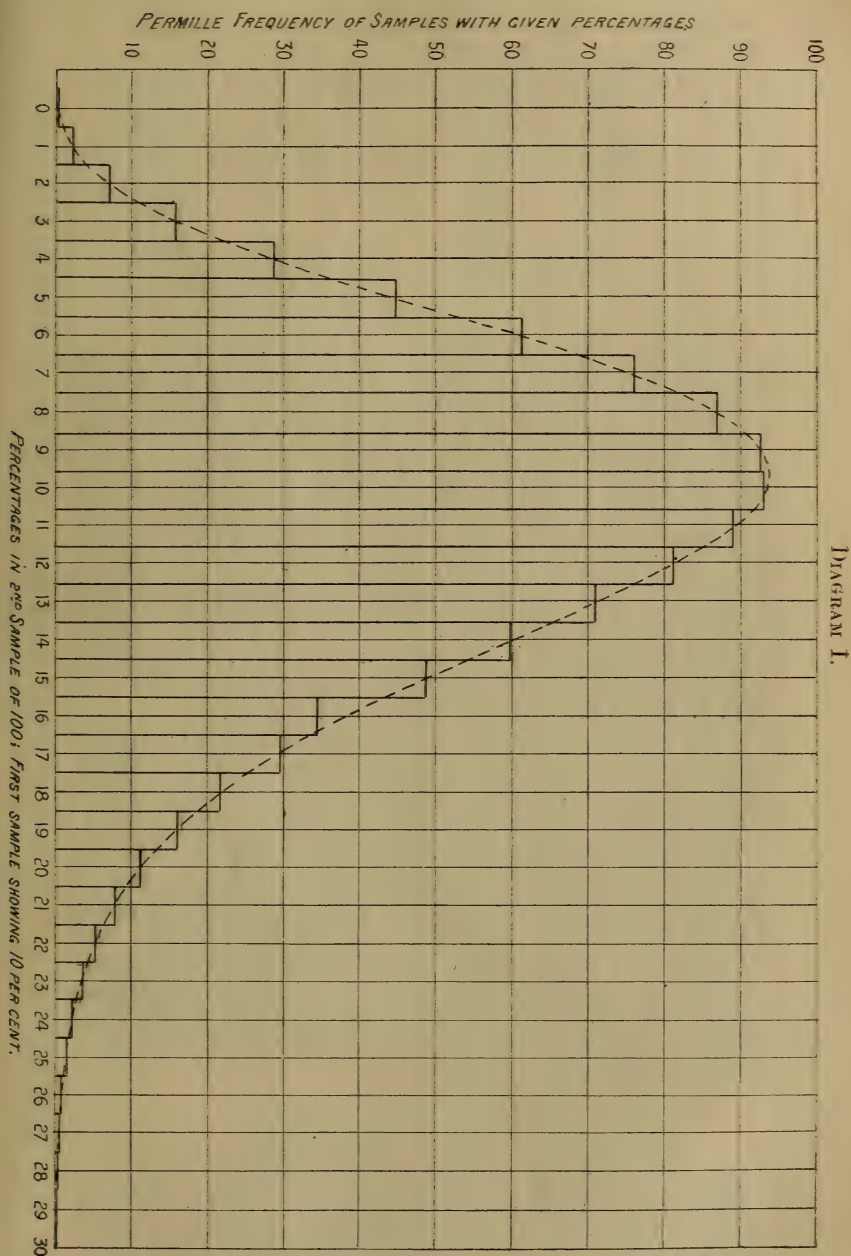


Diagram I.

TABLE II.

Percentage.	Frequency.	Sum of Frequency.	Percentage.	Frequency.	Sum of Frequency.
0	132.8	132.8	6	16.8	992.0
1	258.2	391.0	7	5.7	997.7
2	260.6	651.6	8	1.7	999.4
3	181.6	833.2	9	0.5	999.9
4	98.2	931.4	10	0.1	1000.0
5	43.8	975.2	Total ...	1000.0	—

arranged as in Table I. We see at once that there is no approach to a Gaussian curve. If we proceed by simple interpolation we find the median at 1.919, the first quartile at .954, and the second at 3.042; these give a probable error in defect of .965 and in excess of 1.123, or a mean probable error of 1.044, somewhat greater than the value deduced from the standard deviation or 1.013, and considerably greater than that given by the usual theory. Here, as in the last illustration, we see a sensibly larger chance of the second sample exceeding than of its falling short of the percentage given by the long first sample. We might perhaps express this best by saying that the expectancy is 50 p. c. of samples with a percentage of the character between 0 and 1.92 and 50 p. c. with a percentage between 1.92 and 10.

To show the sort of erroneous inference that may be drawn, I suppose a percentage of 6 to have been observed in a second sample. Table II. shows that a percentage as large as this actually occurs with a frequency of 24.8 in the thousand, or its improbability is only measured by 39 to 1. On the other hand, if we adopt a Gaussian distribution we should only expect a deviation as large as 4 from the mean to occur in excess 3.5 times in the 1000 trials; or the odds are 285 to 1 against its occurrence*. Such long odds would reasonably lead us to suppose some modification in the population, since the first sample was drawn. The error of such a supposition really flows, however, from the assumption of a normal frequency distribution.

Looked at analytically we find:

$$\beta_1 = .5836, \quad \beta_2 = 3.6468;$$

or the curve is very sensibly skew and leptokurtic.

We find $\kappa_1 = .4572$, > 0 , or the curve is of Type I. We get $r = 27.0761$, $\epsilon = 87.3453$, and $m_1 = 2.7435$, $m_2 = 22.3326$;

* 475 to 1 against so large a deviation, if we adopt the old theory, for which such a deviation is 2.86 and not 2.70 times the standard deviation.

whence we deduce as equation to the curve :

$$y = 279.92 \left(1 + \frac{x}{2.5217} \right)^{2.7435} \left(1 - \frac{x}{20.5272} \right)^{22.3326}$$

The origin is at 1.4308, and the range from -1.09 to +22.05. Considering that the moments have not been corrected, the fit is quite reasonable.

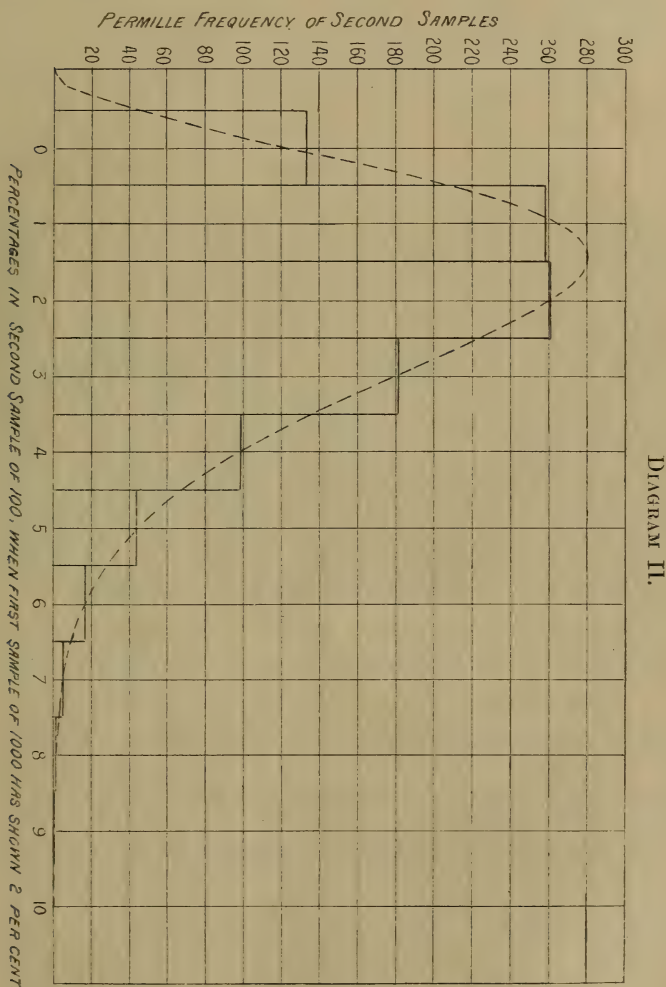


Diagram II. gives the curve set against the histogram of the hypergeometrical series, and the following table gives the

whole number frequencies, those for the curve being found by mechanical integration.

Percentages.	Frequencies.		Percentages.	Frequencies.	
	Hypergeometrical.	Curve.		Hypergeometrical.	Curve.
Less than .5	133	134	6	17	18
1	258	259	7	6	7
2	261	257	8	2	3
3	181	179	9	0	0
4	98	98	10	0	0
5	44	45			
			Sum.....	1000	1000

Conclusions.—The method often adopted for measuring the expectancy of a future sample from past experience is in certain respects defective, when (i.) the size of the future sample is at all comparable with the past experience, (ii.) the characteristic expected occurred only in a small percentage of cases in the first experience. In such cases the mean and probable error of the expectancy can be easily found, and depends upon the relative sizes of the past experience and of the future sample. The frequency of future samples is given by a certain hypergeometrical series, which is not at all closely represented by the Gaussian curve except when the past experience is very large as compared with the proposed sample, and further the characteristic expected does not occur in either a very large or very small percentage of the population.

XXVIII. *On the Radioactive Matter in the Earth and the Atmosphere.* By G. A. BLANC, Ph.D. (Rome) *.

IN a paper published in the Philosophical Magazine (September, 1906), A. S. Eve has given the very interesting results of the investigations made by him on the radioactive matter contained in the soil and atmosphere, concluding that the presence of radium in the earth and radium emanation (with its successive transformation products) in the atmosphere are the principal cause of the ionization which is observed in the open air and inside buildings.

Eve comes to the conclusion that a proportion of 1.8×10^{-11} gr. of radium bromide per c.cm. uniformly distributed in the

* Communicated by Lord Kelvin.

earth's crust is necessary to account for the penetrating radiation observed in the atmosphere, and calls attention to the fact that such a quantity is about four times greater than the average amount which Strutt has found to be contained in rocks.

Now I wish to call attention to a fact which was pointed out by me at the meeting of the "Società Italiana di Fisica" that took place in Rome during the month of April 1906. I then communicated the results of an investigation made by me on the excited activity obtained by exposing a negatively charged wire to the open air, showing that a great part of the active matter contained in the atmosphere (at least in Rome and its surroundings) consists of transformation products of radiothorium.

A simple calculation shows that, owing to the respective values of the constants of decay of radium B and thorium A, while an exposure lasting for 3.1 hours is sufficient to obtain 99 per cent. of the maximum activity in the case of radium, an exposure lasting for 73.5 hours is necessary in order to obtain the corresponding result in the case of radiothorium.

It is interesting to note that already three years ago Sella * had seen that the activity excited on a wire by exposure to electric effluvium in the open air could be considerably increased by prolonging the exposure for many hours; a fact which was not consistent with the hypothesis of radium emanation being the only cause of the observed effects.

Bumstead † later, found that by prolonged exposure of a wire to the open air, in Newhaven, the excited activity of rapid decay was followed by a residual activity which was similar to that obtained by exposure to thorium emanation; and Dadourian ‡ also noticed, in the same locality, a residual activity of the thorium type on a wire exposed to air extracted from the soil.

Burbank § finally, has observed a similar effect for wires made active in the open air, first in Göttingen and afterwards in Washington.

My object was therefore to establish as exactly as possible the relative amount of radium and radiothorium excited activity to be obtained from atmospheric air in Rome and its surroundings.

An insulated brass wire, about twelve metres long, kept

* Sella, *Rendic. Acc. Lincei*, Sem. 1, 1902.

† Bumstead, *Amer. Journ. Science*, xviii. July, 1904.

‡ Dadourian, *Amer. Journ. Science*, xix. January, 1905.

§ Burbank, *Phys. Zeit.* vi. p. 436, 1905; 'Atmosph. Electr. and Terr. Magn.' 1906.

charged at constant potential of about -500 v. by means of a battery of small storage-cells, was repeatedly exposed to the open air for periods lasting three days, after which the activity of the wire, previously wound on a frame, was tested in the usual way, by means of a gold-leaf electroscope.

The results were always similar. After an initial rapid decay, the activity went decreasing according to an exponential law, the constant of decay being, within the limits of experimental error, identical with that of thorium A.

It was easy, by extrapolation, to calculate the effect due to the excited activity of the radiothorium type at the moment in which the activation process had come to an end. Now the important fact is that *the excited activity of the radiothorium type represents generally a large part of the total amount exhibited by the wire*, and precisely from 50 to 70 per cent.

Measurements were made on the terrace of the Physical Institute of Rome, and, to make sure that the effect was not due to the presence of thorium salts in the laboratory, also in a garden at a certain distance from the town. In both cases, results were similar.

A series of experiments was also carried out by activating a wire inside the Catacombs of Sant' Agnese, near Rome, which are dug in the volcanic formation called *pozzolana*. Here also a very strong proportion of radiothorium excited activity was observed, for exposures of the wire lasting three days.

For short exposures of the wire, not exceeding three or four hours, the law of decay was always very similar to that observed by Rutherford and Allan in the case of a wire activated in Montreal; the characteristic time of reduction of the activity to half value varying between 50 and 60 minutes.

It is evident that the large amount of radiothorium excited activity obtained on a wire exposed to the open air at a certain distance from the soil, cannot possibly be due to direct contact with the emanation. The disintegration products thorium A and thorium B, probably fixed on atmospheric dust particles, are evidently carried by the air, and, owing to their slow rate of decay, are able to cause the observed effects. It is probable, on the other hand, that no products such as thorium A or thorium B could, in any appreciable quantity, escape through the capillary conduits of the soil into the atmosphere, owing to the property these products have of remaining fixed on solids with which they come into contact.

Experiments are actually being made with the object of determining as exactly as possible the amount of radiothorium emanation which, by diffusion, escapes from a given area of

soil ; and the results so far obtained confirm my former conclusions as to radiothorium transformation products being, at least in these regions, a most important agent of atmospheric radioactivity.

As I have already mentioned, Eve's estimate of the quantity of radium contained in the earth's crust, based on the observation of the penetrating radiation present in the atmosphere, is four times greater than the average amount found by Strutt in rocks. Now it is clear that the experimental method used by Strutt could only reveal the presence of radium in the materials tested by him. It is therefore not impossible that the reason of such discrepancy should be ascribed to the fact that part of the effects observed by Eve are due to the presence of the product thorium B in the soil and atmosphere.

As I have already said, I shall soon be able to give a detailed account of my researches, which will, I hope, enable me to make an estimate of the amount of radiothorium present in the earth's crust, at least as far as concerns the Roman region.

Rome, Istituto Fisico della R. Università.

XXIX. *Pleochroic Halos*. By J. JOLY, F.R.S.*

[Plate X. †]

CRYSTALS of zircon occurring in certain anisotropic coloured minerals, more especially in biotite, are invariably seen to be surrounded by an area showing strong pleochroism. Thus in biotite, when the section under observation is transverse to the principal cleavage, the darkening of the halo occurs when the vibrations are in the direction of cleavage. The absorption is then often so complete as to produce opacity. When the vibrations are at right angles to this direction the halo almost disappears. The zircon then appears simply as a minute highly refracting speck marking the centre of the halo. In short, the area referred to appears as an intensification of the general pleochroism of the mica. As the term halo is misleading to those unacquainted with the phenomenon referred to, it is well to mention that the darkening takes place uniformly over the circular area around the zircon.

The cause of this curious appearance has long been matter of speculation. It has been assigned to organic (?) pigments, mainly on the strength of the fact that a very high temperature will cause its dissipation. The following considerations

* Communicated by the Author.

† [The plate illustrating this paper will appear in the April number.]

appear to show that it is most probably referable to those radioactive properties of zircon so clearly demonstrated in the Hon. R. J. Strutt's experiments.

The well-developed halo appears never to exceed a certain maximum dimension. The radius may attain 0.05 mm. or even a little more. The average of measurements made on good halos in mica was 0.04 mm. Smaller halos are frequent. In the smaller halos the originating enclosure is often too minute to be discoverable. The halo is the section of a sphere of very considerable truth, judging from its accurate circularity.

Professor Rutherford has shown that 13 layers of aluminium each 0.00031 cm. thick, reduce the velocity of the most energetic α rays from radium to a degree rendering them incapable of effecting photographic action or of exciting phosphorescence. When this result is translated into the equivalent thickness of biotite, the active range is just 0.04 mm. Hence the halo attains just such dimensions as would be expected if due to alterations affected by the α radiation. Halos of smaller radius are probably due to a source of radium so minute in amount as to limit the effect to the inner region which is traversed by rays of every velocity; or the smaller radial dimension may only be apparent, the section being taken in such a plane as to exclude the centre.

Some writers refer these halos to the presence of other minerals. I find sphene, apatite, and allanite mentioned as being occasionally surrounded by halos. As regards the first I believe I have seen associated halos. However this may be, Mr. Strutt has shown that both sphene and apatite are highly radioactive. Not being able to find any determination of radioactivity in the case of allanite, I examined a specimen in the Museum here for radium, using the methods developed by Mr. Strutt; it afforded 270×10^{-12} grams radium per gram. This is comparable with the activity of zircon.

One conspicuous property of these halos might long ago have put any aggregation-theory out of court. The fact that their sphericity is independent of the cleavage or physical structure of the medium in which they are formed. Aggregation or spreading by diffusion would most certainly not take place with equal ease across and along the cleavage; as is shown in all interpositions and alterations in mica. I have, indeed, frequently observed halos extending with perfect circularity across the cleavage of two or more crystals of biotite in mutually inclined positions (Plate X.*). This appears to me to be strong evidence for their radioactive origin.

* [This will be published in the April number.]

Shortly after coming to these views I noticed in Zirkel's fine monograph 'Über Urausscheidungen in Rheinischen Basalten' (p. 52) the remark that the large zircons in these basalts may, by aid of a lens, be seen to be bordered by a thin dark grey margin. He remarks that this border is about 0.05 mm. wide. There can be little doubt that this is of similar origin to the halo in biotite.

The question remains, what is the precise nature of these halos? It is certainly not a colouration similar to the colouration produced in glass by radium. The latter takes place through very considerable thicknesses and is, therefore, ascribed to the β radiation. While it is worth consideration if the deep brown of certain biotites, much spotted by halos, is not a potassium colouration referable to the ionizing influence of β rays (see the interesting paper of Mr. J. C. Maxwell Garnett, Phil. Trans. Roy. Soc. vol. 205), the limited radius of the halo and its particular limiting value point to the α radiation as being responsible. Helium is probably stored in the halo but in very minute quantity. Thus it is easily calculated that if an average halo-sphere is derived from a zircon having a diameter one-tenth of the sphere, a period of 10^9 years must elapse to generate a quantity of helium sufficient to fill one hundred thousandth part of the volume of the sphere.

Halos occur in cordierite which appear to differ somewhat in properties from those in mica. I have no experience of them: the cordierite in my possession showing many inclusions but no halos. The inclusions appear to be non-radioactive substances. On the other hand, Dana makes the remark that halos in cordierite are formed "particularly by zircon." MM. Fouqué and Lévy speak of mica and pleonaste being associated with such halos. The first might well be radioactive. The last mineral, which is very variable in composition, I have not yet investigated.

Geological Laboratory,
Trinity College, Dublin.

XXX. Notices respecting New Books.

La Mécanique des Phénomènes Fondée sur les Analogies. Par M. M. PETROVITCH, Professeur à l'Université de Belgrade. Paris: Gauthier-Villars. 1906. Pp. 96.

THE unity introduced into the investigation of physical phenomena by the use of analytical methods, which enable whole series of physical problems to be thrown into identically the same mathematical form, has long been known as an illustration of the

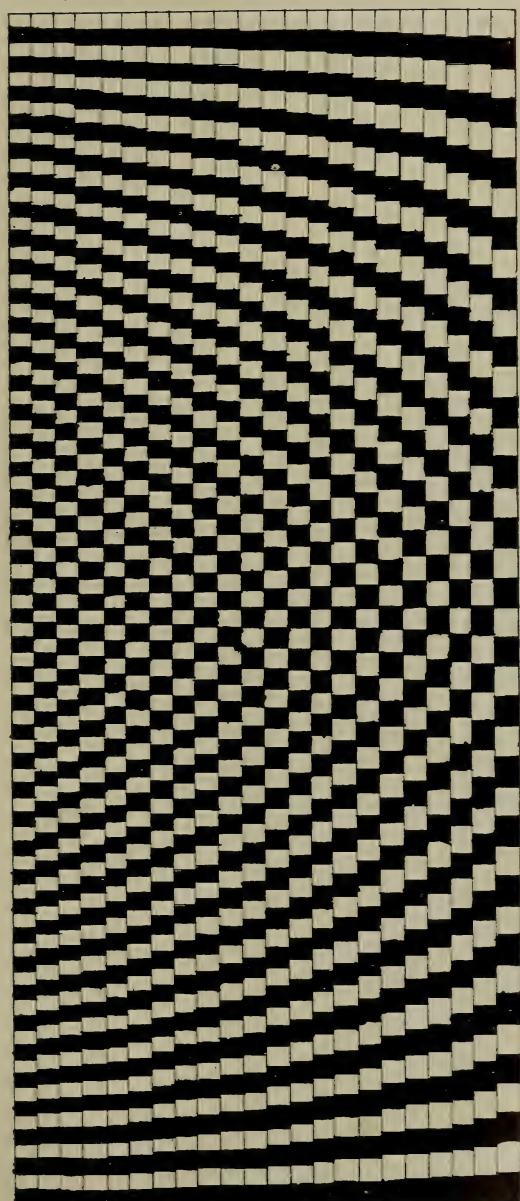
marvellous power and extreme generality of analytical methods. The particular nature of the physical entities dealt with vanishes into thin air by the time we have established our differential equations, and we are then left simply to deal with a number of variables connected by certain relations. A system of "general dynamics" founded on the consideration of the nature of the relationships existing among the variables is the subject-matter of the little volume under review, which forms No. 27 of the "Scientia" series. After a brief introduction and some preliminary considerations regarding the use of analogies, the author outlines a system of general dynamics, and then studies various important groups of problems. In the concluding chapter the possible applications of the system are briefly discussed. The book is a suggestive one, and should prove interesting reading to the mathematical physicist.

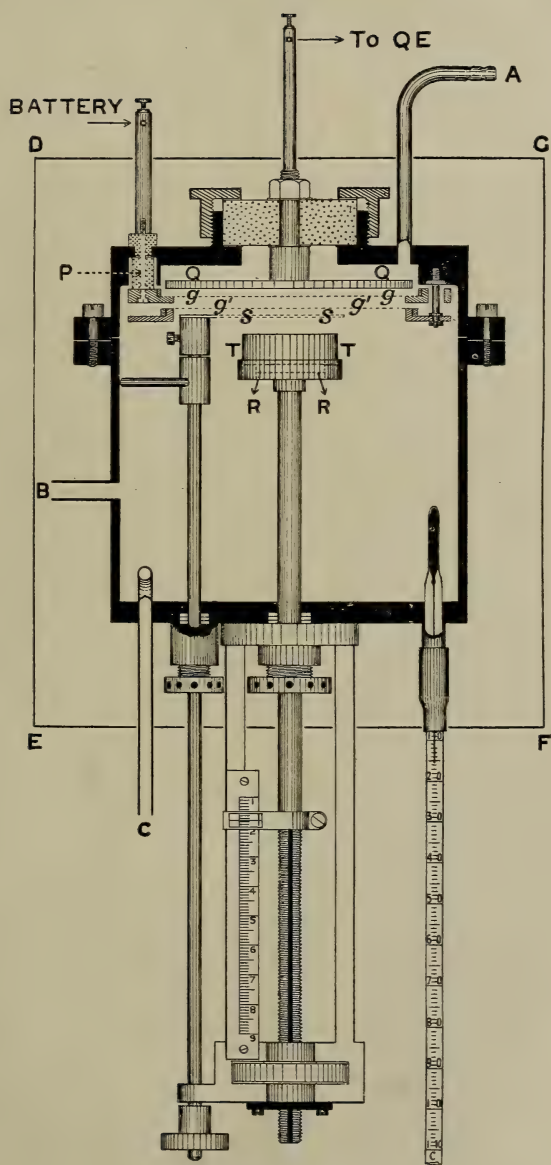
Quadratic Forms and their Classification by means of Invariant Factors. By T. J. I'A. BROMWICH. Cambridge University Press. 1906.

THIS is the third of the Cambridge Tracts in Mathematics and Physics now being published under the editorship of J. G. Leatham and E. T. Whittaker; and both in purpose and method it carries out in an interesting way the object aimed at in these tracts. Rapidly and yet step by step the student is led from simpler to more complex cases. The treatment is arranged in five chapters. In Chapter III., "A Family of Quadratic Forms," the reduction is completely worked out both for three and for four variables, and is used to classify the contacts of conics and quadric surfaces respectively. The last chapter gives further geometrical illustrations, and also the important dynamical case of reduction to normal coordinates. Full references are given to original memoirs on the subject from the early and pioneer work of Sylvester in the middle of last century down to recent times. The reader is also referred to papers dealing with developments which could not be discussed in the limited space of a tract of 100 pages. It is indeed surprising how much Professor Bromwich has managed to pack into the space at his disposal, and how in spite of the conciseness demanded he has presented a clear and logical statement complete so far as it goes. The real student of algebraic form will find the pamphlet of the greatest service to him in many ways.

Annuaire pour l'An 1907. Publié par le Bureau des Longitudes. Avec des Notices scientifiques. Paris: Gauthier-Villars. 1907. Pp. vi+682+8+20+146+45.

THIS well-known annual contains, besides the usual tables, the following special articles:—"Diamètre de Vénus," by Bouquet de la Grye; and "Histoire des idées et des recherches sur le Soleil. Révélation récente de l'atmosphère entière de l'astre," by H. Deslandres. It is published at the moderate price of 1 fr. 50 c.





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XXXI. *The Attraction of Ellipsoidal Shells and of Solid Ellipsoids at External and Internal Points, with some Historical Notes.* By ANDREW GRAY, LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow*.

1. I WROTE out the greater part of the following paper while on a holiday this summer, and was not aware until my return home that the method given below (§§ 24, 28 to 30) for the determination of the potential of a homothetic ellipsoidal shell had been anticipated by a passage in Thomson and Tait's 'Natural Philosophy' (Part ii., § 525). That also, I have since found, had been anticipated in § 24 of the beautiful memoir by Chasles, "Sur l'attraction d'une couche ellipsoïdale," in the *Journal de l'Ecole Polytechnique* (t. xv., 1835). The investigation contained in §§ 14 to 18 below is, however, quite distinct, and I think new. It gives a complete solution of the problem of the shell and the ellipsoid, and leads naturally to the other discussion already referred to, which differs in some respects from the solution of Chasles. I have, moreover, worked out in detail some particular problems. With all due acknowledgment, therefore, I venture to allow my notes to stand. I am aware of course of the recent discussions of the problem, for various laws of density, by Ferrers, Dyson

* From the 'Proceedings of the Royal Philosophical Society of Glasgow,' 1906. Communicated by the Author.

and others; but perhaps this solution of the older and simpler problem may not be without interest.

2. No problem in the Theory of Attractions has received more attention than that of the attraction of a solid ellipsoid on an external particle. The subject engaged the attention of Newton and Maclaurin*, who dealt with it by a combination of the geometrical methods of which these mathematicians had such a perfect mastery, and the results of Newton's own calculus of fluxions and fluents (differential and integral calculus); and all the celebrated analysts of the end of the eighteenth century and the beginning of the nineteenth, Laplace, Lagrange, Ivory, Gauss, Poisson, Chasles, wrote memoirs on the subject which have become classical. To find the attraction of the ellipsoid at an internal point had been soon found to be a comparatively easy matter. The process adopted was to take the attracted point as origin of polar coordinates by which the positions of the particles of the ellipsoid (supposed of uniform density) were specified, to express the volume of an element of the attracting body by these coordinates, and then the components of attraction on the particle as triple integrals with respect to the radius-vector and the two angular coordinates. As Poisson puts it in the introduction to the very remarkable memoir† read to the Académie des Sciences on October 3, 1833:—"The integration with reference to the radius-vector can be carried out without any difficulty, and in the case of an internal point a second integration is easily effected, so that the three components of attraction are expressed by single integrals, reducible to elliptic integrals of the first and second kinds. These integrals are obtainable in a finite form when the ellipsoid is one of revolution. When, however, the attracted particle lies outside the ellipsoid, the double integrals contain a radical, and have limits which render them much more complicated, so that, instead of carrying out the second integration directly, we have to turn the difficulty by reducing the problem for the external point to that for an internal point, which leaves the problem of the direct integration unsolved. For this reduction the theorem of Ivory leaves nothing to be desired."

3. Ivory's theorem depends on his notion of corresponding points on two ellipsoidal surfaces, the axes of which are coincident. Let a, b, c be the lengths of the semi-axes of one, say the smaller, ellipsoid, x, y, z the coordinates of a

* See Newton's *Principia*, Lib. I., ss. xii. and xiii., and Maclaurin's 'Treatise on Fluxions,' vol. ii.

† *Mémoires de l'Académie Royale des Sciences de l'Institut*, t. xiii., 1835.

point P on the surface, and similarly a', b', c', x', y', z' the lengths of the semi-axes of the larger ellipsoid, and the co-ordinates of a point Q upon it. If the coordinates fulfil the relations $x/a = x'/a'$, $y/b = y'/b'$, $z/c = z'/c'$, the points P, Q are corresponding points. It is easy to show that if P, P' be two points on the first ellipsoid, Q, Q' the corresponding points on the other, and the ellipsoids be confocal, the distances PQ', P'Q are equal.

Now considering attractions in the direction of the principal axes, and taking first the axes a, a' , it can be proved that the attraction X of the first ellipsoid, A say, on a particle of unit mass at Q on the surface of the other ellipsoid, B, is to the attraction X' of the ellipsoid B on a unit particle at the corresponding point P on the surface of A, in the ratio $bc/b'c'$. Similarly $Y/Y' = ca/c'a'$, $Z/Z' = ab/a'b'$. The ellipsoids are here supposed to be solid and of uniform density, ρ say, and confocal. If we call the masses of the ellipsoids M, M', then since $M/M' = abc/a'b'c'$, we may express the theorem in the form (not given by Ivory)

$$\frac{X}{X'} = \frac{a' M}{a M'}, \quad \frac{Y}{Y'} = \frac{b' M}{b M'}, \quad \frac{Z}{Z'} = \frac{c' M}{c M'} \quad \dots \quad (1)$$

When expressed in this form the theorem is evidently true whether the two ellipsoids have the same density or not, provided that each is homogeneous. For the components of force on a unit particle evidently vary with the masses of the ellipsoids, when their dimensions remain unchanged, and therefore a change in the ratio M/M' , caused by varying the density of either ellipsoid, is represented by a corresponding change in each of the ratios X/X' , Y/Y' , Z/Z' .

In the particular case of $M = M'$, the theorem takes the form

$$\frac{X}{X'} = \frac{a'}{a}, \quad \frac{Y}{Y'} = \frac{b'}{b}, \quad \frac{Z}{Z'} = \frac{c'}{c} \quad \dots \quad (2)$$

It was first pointed out by Poisson that Ivory's theorem is true for every law of attraction, provided the law is a function of the distance only.

4. Let us now suppose that the problem of finding the attraction of a homogeneous ellipsoid at an internal point has been solved, and that it is required to find the attraction at an external point, Q say. It is only necessary to find for the corresponding point P on the surface of the given ellipsoid the attraction exerted on a unit particle by the confocal ellipsoid, the surface of which contains the point Q. The

components are given by (1), and thus the so-called external problem is reduced to the internal problem of which the solution is known.

5. The external problem was, however, solved directly by Poisson in the memoir above referred to, by the device, which he appears to have been the first to adopt, of imagining the ellipsoid divided into infinitely thin similar and similarly situated ellipsoidal shells, or "elliptic homœoids" as they have been called by Thomson and Tait*, then finding by direct integration the attraction exerted by such a shell on a unit particle at the given external point, and finally passing, by another integration from shell to shell, to the attraction exerted by the solid ellipsoid.

[The term homothetic ellipsoidal shell is used by many writers for the "couche elliptique" on which Poisson based his solution, and is perhaps less open than the term elliptic homœoid to objection on the ground of derivation; but we shall adopt the name elliptic homœoid, or simply homœoid, where there is no risk of ambiguity. Thomson and Tait also give the name "focaloid" to a shell bounded by two confocal ellipsoidal surfaces.]

6. On Ivory's notion of corresponding points Chasles based a very important and elegant theorem of the attraction of confocal homœoids. Imagine two elliptic homœoids, A, B, of infinitesimal thickness, and each of uniform (not necessarily the same) density, the two outer surfaces and the two inner surfaces of which are confocal. Let a, b, c be the lengths of the semi-axes of A, and a', b', c' the same quantities for B. Also let P and Q be corresponding points on the two shells. The theorem of Chasles affirms that the potential at the point Q due to the homœoid A is to the potential at P due to the homœoid B, as the mass of A is to the mass of B.

For let ds be an element of A and ds' the corresponding element of B, and p, p' the length of the perpendiculars from the centre on the tangent plane at the elements. It is easy to prove that

$$\frac{p'ds'}{pds} = \frac{a'b'c'}{abc} \cdot \cdot \cdot \cdot \cdot \cdot \quad (3)$$

The masses of corresponding elements of the shell are βpds , $\beta' p'ds'$, where β, β' are constants depending on the density and scale of thickness in the two cases. The total masses are $4\pi\beta abc$, $4\pi\beta' a'b'c'$ respectively. Hence it follows that the masses of corresponding elements are in the ratio of the total masses in the two cases. [See § 13 below for the value of β, β' .]

* Nat. Phil. Part II. § 494 g.

Now, if r be the distance of Q from any element of the shell A , it is also the distance of P from the corresponding element of B . If β, β' be constant multipliers, as already explained, the potential at Q due to the matter at ds is $\beta p ds/r$, and the potential at P due to the matter at ds' is $\beta' p' ds'/r^*$. The former bears to the latter the ratio $\beta abc/\beta' a'b'c'$. Since this is true of every element the total potentials have the ratio just stated, that is the ratio of the masses of the shells.

It follows from this theorem that if the potential at an internal point is known for a thin elliptic homœoid, the potential at an external point can be found, and *vice versa*, by considering a confocal shell.

7. These results are true whatever the law of attraction may be, if only it is a function of the distance, as may be seen by substituting $f(r)$ for $1/r$ in the expression for the potential of an element. In the case of ordinary matter the law of attraction is that of the inverse square of the distance, and the potential determined is commonly called the Newtonian potential.

It is well known (and it will be referred to again presently) that the Newtonian potential of a homœoid is the same at every internal point, and therefore also at every point of the shell, since the potential is not discontinuous at points within attracting matter of finite volume density, or even at passage across a surface of finite density. Thus in order to find the potential at an external point P , due to a given elliptic homœoid, it is only necessary to imagine a confocal homœoid of equal mass constructed so as to have P on its surface, and find the uniform potential which it produces at every internal point. This is the potential required.

8. It follows from this result that the external confocal ellipsoidal surfaces are the equipotential surfaces of a uniform homœoid, and that such a shell is itself at uniform potential.

* We take here as the specification of the potential

$$V = \sum \frac{dm}{r},$$

where r is the distance of the point for which the potential is defined from an element dm of the attracting matter, and $\sum$ denotes summation for all elements. Here the unit of mass is that which concentrated at unit distance from an equal mass, also concentrated at a point, is attracted with unit force. When the ordinary unit of mass, the gramme, say, is used, the right-hand side of the equation for V must be multiplied by the value of the force of attraction which exists between two such units placed at unit distance, a centimetre, say, apart. This multiplier is called the "gravitation constant." It is generally omitted (that is taken as unity) in what follows: where it is inserted it is denoted by κ .

The resultant attraction due to such a shell exerted on a particle at an external point is along the normal through the point to the confocal surface on which the point lies.

It can be seen at once without analysis that an elliptic homœoid exerts no force at any point in the internal hollow, that is, that the potential has there a uniform value. For the shell may be imagined constructed by homogeneously straining a uniform thin spherical shell, within which, of course, the potential is uniform. Such strain is that which elongates all parallel dimensions of the shell in the same ratio, and therefore leaves it of uniform density, though of varying thickness, proportional to the length of the perpendicular from the centre on the tangent plane at each point. If then a cone of small solid angle be drawn with its vertex at any point in the hollow of the spherical shell, so as to intercept two elements of the surface, these two elements exert equal and opposite forces on a particle at the vertex. By the strain the masses of these two elements are not altered, nor the ratio of their distances from the vertex of the new cone into which the former one is changed. The elements, therefore, still exert equal and opposite forces on a particle at the vertex. Hence, as cones can be thus drawn so as to exhaust the shell by pairs of elements, the shell as a whole exerts no force at the common vertex.

9. The same idea of division of a solid ellipsoid into homœoids had, however, occurred to O. Rodrigues, and been used by him in a "*Mémoire sur l'attraction des Sphéroïdes*," published in the *Correspondance sur l'Ecole Polytechnique* (t. iii., 1816). The method adopted for the solution of the problem of the attraction of a solid ellipsoid seems to have been suggested by a previous paper by Gauss, and consists in finding the variation, δW , say, in the ratio of the potential of the ellipsoid at the given point, h, k, l , say, to the mass of the solid, when the squares of the semi-axes a^2, b^2, c^2 are altered by the same small amount, $\delta\phi$, say, that is by the passage from the given ellipsoid to an adjacent confocal ellipsoid. It is shown that for an external point $\delta W=0$, and for an internal point

$$\delta W = \frac{3}{4abc} \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2} - 1 \right). \quad \dots \quad (4)$$

From these results Rodrigues deduced the attraction of the ellipsoid at the given point. It is easy to find from them an expression for the potential. (See § 18 below.)

The determination of δW depends on the evaluation of a certain integral taken throughout the volume of the ellipsoid

in which the element of volume is expressed in ordinary polar coordinates. The integral is transformed by considering a homothetic shell within the ellipsoid, and taking as the element of volume that intercepted between the two surfaces at an element ds of one of them. Then the integration is effected for the whole shell to which ds belongs, and then carried from shell to shell for the whole ellipsoid.

It is to be noticed that Rodrigues did not determine the attraction or the potential due to a single homothetic shell, but merely imagined the solid ellipsoid divided into such shells in his process of integration.

10. In the volume of the *Mémoires de l'Institut* (t. xv., 1835) already referred to and immediately preceding the memoir cited, is a "Mémoire sur l'attraction des Ellipsoïdes," also by Chasles, in which the mode of division into homothetic shells is used, and is attributed to Poisson. There can be no doubt that Poisson was the first to calculate explicitly the attraction of such a shell at an external point, and to apply it to the problem of the solid ellipsoid; but it is equally clear that the idea of this mode of division is of earlier date. Over this point arose in 1837 a curious reclamation. Another memoir by Chasles, in which the same method was used, was referred by the Académie des Sciences to Libri and Poinot. The latter reported without mentioning Poisson's memoir of 1835, and thereupon Poisson in the *Comptes Rendus* (t. vi., pp. 83-840) called attention to this mode of decomposition of a solid ellipsoid, and affirmed that it offered the only means of reducing the double integrals of the problem to single integrals. Poinot rejoined re-affirming the priority of Rodrigues in this matter, and the discussion was closed by some further remarks by Poisson, and a second rejoinder from Poinot. These are to be found at the beginning of the next volume of the *Comptes Rendus*: a fairly full account of the controversy is given also by Todhunter in his 'History.'

11. In Poisson's memoir of 1835 it is proved that the resultant attraction of an elliptic homœoid at an external point f, g, h , is directed along the internal axis of the cone drawn from the external point as vertex to touch the homœoid. This is a very remarkable theorem, and attracted very considerable attention. For that axis of the cone is the normal to the confocal ellipsoidal surface through f, g, h ; and the theorem at once gives the family of external confocal ellipsoidal surfaces as the equipotential surfaces of the homœoid. The importance of these surfaces was not perceived until later, when the researches of Green, Gauss, Chasles, and others had become known, and had led to new methods of treating

problems of attraction—methods which had become essential for the progress of the theory of electricity.

Poisson resolved the force due to each element of the homœoid along the axis of the cone, and by expressing the component in terms of polar coordinates referred to f, g, h as origin, was able to obtain the resultant force in an integrable form. His process is one of direct integration of the expressions obtained, and involves some troublesome considerations as to the limits of the integration with respect to θ , the angle between the axis of the cone and the line drawn to the element considered, and therefore runs to considerable length.

12. A very different process of calculating this integral is followed by Charles in his memoir already cited, "*Sur l'attraction d'une couche ellipsoïdale.*" There the theorem of Lamé (given for the steady motion of heat in a uniform solid), that Laplace's differential equation of the potential is integrable when the equipotential surfaces are known, is employed for the family of equipotential surfaces revealed by Poisson's theorem, and the attraction is reduced to the evaluation of a constant left undetermined by the integration of the specialized form of the differential equation for this case. This is effected by considering the particular case of the attracted point on the surface, evaluating the integral for this case, and comparing with the expression obtained from the integration of the differential equation.

13. I shall now show how the integral for the resultant force at f, g, h , expressed in terms of the element ds , its coordinates x, y, z , the perpendicular p from the centre on the tangent plane at ds , the distance r from the point f, g, h to ds , and the angle θ between this line and the axis of the cone, can, by means of a simple geometrical theorem of confocal surfaces, which I have not before seen remarked, be transformed to an immediately integrable form, so that the whole calculation can be set forth very briefly.

In order that the result may be at once applicable to the calculation of the potential of a solid ellipsoid, I take as the equation of the outer surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = k$$

or in the usual abridged notation

$$\sum \frac{x^2}{a^2} = k \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This theorem, it is to be remarked, is not confined to confocal ellipsoidal surfaces.

15. Now, imagine drawn from P as vertex a cone of small solid angle $d\omega$, intercepting two elements of the homœoid at A and B: let ds be the area of that at A. The length of the perpendicular from the centre on the tangent plane at A being p , the mass of the element is $\frac{1}{2}ppdkds/k$; and if ds' be the element at A' of the surface of the confocal corresponding to ds , that is, containing the points corresponding to those contained in ds , we know that

$$pds = \frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} p'ds'. \quad (11)$$

The attraction at right angles to the axis PQ exerted by the element on a unit particle at P is when $\kappa=1$ (see footnote § 6)

$$\frac{1}{2}\rho \frac{dk p ds}{k r^2} \sin \theta_0$$

which by (11) and (10) may be written

$$\frac{1}{2}\rho p_0 \frac{dk}{k} \frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \frac{\cos \theta' ds'}{r^2} \tan \theta_0.$$

In this the only factor which varies from element to element is

$$\frac{\cos \theta' ds'}{r^2} \tan \theta_0.$$

Now $\cos \theta' ds'/r^2$ is the solid angle $d\omega$ subtended at P_1 by the element ds' at A'. We can exhaust the whole of the homœoid by means of elements intercepted by small cones drawn from P as vertex; and to this corresponds precisely an exhaustion of the confocal by small cones radiating from the internal point P_1 as vertex. And clearly for every elementary cone of solid angle $d\omega$, there exists another in the same plane through PQ, for which the factor just referred to has the same value with opposite sign. Hence in no plane through PQ is there any force perpendicular to PQ on a unit particle at P; that is the resultant is along PQ. This is Poisson's theorem.

16. The total force F along PQ (in the direction from P to Q) is given by

$$F = \frac{1}{2}\rho \frac{dk}{k} \int p \frac{\cos \theta_0 ds}{r^2}, \quad (12)$$

in which the expression to be integrated over the homœoid is the component of attraction along PQ due to the single

element ds . If now we use the transformation already employed above, that is, express the coordinates of A and P in terms of the coordinates of their corresponding points A' and P₁, A' on the confocal, P₁ on the given shell, and substitute for ds its proper value in terms of the area ds' of the corresponding element on the confocal as given by (11), and use the theorem (10), the transformed equation is

$$F = \frac{1}{2}\rho \frac{dk}{k} \frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} p_0 \int \frac{\cos \theta'}{r^2} ds' \quad (13)$$

in which the integral is now taken over the confocal. The only variable factor is now $ds' \cos \theta' / r^2$, and it is well known that for the complete confocal shell

$$\int \frac{ds' \cos \theta'}{r^2} = 4\pi$$

since the point P₁ is within it. Hence, if V be the potential at P, (13) becomes

$$F = -\frac{dV}{dn} = 2\pi\rho \frac{dk}{k} \frac{abc}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} p_0 \quad (14)$$

where dn is an infinitesimal step outward along the normal at P to the confocal. But by the equation of the confocal [see also § 29]

$$dn = \frac{1}{2} \frac{k}{p_0} du$$

and (14) becomes

$$-dV = \pi pabc \frac{dkdu}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (15)$$

The potential V of the homœoid at P, and at any other point of the confocal surface, is thus given by

$$V = \pi pabcdk \int_{\lambda'}^{\infty} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \quad (16)$$

where u is now supposed to vary in value as we go from confocal to confocal outward from P in the integration: the confocal from which the integration starts is that on which P lies, and λ' is the positive root of (7) regarded as a cubic in u . The value of the expression on the right is $-V_{\infty} + V$; and as $V_{\infty} = 0$, we have (16).

17. We may proceed in precisely the same way when P is within the homœoid. A confocal ellipsoidal surface of

equation

$$\sum \frac{f^2}{a^2 - u} = k \quad . \quad . \quad . \quad . \quad . \quad (17)$$

is described through P and the corresponding point P_1 on the shell is taken as before. P is joined to the point A on the shell and P_1 to the corresponding point A' on the confocal. The angle θ_0 is as before between AP and the normal at P, and θ' between P_1A' and the normal at A' ; also $AP = A'P_1 = r$. If p_0, p' be the lengths of the perpendiculars from the centre on the tangent planes at P and A' , we have as before

$$p' \cos \theta_0 = p_0 \cos \theta'.$$

Precisely as in the former case we get for the attraction on a unit particle at P

$$F = \frac{1}{2} \rho \frac{dk}{k} \frac{abc}{\sqrt{(a^2 - u)(b^2 - u)(c^2 - u)}} p_0 \int \frac{\cos \theta'}{r^2} ds', \quad (18)$$

where the integral is to be taken over the confocal. The integral is the total solid angle at P_1 subtended by the confocal, and as P_1 is outside that surface, the solid angle is zero. Hence the force is zero at every point within the shell, and the potential is there uniform. Thus the problem of the attraction of a homœoid is completely solved.

We have, in the result stated in (13), the curious theorem that the value of F at P is, to a constant factor, equal to the potential produced, at any point internal to itself, by a uniform magnetic shell coinciding with the confocal surface. The strength of this shell is proportional to the length of the perpendicular from the centre on the tangent plane to the confocal at the point P, and therefore varies with the position of P on the surface, as does the length of that perpendicular.

A similar theorem holds for a distribution upon any surface whatever, which maintains that surface at uniform potential. The potential which that distribution produces at an external point P is equal to the potential which a magnetic shell, coinciding with the equipotential surface through P for the same distribution, produces at any point internal to itself. The strength of the shell varies with the position of P, and is inversely proportional to the distance, dn , of P measured along the normal to a chosen neighbouring equipotential surface. This expresses, in fact, the relation (32) below, namely,

$$-\frac{dV}{dn} = 4\pi\sigma.$$

This shows that σ is inversely proportional to dn for a constant dV . [See § 27.]

18. For a solid homogeneous ellipsoid, which has

$$\sum \frac{x^2}{a^2} = 1$$

for the equation of its surface, (15) becomes

$$V = \pi \rho abc \int_{\lambda}^{\infty} \int_{f(u)}^1 \frac{du dk}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \quad (19)$$

where λ [the positive root of the equation in u

$$\sum \frac{f^2}{a^2+u} = 1]$$

is the lower limit of the integration with respect to u , and $f(u)$ is the value of k that fulfils equation (7) for any one particular value that u may have in the range of integration. This value of k is given by

$$k = \sum \frac{f^2}{a^2+u}.$$

Thus

$$V = \pi \rho abc \int_{\lambda}^{\infty} \frac{\left(1 - \sum \frac{f^2}{a^2+u}\right) du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \quad (20)$$

There is no difficulty in the application of this method of integration by shells to the formal determination of the potential of solid ellipsoids or of thick shells of varying density, if each homœoidal film is of uniform density throughout. If the density ρ vary from shell to shell then for the ellipsoid by (16)

$$V = \pi abc \int_0^1 \rho dk \int_{\lambda'}^{\infty} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}},$$

where λ' is again the positive root of (7) for any given value of k . But

$$dk = - \left\{ \sum \frac{f^2}{(a^2+u)^2} \right\} du$$

and when $k=0$, $u=\infty$; likewise when $k=1$ $u=\lambda$, where λ is the positive root of

$$\sum \frac{f^2}{a^2+u} = 1$$

regarded as a cubic in u . Therefore

$$V = \pi abc \int_{\lambda}^{\infty} \rho \left\{ \sum \frac{f^2}{(a^2+u)^2} \right\} du \int_u^{\infty} \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (20')$$

19. The result stated in (20) seems to have been first given by Plana in a "Note sur l'intégrale $\int \frac{dM}{r} = V$ " which appeared in *Crelle's Journal*, vol. xx. (1840). It was given also by Lejeune Dirichlet in the same journal in 1846. His process of demonstration is, however, indirect. The result in (20) is first assumed, and then verified by showing that this value of V satisfies Laplace's differential equation of the potential

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad (21)$$

at every point external to the ellipsoid, and that V vanishes at infinity. This latter fact is important. It is fairly obvious for both the homœoid and the solid ellipsoid. The integral (16) for the homœoid, for example, may be compared with the greater integral obtained by putting for each of a^2, b^2, c^2 the value of the smallest. This integral is then evaluated, and it is seen that it vanishes at infinity; *a fortiori* so does the integral for the homœoid.

20. If the point f, g, h be on the surface of the ellipsoid the value of λ is zero in (16) and (20). In the former case the modified equation gives the potential at every internal point for the homœoid; in the latter case

$$V = \pi p a b c \int_0^\infty \frac{\left(1 - \Sigma \frac{f^2}{a^2 + u}\right) du}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} \quad (22)$$

for a surface point on the solid ellipsoid. The case of an internal point requires examination in the latter case.

It is easy to show that equation (22) is applicable without change of form to the case in which the point f, g, h lies within the surface of the solid ellipsoid. For let the point be on the homœoidal surface given by

$$\Sigma \frac{f^2}{a^2} = \mu \quad (23)$$

then the potential is made up of two parts, V_1 due to the ellipsoid internal to the surface (23), and V_2 due to the homœoid of finite thickness external to the point. By (22)

$$V_1 = \pi p a b c \mu^{\frac{3}{2}} \int_0^\infty \frac{\left(1 - \Sigma \frac{f^2}{a^2 \mu + u}\right) du}{\sqrt{(a^2 \mu + u)(b^2 \mu + u)(c^2 \mu + u)}},$$

which, if we write u' for u/μ and substitute, becomes (accents omitted)

$$V_1 = \pi \rho abc \int_0^\infty \frac{\left(\mu - \Sigma \frac{f^2}{a^2 + u}\right) du}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} \quad . \quad . \quad (24)$$

Again, as may easily be verified,

$$\begin{aligned} V_2 &= \pi \rho abc \int_\mu^1 dk \int_0^\infty \frac{du}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} \\ &= \pi \rho abc (1 - \mu) \int_0^\infty \frac{du}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} \quad . \quad . \quad (25) \end{aligned}$$

Hence

$$V = V_1 + V_2 = \pi \rho abc \int_0^\infty \frac{\left(1 - \Sigma \frac{f^2}{a^2 + u}\right) du}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} \quad . \quad (26)$$

The value of V given by (20) and (26) was verified by Dirichlet by showing that it satisfies Poisson's differential equation

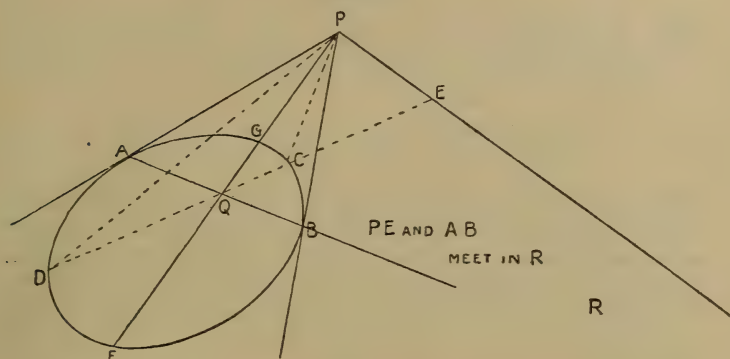
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho \quad . \quad . \quad . \quad (27)$$

within the attracting matter, and Laplace's equation elsewhere, gives continuous values of the force-components $-dV/dx$, $-dV/dy$, $-dV/dz$ at the surface, and vanishes for $u=\infty$. Thus (20) and (26) give the solution of the differential equation of the potential for the given distribution of matter, and the known family of equipotential surfaces possessed by each homœoidal part. As has already been remarked, it was shown by Lamé that the differential equation could be integrated in these circumstances. It would, however, be outside the scope of the present paper to enter into a discussion of the process. Suffice it to say that any solution which fulfils the conditions indicated above can be proved to be the only one.

21. The lemma stated in § 14 above enables the whole problem of the ellipsoid to be disposed of very simply; but, so far as merely proving Poisson's theorem of the direction of the attraction of a thin homœoid is concerned, nothing more elegant has ever been invented than the demonstration published in *Crelle's Journal* (Bd. 12, 1834) by Steiner, immediately after the theorem was announced by Poisson in the memoir of 1833, to which reference has already been made. Steiner's construction is shown in the adjoining diagram.

P is the external point and PQ is the normal to the confocal through P to the homœoid of which a section through PQ is given in fig. 2. A, B are two of the points in which the enveloping cone touches the homœoid, and therefore the line

Fig. 2.



AB is the polar of P with respect to the elliptic section made by the plane BPA. The line PQ meets the curve in G and produced again meets it in F. G, F are points in which the line PQ, starting from P and ending in F, is divided harmonically. Similarly, if PE be drawn at right angles to PG and meet AQB produced in R, the line RQ is also harmonically divided in B and A. It follows that if EQ meet the curve in C and D, EQ is divided harmonically in C and D. Thus PQ not only bisects the angle APB, but also the angle CPD. Hence $DQ/QC = PD/PC$.

Now let a cone of small vertical angle be drawn from Q as vertex with its axis along CD. It will intercept two elements of the homœoid at C and D, the masses of which are directly as the squares of their distances from Q, while their attractions, per unit of their mass in each case, are inversely as the squares of these distances. Hence the total attractions on a unit particle at Q are equal and opposite. But it has been seen that $PD/PC = DQ/QC$; hence the attractions of the same pair of elements on a particle at P must be of equal amount, and being along PD and PC are equally inclined to PQ, and have therefore a resultant along that line. The same thing is true for any other pair of elements intercepted by a cone with vertex at Q, and the whole homœoid may be exhausted by pairs of elements in this way. Any plane through PQ thus divides the homœoid into two portions which exert attractions at P equal in amount and equally inclined to PQ.

22. Returning now to the results expressed by equations (16), (20), with lower limit λ or 0, as the case may be, it is obvious that we can change the lower limit of the integral. Thus, taking (16) we write

$$u + \lambda'_1 = u' + \lambda'$$

where λ'_1 is positive as well as λ . Then when u is equal to λ' , u' is equal to λ'_1 , and the equation becomes

$$V = \pi p a b c d k \int_{\lambda'_1}^{\infty} \frac{du'}{\sqrt{(a^2 + u')(b^2 + u')(c^2 + u')}}, \quad (28)$$

which is of the same form as (16) and the accents on the u 's may be omitted. This proves that two confocal homœoids of equal mass produce the same potential at any point P external to both, that is, that their external fields are identical. If the masses are different, the potentials (and therefore also the field-intensities) at the different points are proportional to the masses.

This of course is a particular case of the very general theorem of the potentials of distributions which was given by George Green, of Nottingham, in his celebrated "Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," published by subscription at Nottingham in 1828.

The substitution used above is also applicable to (20), and proves that any two confocal solid ellipsoids of equal mass produce the same potential at every point external to both. If the masses are different in the two ellipsoids, the potentials at the same external point are proportional to the masses. This is what is usually called Maclaurin's* theorem; but it was only given in its full generality by Laplace many years after Maclaurin's death. It is stated in Maclaurin's 'Treatise of Fluxions' (Edinburgh, 1742), § 653, that the attractions of two confocal ellipsoids are the same at all external points which are on the prolongation of the axes. This was a very remarkable result for the time, and though the theorem was only fully generalised by Laplace in his book entitled 'Théorie du Mouvement et de la Figure Elliptique des Planètes'

* Colin Maclaurin, 1698-1746, Professor of Mathematics in the University of Edinburgh, appointed as assistant and (apparently) successor to James Gregory in 1725. There being a difficulty, through want of funds, in making this arrangement, Newton offered to pay £20 a year if Maclaurin were appointed. Thus Maclaurin was appointed, *Newtono suadente*, as stated in the inscription on his monument.

(Paris, 1784)*, there is some justification for continuing, as has been very generally done, to associate it, even in its extended form, with the name of Maclaurin.

23. From (14) in § 16 above we obtain the components of attraction at the point P. The cosine of the angle which the normal at P to the confocal makes with the semi-axis, of which the length is a , is $p_0 f / (a^2 + u)k$. [See (7) § 13.] Hence for the component X of force in the direction of x increasing on a unit particle at P (14) gives

$$X = -2\pi\rho \frac{dk}{k^2} \frac{abcf}{\sqrt{(a^2+u)^3(b^2+u)(c^2+u)}} p_0^2 \quad (29)$$

Y and Z are obtained by substituting $g/(b^2+u)$, $h/(c^2+u)$ for $f/(a^2+u)$ in this equation. The same results are deducible from (16) by differentiating with respect to u , and multiplying the result by the value of du/df drawn from the equation

$$\Sigma \frac{f^2}{a^2+u} = k$$

of the confocal.

In the same way (20) gives

$$X = -2\pi\rho abcf \int_{\lambda}^{\infty} \frac{du}{\sqrt{(a^2+u)^3(b^2+u)(c^2+u)}} \quad (30)$$

and similar expressions for Y, Z, which may be written down by symmetry. In the case of $\lambda=0$, the factors which multiply f , g , h in these expressions for X, Y, Z are independent of the values of these coordinates. Hence for an internal point f , g , h of a solid ellipsoid

$$X = Af, \quad Y = Bg, \quad Z = Ch \quad \dots \quad (31)$$

where A, B, C are constants, the values of which are given by (30) and the other two similar equations

24. I shall now indicate the method which the theory of equivalent distributions affords for the solution of the problem of the ellipsoid. It was shown by Coulomb, for the case of an electrical distribution, that the normal force just outside a closed conductor is proportional to the surface

* This book is referred to by Todhunter in his 'History of Attractions,' and he quotes Professor de Morgan as to its rarity. I have not seen it, and give the reference above only at second hand. The University Library has no copy, and though the magnificent edition of Laplace's works, which is now being published in Paris, has reached vol. xiii., this book has not been included, though much of later date and on similar subjects has already appeared.

density (amount of attracting or repelling matter per unit area) in the neighbourhood. In the language of the potential this is expressed by the equation

$$-\frac{dV}{dn} = 4\pi\sigma \quad \dots \quad (32)$$

where σ is the surface-density, and dV/dn denotes the rate of variation of potential per unit of distance outwards along the normal. This equation is at once transferable to gravitational attraction, and σ becomes the density of a thin stratum of ordinary matter. Here σ of course denotes the surface-density of ordinary matter; *e. g.* the mass $\frac{1}{2}\rho pdk/k$ of a homœoid taken per unit area at an element just outside which the normal force $-dV/dn$ is taken.

25. I shall not discuss the properties of level surfaces here; but merely apply some of the properties I have mentioned to the problem of the ellipsoid. But a theorem of Bertrand may be referred to of which our process will afford an illustration. Let there be a family of surfaces represented by the equation

$$f(x, y, z) = \alpha \quad \dots \quad (33)$$

where α is a variable parameter; and let them be such that if a distribution of matter be placed on the surface, S say, characterized by any chosen value of α , and be made of surface-density inversely proportional to the distance from that surface to an adjacent one of the family, the whole space within the surface is at uniform potential. Then the surfaces external to S are level surfaces for the distribution specified.

The truth of this theorem may be seen as follows. Let the distribution specified be made on an inner surface, S_1 say, of the family: the space within is at uniform potential, and therefore so also are all points of the surface. But if σ be the density at any point of the surface, then just outside

$$-\frac{dV}{dn} = 4\pi\sigma \quad \dots \quad (34)$$

Now the step from the surface S_1 to an adjacent one S_2 may be taken as dn , and being inversely proportional to σ gives a constant difference of potential between S_1 and S_2 . Hence S_2 is a level surface for the distribution on S_1 . Let now the distribution be transferred to S_2 , and be made according to the law set forth in (32) for the values of dV/dn which exist at S_2 with the distribution on S_1 . Since S_2 is a level surface for the distribution on S_1 , the transference thus effected will bring the whole space within S_2 to uniform

potential equal to that which existed at S_2 before the transference. But the surface S_2 is by hypothesis one over which a distribution can be made of density inversely proportional at the different points to the normal step to an adjacent surface S_3 , and producing uniform potential in the interior. If the mass in this latter distribution be made the same as that which has been transferred from S_1 , the potentials at S_2 and within it produced in the two cases will be the same. For it can be proved * that there cannot be two distributions of a given charge of matter over a surface so as to produce uniform potential at all parts on or within the surface.

The distribution therefore transferred from S_1 must have the same density as in the other case supposed, that is σ must be inversely proportional to the step from S_2 to S_3 . S_3 is a level surface and the charge can now be transferred to S_3 , when S_4 will be found to be a level surface and so on. Hence Bertrand's theorem is proved. It is easy to construct (see Picard, *Traité d'Analyse*, tome i.) an analytical proof of the theorem, founding on Lamé's theorem of the integration of Laplace's equation for a given system of level surfaces.

26. In the transference of matter imagined in the last article the particles may be regarded as carried out along trajectories, cutting the successive surfaces at right angles. Thus, if we draw these trajectories from points in the periphery of ds_1 , they will mark out elements ds_2, ds_3 , etc., on the successive surfaces. The matter first on ds_1 will be carried to ds_2 , then to ds_3 , and so on; and this law will hold however small ds_1, ds_2 , etc. may be made.

Let now this process be applied to the elliptic homœoid discussed above. It is plain that we may take as σ the value $\frac{1}{2}\rho\rho dk/k$, which gives us the result

$$-\frac{dV}{dn} = 2\pi\rho\rho \frac{dk}{k}. \quad . \quad . \quad . \quad (35)$$

But if m denote the total mass of the homœoid

$$m = \frac{1}{2}\rho \int \frac{dk}{k} p ds = 2\pi\rho abck^{\frac{1}{2}} dk, \quad . \quad . \quad . \quad (36)$$

since $\int p ds$ is three times the volume of the homœoid, that is, $4\pi abck^{\frac{3}{2}}$. Thus we obtain $dk/k = m/2\pi\rho abck^{\frac{3}{2}}$ and

$$-\frac{dV}{dn} = \frac{m}{k^{\frac{3}{2}}} \frac{p}{abc}. \quad . \quad . \quad . \quad (37)$$

* This is one of a set of theorems as to the uniqueness of solutions of potential problems. The proof will be found in treatises on Electricity or on Gravitational Attraction.

From (37) it follows that the work done against attraction in carrying a unit particle outwards through a small distance dn along the normal is

$$-dV = \frac{1}{k^{\frac{3}{2}}} \frac{m}{abc} p dn. \quad . \quad . \quad . \quad . \quad (38)$$

27. If then dn be taken for different points of the surface so that $p dn$ is constant, there will be the same step of potential at every point, and the surface on which lie the extremities of these elements, dn , of the normals will be, like the surface of the homœoid, an equipotential or level surface. The distance between this surface and that of the homœoid is inversely proportional to p . It can in fact be shown very easily that the shell of space between the surfaces is a *focaloid*, to use the name given by Thomson and Tait to a space bounded by confocal ellipsoidal surfaces. For the equation of an ellipsoidal surface external to and near to the external surface of the homœoid $\Sigma(x^2/a^2) = k$, is

$$\Sigma \frac{x^2}{a^2 + du} = k,$$

where du is small, and x, y, z are the coordinates of a point on the new surface. If for x, y, z in the last equation we write $x+dx, y+dy, z+dz$, and subtract from the result $\Sigma(x^2/a^2) = k$, we obtain for the thickness dn at x, y, z of the shell of space

$$\frac{p}{k} \Sigma \left(\frac{x dx}{a^2} \right) = dn = \frac{1}{2} \frac{k}{p} du. \quad . \quad . \quad . \quad . \quad (39)$$

Thus the equipotential surface given by dn thus chosen is confocal with the surface of the homœoid.

28. Let us now imagine the mass of the homœoidal shell carried out along the normal at each point, and distributed on the near confocal surface, so that the mass on any element ds of the shell is placed on the element ds' which is marked off by normals drawn from the periphery of ds . The shell thus formed will, by Green's principle of equivalent distributions, be a new homœoid which will give the same field external to itself as was produced by the original shell. For, take the tubular space marked out by normals drawn from the periphery of ds , and terminated by two caps, one just inside ds , but otherwise coinciding with it, the other outside ds' , and fitting closely to that element. The surface of this portion of space is the lateral surface, the inner cap, and the outer. Now take the integral of normal force $\int N ds$ over the whole surface of this space. The inner cap contributes nothing to it since there is no force within the homœoid,

the lateral surface also contributes nothing, the outer cap gives $F'ds'$ if F' be the field-intensity there. The matter within the space is σds , and therefore by Green's (or Gauss's) theorem of the surface integral of normal force over a closed surface $F'ds' = -4\pi\sigma ds$. But if matter were distributed over ds' , and similarly over the rest of the surface of the confocal, so that the field remained unaltered, the surface integral over the short tube just described would still be $F'ds'$, and the matter on ds' would now be σds . This distribution is unique for the given level surface, and the given field external to it. Since the surface S' is ellipsoidal, and the potential is constant within it, the distribution upon it effected as described, by carrying the matter out from the initial homœoid, is also homœoidal; otherwise the distribution over the surface producing uniformity of potential would not be, as it can be proved to be, unique.

It is easy to verify this latter point as to the nature of the distribution. The matter on ds' is now σds , and for σ we may write βp where β is a constant. Hence the new surface-density $\sigma' = \beta p ds/ds'$. But if p' be the length of the perpendicular from the centre for ds' , and $a', b', c' = \sqrt{a^2 + du}, \sqrt{b^2 + du}, \sqrt{c^2 + du}$, we have $p'ds' = pds \cdot a'b'c'/abc$, so that $\sigma' = \beta p'abc/a'b'c'$, thus σ' varies as p' , that is, the distribution is homœoidal.

We can now imagine a further step of potential taken from the surface S' to a succeeding confocal and so on, until we have carried the whole distribution of matter to a surface, every point of which is at an infinite distance from the original surface. There the surface-density will be zero, and the potential at infinity, which has not been altered by the transference, will be zero. It is thus seen that the equipotential surfaces of the external field of the original homœoid are ellipsoids confocal with the original homœoid, a well-known result which has been otherwise established above.

The matter in the transference passes from confocal to confocal, so that the matter on an element of one is carried to the corresponding element on the next, and so on. Thus at any stage of the transference when the distribution is on a given confocal the matter which was originally on the element ds of the original homœoid is situated on the element of the confocal which corresponds to ds . The transference is along the hyperbolas which are the orthogonal trajectories of the confocals, or, as they are often called, the lines of force of the field.

29. Another way of dealing with this problem of equivalent

distributions is to apply the theorem of Bertrand discussed in § 27. The initial distribution is an elliptic homœoid, and we know that the family of confocal surfaces surrounding it fulfil the condition stated in Bertrand's theorem. For upon any one of them a homœoidal distribution can be placed so as to produce a constant potential throughout its interior. Hence the confocals are the level surfaces of the field of the original homœoid. Hence also we can suppose the whole distribution carried out from confocal to confocal, so that at each instant the distribution is homœoidal on one of the level surfaces, and the path of each particle is along the line of force at the inner extremity of which it was originally situated.

30. We have found [§ 28, equation (38)] the step of potential from the initial surface to an adjacent one of which the equation is

$$\Sigma \frac{x^2}{a^2 + du} = k$$

and it is proved in § 29 that $dn = \frac{1}{2}kdu/p$. Thus (38) becomes

$$-dV = \frac{1}{2\sqrt{k}} \frac{m}{abc} du$$

In the same way the step of potential from the level surface

$$\Sigma \frac{x^2}{a^2 + u} = k$$

to the adjoining surface for which u has been increased by du is

$$-dV = \frac{1}{2\sqrt{k}} \frac{mdu}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} \quad \dots \quad (40)$$

Integrating from $u=0$ to $u=\infty$, and observing that the integral must vanish at infinity, we have

$$V = \frac{m}{2\sqrt{k}} \int_0^\infty \frac{du}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} \quad \dots \quad (41)$$

If in this we insert the value of m stated in (36) it becomes

$$V = \pi pabcdk \int_0^\infty \frac{du}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} \quad \dots \quad (42)$$

If in (42) we change u to $u-u'$ we obtain

$$V = \pi pabcdk \int_{u'}^\infty \frac{du}{\sqrt{(a^2 + u - u')(b^2 + u - u')(c^2 + u - u')}} \quad (43)$$

which is to be interpreted as the total step of potential involved in carrying the matter, supposed initially in a thin homœoid (mass m) of which the equation is $\Sigma\{x^2/(a^2-u')\}=k$ from the level surface $\Sigma x^2/a^2=k$ to infinity in the manner described. Thus matter m distributed in a homœoid on the latter surface has the same potential at the surface and at all external points as the same matter had when in the original homœoid. This is for confocal homœoids what Maclaurin's theorem is for solid ellipsoids, and indeed Maclaurin's theorem flows from it at once since a solid ellipsoid may be supposed built up of a succession of homœoids. It is to be observed, however, that this theorem of equivalence of homœoids on confocal surfaces is only a particular case of Green's very general theorem of equivalence. Maclaurin's theorem and the analogous theorems for shells have been explained in § 22, and it is not necessary to pursue the subject here. In § 18 the extension for a shell to a solid ellipsoid has also been fully discussed.

31. The potential of a thin homœoid at a point on its surface or anywhere in the interior is given in (42). From this we can obtain the potential of a thick homœoid at a point within the interior hollow. We shall suppose that the equations of the outer and inner surfaces are respectively

$$\Sigma \frac{x^2}{a^2} = 1, \quad \text{and} \quad \Sigma \frac{x^2}{a^2} = h.$$

We have therefore only to integrate (42) with regard to k from $k=h$ to $k=1$. Thus for the thick homœoid

$$V = \pi pabc(1-h) \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \quad (44)$$

The potential produced by this homœoid at an external point may be found as follows: The squares of the semi-axes of the interior ellipsoidal hollow are a^2h , b^2h , c^2h , where $h < 1$. The potential V' at the external point f , g , h (here h is a co-ordinate), due to an ellipsoid of density ρ filling the hollow is

$$V' = \pi pabch^{\frac{3}{2}} \int_{\lambda_1}^\infty \frac{\left(1 - \Sigma \frac{f^2}{a^2h + u_1}\right) du_1}{\sqrt{(a^2h + u_1)(b^2h + u_1)(c^2h + u_1)}} \quad (45)$$

where λ_1 is the positive root of the equation

$$\Sigma \frac{f^2}{a^2h + u_1} = 1.$$

If in (45) we write uh for u_1 , we get

$$V' = \pi pabc \int_{\lambda'}^{\infty} \frac{\left(h - \Sigma \frac{f^2}{a^2 + u}\right) du}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} \quad \dots \quad (46)$$

where $\lambda' = \lambda_1/h$. The potential at the external point f, g, h due to the complete ellipsoid is given by (20) [§ 18]. Subtracting (46) from (20) we obtain for the thick homœoid

$$V = \pi pabc \left[\int_{\lambda}^{\lambda'} \frac{\left(1 - \Sigma \frac{f^2}{a^2 + u}\right) du}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} + (1 - h) \int_{\lambda'}^{\infty} \frac{du}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} \right]. \quad (47)$$

When the internal hollow is contracted to zero, that is when $h=0$, λ' becomes ∞ , and the second integral on the right vanishes. The equation then coincides with (20) as it ought to do.

32. We can now find the potential produced by a thick focaloid at an external or internal point. First for an external point: let two confocal ellipsoids have the same density, and let the shell between their surfaces be the focaloid to be considered. The potentials which they produce at a point external to both are in the ratio of the masses of the ellipsoids. For the potential of that which has $\Sigma(x^2/a^2)=1$ for the equation of its surface the equation is (20) [§ 18 above], and for the potential of the other which has, let us say, $\Sigma\{x^2/(a^2-s)\}=1$ for the equation of surface, the equation is the same with as multiplier of the integral $\pi\rho\sqrt{(a^2-s)(b^2-s)(c^2-s)}$ instead of $\pi pabc$. Thus for the potential of the focaloid at an external point f, g, h we obtain

$$V = \frac{4}{3}m' \int_{\gamma}^{\infty} \frac{\left(1 - \Sigma \frac{f^2}{a^2 + u}\right) du}{\sqrt{(a^2 + u)(b^2 + u)(c^2 + u)}} \quad \dots \quad (48)$$

where m' is put for the mass of the focaloid, that is

$$\frac{4}{3}\pi\rho\{abc - \sqrt{(a^2-s)(b^2-s)(c^2-s)}\}.$$

This equation is precisely the same as (20) which gives the potential at an external point for a solid ellipsoid: and just as in the case of ellipsoids it follows that:—any two uniform focaloids which are confocal with the same ellipsoid produce, at any point external to both, potentials which are in the ratio of the masses of the focaloids.

This theorem was practically given for points in the axes of an ellipsoid of revolution by Maclaurin. For he states [*Fluxions*, Art. 650], with the same limitations, the remarkable result that an ellipsoid made up of confocal shells each of uniform density, differing from shell to shell, and an ellipsoid of the same size and of uniform density, exert attractions on an external particle which are in the same direction, and have values in the same ratio as the masses of the ellipsoids.

The theorem becomes generalized by the extension of Maclaurin's theorem by Laplace to any form of ellipsoid and any external point. Obviously in the theorem just stated for a heterogeneous ellipsoid, the ellipsoids compared need not be of the same size but only confocal.

33. If the point f, g, h considered be in the hollow within the focaloid, the potential can be found by subtracting from the expression for the potential at the point due to the complete ellipsoid, the potential at the same point due to the solid ellipsoid of the same density bounded by the surface $\Sigma\{x^2/(a^2-s)\}=1$, the internal surface of the focaloid. Making this calculation by (22), and putting in the result m' (see § 32) for the mass of the focaloid, we get

$$V = \frac{3}{4}m' \int_0^\infty \frac{\left(1 - \Sigma \frac{f^2}{a^2+u}\right) du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} - \frac{3}{4}(m-m') \int_{-s}^0 \frac{\left(1 - \Sigma \frac{f^2}{a^2+u}\right) du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \quad (49)$$

where m is the mass of the complete ellipsoid.

In the remaining case, that in which the point f, g, h is within the mass of the focaloid, the procedure is exactly the same as that just described. The form of the result is slightly different: it is

$$V = \frac{3}{4}m' \int_0^\infty \frac{\left(1 - \Sigma \frac{f^2}{a^2+u}\right) du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} - \frac{3}{4}(m-m') \int_{\lambda'-s}^0 \frac{\left(1 - \Sigma \frac{f^2}{a^2+u}\right) du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}, \quad (50)$$

where λ' is the positive root of $\Sigma\{f^2/(a^2-s+u)\}=1$, regarded

as an equation in u . When $\lambda' = 0$, the point f, g, h lies on the internal surface of the focaloid. Then (50) agrees with (49).

34. It is interesting to compare equations (42) and (49). The first gives the potential produced at any internal point by a thin elliptic homœoid of mass m , and (49) gives that of a focaloid of mass m at an internal point f, g, h . If we put $f = g = h = 0$, we get the potentials at the centre in the two cases. Let it be supposed that both shells are thin, and that both have the same external surface. The thickness of the homœoid being directly, and that of the focaloid inversely, as the length of the perpendicular let fall from the centre on the tangent plane at the point considered, the potential at the centre must be greater for the focaloid than for the homœoid. The first term on the right of (49) involves for the centre the same integral as does (41) for the homœoid, but this integral is multiplied by $\frac{3}{4}m'$ in (49) as against $\frac{1}{2}m$ in (41). The excess is diminished by the second term in (49) which varies with the deviation of the surface from sphericity; and also with the thickness of the focaloid on the whole.

In the particular case in which the surface is spherical the second term just makes the potential at the centre the same for a thin focaloid as for a thin homœoid, as the reader may verify by evaluation. If we take the case of a solid ellipsoid the second term in (49) vanishes, and the potential at the centre is $\frac{3}{2}$ of that which would be produced in the interior of a thin homœoid of the same mass and coincident with the surface of the ellipsoid.

35. From the expression given in (44) for the potential in the interior of a homœoid of any thickness, we can readily calculate the work done by gravitational attraction in bringing together from infinite dispersion in space the matter composing an ellipsoid, or a homœoid of finite thickness, whether uniform or made up of homœoidal shells of different densities. For the case of uniform density, let mass of amount $2\pi\rho abch^{\frac{1}{2}}dh$, be brought from infinity to the homœoid to which (44) refers, and be placed as an additional thin homœoid on the interior surface. The work done by gravitational attraction in bringing this matter into position is Vm . Hence

$$Vm = 2\pi^2\rho^2a^2b^2c^2h^{\frac{1}{2}}(1-h)dh \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (51)$$

If then W denote the whole work done in building up the

homœoid by adding to the interior until h is changed from 1 to H ,

$$W = -2\pi^2 \rho^2 a^2 b^2 c^2 \int_H^1 h^{\frac{1}{2}} (1-h) dh \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \\ = 2\pi^2 \rho^2 a^2 b^2 c^2 \left\{ \frac{4}{15} - \frac{2}{3} H^{\frac{1}{2}} (1 - \frac{2}{3} H) \right\} \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (52)$$

If $H=0$, this becomes the result for a solid ellipsoid.

For this case

$$W = \frac{8}{15} \pi^2 \rho^2 a^2 b^2 c^2 \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (53)$$

In the particular case of a uniform sphere this becomes

$$W = \frac{3}{5} \frac{M^2}{a}$$

if the unit of mass is gravitational (see § 6, footnote), or

$$W = \frac{3}{5} \kappa \frac{M^2}{a}$$

if the ordinary unit of mass is used and κ is the proper value of the gravitational constant. This is the result given by Helmholtz, from which the rate of shrinking of the sun necessary to supply the energy radiated may be calculated.

If the density vary from shell to shell (53) becomes

$$W = 2\pi^2 a^2 b^2 c^2 \int_0^1 \rho \frac{dh}{h} \int_h^1 \rho dk \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (54)$$

as the reader may verify by finding the work done in building up a thick homœoid by adding shells of varying density ρ , and then varying the constant h of this homœoid from 0 to 1.

XXXII. *On a New Principle of Relativity in Electromagnetism.* By A. H. BUCHERER, D.Sc., Privatdocent in the Bonn University*.

§ 1. IT is needless to dwell on the serious difficulties which the Maxwellian theory has encountered by the well established experimental fact that terrestrial optics is not influenced by the earth's motion. The endeavours of some distinguished physicists, notably of H. A. Lorentz, to

* Communicated by the Author. A short note on the same subject was published in the *Physik. Zeitschr.* vii. p. 556 (1906).

modify the Maxwellian theory in such a manner as to eliminate the effects of translatory motion have admittedly failed. Now the author, guided by the feeling that the *form* of the Maxwellian equations must correspond somehow to the true laws of electromagnetism, has attempted a new *interpretation* of these equations that should harmonize with facts.

In the following the author will show how this task can be accomplished. We consider two electromagnetic systems A and B in uniform rectilinear motion relatively to each other. Then, whenever we speak of the dynamical interaction of the systems we stipulate that the *system acted upon*—we will call this henceforth the *passive* system, and accordingly the other system the *active* one—experiences the same force as it would in the Maxwellian theory on the assumption *that it were at rest in the aether* and the other system moving relatively to it. As will be shown later on, the stipulation that the passive system is invariably the system “at rest” implies the principle of relativity.

§ 2. The Maxwellian equations as adapted to the electron theory have this form for empty space :

$$(I.) \quad -\frac{d\mathbf{H}}{dt} = \text{curl } \mathbf{E},$$

$$(II.) \quad \frac{d\mathbf{E}}{dt} = v^2 \text{curl } \mathbf{H} - 4\pi v^2 \rho \mathbf{u},$$

$$(III.) \quad \nabla \mathbf{H} = 0,$$

$$(IV.) \quad \nabla \mathbf{E} = 0, \quad \text{or} \quad = 4\pi v^2 \rho.$$

As in the Maxwellian theory, $\mathbf{H}$ and $\mathbf{E}$ are the field intensities measured in the system “at rest.” The forces exerted by the active system are purely electrical or magnetic. There are no electrodynamic forces on a passive system. We will limit our investigation for simplicity to rigid systems. The total force exerted on the passive system takes the form

$$(V.) \quad \mathbf{F} = \iint \mathbf{H} \sigma_m dg + \iiint \mathbf{E} \rho d\tau.$$

Here σ_m means the surface-density of magnetism ; dg is an element of surface, and $d\tau$ an element of volume. While $\mathbf{H}$ and $\mathbf{E}$ are due to the active system, σ_m and ρ belong to the passive system.

We will now proceed to apply our equations to the relative motion of electrons and fictitious magnetic poles. We thus obtain so-called point laws which by suitable integrations can be employed to find the ponderomotive forces for any distribution of electrical and magnetic masses of systems

in relative motion. The following point laws are partly known from the Maxwellian theory. They can be easily verified and found in agreement with the differential equations.

When two electrons move relatively to each other the passive one is acted upon by the force

$$\mathbf{F} = \frac{\mathbf{r}_1 q^2 v^2 s}{r^2 \left(1 - \frac{u^2}{v^2} \sin^2 \gamma\right)^{\frac{3}{2}}} \cdot \cdot \cdot \cdot (1)$$

Here $\mathbf{r}_1$ signifies a unit radius vector drawn from the active to the passive electron; q is the charge of the electron, γ is the angle which the direction of motion makes with $\mathbf{r}$, $\mathbf{u}$ is the velocity of motion, and $s = 1 - \frac{u^2}{v^2}$.

Of two moving fictitious magnet-poles of unit strength the passive one is acted upon by the force

$$\mathbf{F} = \mathbf{r}_1 \frac{s}{r^2 \left(1 - \frac{u^2}{c^2} \sin^2 \gamma\right)^{\frac{3}{2}}} \cdot \cdot \cdot \cdot (2)$$

Further, when an electron and a unit magnet-pole are moving relatively to each other, the force on the passive electron is

$$\mathbf{F} = V \mathbf{u} \mathbf{r}_1 \frac{sq}{r^2 \left(1 - \frac{u^2}{v^2} \sin^2 \gamma\right)^{\frac{3}{2}}}, \cdot \cdot \cdot (3)$$

whereas the force on the passive magnet-pole is

$$\mathbf{F} = V \mathbf{r}_1 \mathbf{u} \frac{sq}{r^2 \left(1 - \frac{u^2}{v^2} \sin^2 \gamma\right)^{\frac{3}{2}}}. \cdot \cdot \cdot (4)$$

Here as elsewhere the radius vector is always drawn to the passive system. *Now an inspection of these four fundamental equations shows that the choice of the passive one of the two systems in relative motion is arbitrary; i. e., a moving magnetic pole exerts exactly the same force on a "resting" electron as a moving electron on a "resting" pole, and we can at once conclude that with any distribution of magnetic and electric masses in the two systems, the force exerted by A on B is the same as that exerted by B on A. This proves the validity of the third law of Newton and implies the principle of relativity.*

Suppose a system A contains only magnets and B only electric charges; then the observer, whom we imagine always to be on the passive system, will interpret the action exerted on his system as electric or magnetic according as he stays on B or on A. Evidently the electric action must equal the magnetic action, and we have the relation

$$\iiint \mathbf{E} \rho_B d\tau = - \iint \mathbf{H} \sigma_{m_A} dg. \quad . \quad . \quad . \quad (5)$$

Of particular interest are the forces which uniform magnetic and electric field exert on moving electrons. They are different from those of the Maxwellian theory. Suppose an electron moves between the poles of an electromagnet, then, according to equation (3), the force on the passive electron is

$$\begin{aligned} \mathbf{F} &= \iint \frac{\mathbf{V} \mathbf{u} \mathbf{r}_1 s q \sigma_m dg}{r^2 \left(1 - \frac{u^2}{v^2} \sin^2 \gamma\right)^{\frac{3}{2}}} \\ &= \frac{4\pi \sigma_m q}{1 - \frac{u^2}{v^2} (\mathbf{g}_1 \mathbf{u}_1)^2} \mathbf{V} \mathbf{g}_1 \mathbf{u}. \quad . \quad . \quad . \quad (6) \end{aligned}$$

Here $\mathbf{g}$ is the outward normal erected on the positive surface of the pole. For $4\pi\sigma_m$ we can put $\mathbf{H}_0$, the field intensity as measured by an observer at rest with the electromagnet. We then can write for the force exerted on the electron:

$$\mathbf{F} = \mathbf{V} \mathbf{H}_0 \mathbf{u} \frac{q}{1 - \frac{u^2}{v^2} (\mathbf{g}_1 \mathbf{u}_1)^2}, \quad . \quad . \quad . \quad (7)$$

whereas the Maxwellian theory furnishes

$$q \mathbf{V} \mathbf{H}_0 \mathbf{u}.$$

Equation (7) can be tested by the deviation of Becquerel rays in a magnetic field.

Equation (4) furnishes the force which an electron experiences in a uniform electric field, for instance in that of a condenser. Let the surface-density of electricity on the condenser-plate be σ , then evidently

$$\begin{aligned} \mathbf{F} &= \iint \frac{\mathbf{r}_1 q v^2 s \sigma dg}{r^2 \left[1 - \frac{u^2}{v^2} \sin^2 \gamma\right]^{\frac{3}{2}}} \\ &= \frac{4\pi \sigma v^2 q}{1 - \frac{u^2}{v^2} (\mathbf{g}_1 \mathbf{u}_1)^2} \left\{ \mathbf{u}_1 \frac{u^2}{v^2} (\mathbf{g}_1 \mathbf{u}_1) - \mathbf{g}_1 \right\}. \quad . \quad . \quad (8) \end{aligned}$$

$\mathbf{g}_1$ is the unit normal erected on the positive plate of the condenser; $(\mathbf{g}_1 \mathbf{u}_1)$ is the cosine of the angle which $\mathbf{u}$ makes with $\mathbf{g}_1$.

In case the electrons move parallel to the plates the force takes the same form as in the Maxwellian theory.

§ 3. We will now proceed to consider the energy relations that follow from our principle of relativity. Suppose we have two systems A and B in relative motion. External forces are required to keep the motion steady. We denote the field intensities due to the active system A by $\mathbf{E}_A$ and $\mathbf{H}_A$, and those due to the active system B by $\mathbf{E}_B$ and $\mathbf{H}_B$, where it is understood that $\mathbf{E}_A$ and $\mathbf{H}_A$ are measured on B and $\mathbf{E}_B$ and $\mathbf{H}_B$ on A. Further, the volume-density of electricity of A is ρ_A , and that of B is ρ_B . As to the electromagnetic nature of the two systems, we will assume for simplicity that the system A contains only electrical masses, while B contains electrical and magnetic masses. Now remembering that the relative motion of electric and magnetic masses produces forces that are perpendicular to the direction of motion, and hence do not perform work, we find for the energy change in the element of time :

$$dW = dt \iiint \rho_B \mathbf{u} \mathbf{E}_A d\tau = dt \iiint \rho_A \mathbf{u} \mathbf{E}_B d\tau.$$

By the aid of the equations I. to IV. and by the general rule

$$\mathbf{E} \text{ curl } \mathbf{H} = \text{div } \mathbf{V} \mathbf{H} \mathbf{E} + \mathbf{H} \text{ curl } \mathbf{E},$$

we can easily transform this expression and obtain:

$$\begin{aligned} \frac{dW}{dt} = & -\frac{1}{8\pi} \frac{d}{dt} \iint \{ \mathbf{V} \mathbf{E}_A \mathbf{H}_B + \mathbf{V} \mathbf{E}_B \mathbf{H}_A \} d\mathbf{g} \\ & - \frac{1}{8\pi} \frac{d}{dt} \iiint \mathbf{H}_A \mathbf{H}_B d\tau - \frac{1}{8\pi v^2} \frac{d}{dt} \iiint \mathbf{E}_A \mathbf{E}_B d\tau. \quad (9) \end{aligned}$$

In this equation $d\mathbf{g}$ denotes a vectorial element of surface which has the direction of the outward normal. The surface integral vanishes by removing the surface to infinity since radiation is excluded. Thus:

$$\frac{dW}{dt} = -\frac{1}{8\pi} \frac{d}{dt} \iiint \mathbf{H}_A \mathbf{H}_B d\tau - \frac{1}{8\pi v^2} \frac{d}{dt} \iiint \mathbf{E}_A \mathbf{E}_B d\tau. \quad (10)$$

We examine now what happens when the external forces which kept the motion steady are removed. Acceleration will set in, and if the inertia of the system B is infinitely larger than that of A, the latter system alone will be

accelerated. The form of equation (10) suggests putting for the increase of the electromagnetic energy of A in virtue of its acceleration :

$$\frac{dW_A}{dt} = \frac{1}{8\pi} \frac{d}{dt} \iiint \mathbf{H}_A^2 d\tau + \frac{1}{8\pi v^2} \frac{d}{dt} \iiint \mathbf{E}_A^2 d\tau. \quad (11)$$

This equation, which holds for quasistationary motions, is compatible with our principle of relativity but does not follow from it.

§ 4. We consider an electron in motion, and we inquire after its electromagnetic masses. Let μ_l denote its longitudinal and μ_t its transversal mass. External forces acting in the direction of motion will increase its electromagnetic energy W. Proceeding exactly as in mechanics we can put

$$\mu_l = \frac{1}{u} \frac{\partial W}{\partial u} = \frac{1}{\pi u} \frac{\partial}{\partial u} \iiint H^2 d\tau + \frac{1}{8\pi u v^2} \frac{\partial}{\partial u} \iiint E^2 d\tau. \quad (12)$$

Now the right member of this equation has been evaluated by G. F. C. Searle for the case of a charged sphere. Supposing an electron to be a rigid charged sphere, we find by an easy calculation the same expression as Abraham derived for the longitudinal mass.

We can connect the electromagnetic masses of the electron with the quantity of motion $\mathbf{M}$

$$\frac{d\mathbf{M}}{dt} = \mathbf{M}_1 \frac{d\mathbf{M}}{dt} + \mathbf{M} \frac{d\mathbf{M}_1}{dt}. \quad (13)$$

The first term of the right member represents the longitudinal force, while the second member represents the transverse force.

From mechanics we know

$$\mathbf{M} = \int_0^u \frac{1}{u} \frac{\partial W}{\partial u} du = \int_0^u \mu_l du$$

But the general expression for the transverse force is

$$\mathbf{M} \frac{d\mathbf{M}_1}{dt} = - \frac{u^2 \mathbf{R}_1}{R} \mu_t,$$

where $\mathbf{R}$ denotes the radius of curvature.

And

$$\frac{d\mathbf{M}_1}{dt} = - \mathbf{R}_1 \frac{u}{R}.$$

Therefore

$$\mu_t = \frac{1}{u} \int_0^u \mu_l du. \quad (14)$$

Substituting the value of μ_l we find the same value for μ_t as was obtained by Abraham.

It is worthy of note that whereas in general the forces acting on the electron in a uniform magnetic or electric field are different from those of the Maxwellian theory, yet the masses acted upon are identical. Becquerel rays moving obliquely to the field should therefore exhibit a different law of deviation from that expected on the basis of the Maxwellian theory. Here there is the possibility for carrying out an *experimentum crucis* to decide for or against the principle of relativity as here introduced.

§ 5. It is of especial interest to investigate the bearing of our principle on the phenomena of radiation. It is evident that relativity excludes any influence of the translatory motion of the earth on terrestrial optics; and it is likewise apparent that for relative motions of a source of light and an observer the principle of Doppler holds and that the phenomena of aberration are easily accounted for.

The phenomena hitherto considered were governed by the Maxwellian equations for *vacuous space*; and the question imposes itself whether the adopted principle of the relativity of motion is capable of accounting for phenomena in which the structure of ponderable matter plays an essential rôle.

The electron theory, as is well known, interprets the properties of matter by assuming that it is made up of complex assemblages of negative and positive charges. Thus, while our equations still hold in the interstices of matter for the single electrons, the observable properties are due to the combined effects of numerous assemblages of electrons. We obtain these effects by a process of averaging, and as Lorentz has shown the result is a set of equations for ponderable matter which is identical in form with that of Maxwell,—leaving out of regard phenomena of dispersion.

Now it is evident that with our principle of relativity we arrive at the same results as Lorentz as long as we restrict our investigation to bodies at rest. But there are two well-known experiments which have always been adduced as proving the existence of an æther at rest. These experiments are that of Röntgen on the magnetic effect of the rotation of a dielectric between charged condenser-plates, and that of Fizeau with the propagation of radiation in streaming water. It will suffice to show that the former experiment is in perfect accord with our hypothesis. For let a liquid dielectric flow between charged condenser-plates. At the boundary of the dielectric free charges will form and will take part in the motion of the liquid. This motion constitutes a convection current, and as such will be proportional to the free charge; and the magnetic field outside the condenser will therefore likewise be proportionate to the free charge. The latter for a given

condenser charge, as follows from Maxwell's theory and from Lorentz's method, is proportionate to $\frac{K-1}{K}$, where K denotes the dielectric constant. And this is in perfect accord with the experimental result that the magnetic force is proportionate to the fraction $\frac{K-1}{K}$.

Now any theory which yields this coefficient for the Röntgen current, leads to the same coefficient called Fresnel's coefficient for the Fizeau experiment. Evidently according to Maxwell $\frac{K-1}{K}$ equals $1 - \frac{1}{n^2}$ where n is the refractive index, if we disregard dispersion effects. I intend working out a complete theory of the Fizeau experiment on the basis of the principle of relativity.

I hope to have shown by this short survey that this principle can account for all the known facts of electro-magnetism.

Bonn University,
January 1907.

XXXIII. *The Magnetic Field and Inductance Coefficients of Circular, Cylindrical, and Helical Currents.* By ALEXANDER RUSSELL, M.A., M.I.E.E.*

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1. *Introduction.*

SERIES formulæ † are usually given for the components of the magnetic force at a point in the neighbourhood of a current flowing in a circular filament. They are generally

* Communicated by the Physical Society: read February 8, 1907.

† E. Mascart and J. Joubert, *Leçons sur l'Electricité et le Magnetisme*, vol. ii. § 566 (1897).

deduced from the expression for the magnetic potential at the point expressed in a series of spherical harmonics. In many cases the series converge very slowly. It seemed desirable, therefore, to attempt to find formulæ which can be evaluated more readily.

By means of Laplace's formula* for the magnetic force at any point due to an element of current, formulæ for the component forces can be found very easily in terms of elliptic integrals, the values of which are given in Mathematical Tables†. The results found are also useful in hydrodynamics in connexion with the theory of the circular vortex filament. The formulæ give directly the mutual inductance between two coaxial circular currents and a close approximation to the self-inductance of a thin circular current. The author has not found it necessary to assume Neumann's theorem in proving any formula for self or mutual inductance. An expression is found for the axial magnetic force at any point due to a cylindrical current sheet, and particular cases are noticed. It is proved that the mutual inductance between two cylindrical current sheets is the same as that between one of them and a certain helical current of the same diameter and axial length as the other sheet. This theorem can also be immediately deduced from a formula given by Viriamu Jones‡.

The exact formula for the mutual inductance between a cylindrical current sheet and a coaxial helical filament is expressed both in terms of elliptic integrals and in a series which in general converges rapidly. Viriamu Jones left the solution in the form of a definite integral. By the formulæ given in his paper§ this can be expressed without difficulty in terms of complete and incomplete elliptic integrals. By utilising Jones's results, Professor Coffin in the 'Bulletin of the Bureau of Standards' (p. 118, June 1906) has done this when the axial length of the two helices is the same. The author, however, starting from Laplace's formula, directly deduces the complete solution in forms adapted for easy computation. Lorenz's formula|| for the self-inductance of a helical filament is a particular case of the general formula for mutual inductance given in this paper. It is shown also that a well-known formula due to Lord Rayleigh is a particular case of Lorenz's formula.

* A. Russell, 'Alternating Currents,' vol. i. p. 29. This formula is sometimes attributed to Ampère in English text-books.

† For instance, Dale's 'Mathematical Tables,' p. 76.

‡ Proc. Roy. Soc. p. 203 (1898).

§ *L. c. ante.*

|| Wiedemann's *Annalen*, vii. p. 170 (1879).

2. *Mathematical Formulæ.*

It is convenient to collect together for reference the mathematical definitions and theorems in connexion with Elliptic Functions which we shall require in proving the theorems which follow. Most of them are well known, and they are nearly all to be found in Legendre's *Traité des Fonctions Elliptiques* (1825). For proofs the reader is referred to A. Cayley's 'Elementary Treatise on Elliptic Functions.'

The definitions of the complete elliptic integrals E and F are

$$E = \int_0^{\pi/2} \Delta d\phi, \quad \text{and} \quad F = \int_0^{\pi/2} \frac{d\phi}{\Delta},$$

where $\Delta = (1 - k^2 \sin^2 \phi)^{1/2}$, and k is the modulus.

It follows at once from the Integral Calculus that

$$E = \frac{\pi}{2} \left\{ 1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{k^6}{5} - \dots \right\}$$

$$F = \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \dots \right\}.$$

Hence when k is small E and F can be readily computed. When k is nearly equal to unity (which is its maximum value) Legendre* has shown that we can use the formulæ

$$E = 1 + \frac{1}{2} k_1^2 \left(\log_{\epsilon} \frac{4}{k_1} - \frac{1}{1 \cdot 2} \right) + \frac{1^2 \cdot 3}{2^2 \cdot 4} k_1^4 \left(\log_{\epsilon} \frac{4}{k_1} - \frac{2}{1 \cdot 2} - \frac{1}{3 \cdot 4} \right) + \dots, \quad (1)$$

$$F = \log_{\epsilon} \frac{4}{k_1} + \left(\frac{1}{2}\right)^2 k_1^2 \left(\log_{\epsilon} \frac{4}{k_1} - \frac{2}{1 \cdot 2} \right) + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k_1^4 \left(\log_{\epsilon} \frac{4}{k_1} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} \right) + \dots, \quad (2)$$

where $k_1^2 = 1 - k^2$, and so k_1 is a small quantity. For instance, when $k = 0.99$, $k_1 = 0.14$ approximately. For a proof of (1) and (2) see A. Cayley, 'Elliptic Functions,' 2nd edition, p. 54.

* A. le Gendre (A. M. Legendre), "*Mémoire sur les Intégrations par arcs d'ellipse*," Histoire de l'Académie Royale des Sciences, 1786.

The following theorems can be easily proved by elementary Integral Calculus:—

$$\int_0^{\pi/2} \frac{\sin^2 \phi}{\Delta} d\phi = \frac{1}{k^2} (F - E), \quad . \quad . \quad . \quad (3)$$

$$\int_0^{\pi/2} \frac{\cos 2\phi}{\Delta} d\phi = \frac{2}{k^2} (E - F) + F, \quad . \quad . \quad . \quad (4)$$

$$\int_0^{\pi/2} \frac{\sin^2 \phi \cos^2 \phi}{\Delta} d\phi = \frac{2 - k^2}{3k^2} E - \frac{2 - 2k^2}{3k^2} F, \quad . \quad . \quad . \quad (5)$$

$$\int_0^{\pi/2} \frac{d\phi}{\Delta^3} = \frac{E}{1 - k^2}, \quad . \quad . \quad . \quad (6)$$

$$\int_0^{\pi/2} \frac{\sin^2 \phi d\phi}{\Delta^3} = \frac{E}{k^2(1 - k^2)} - \frac{F}{k^2}, \quad . \quad . \quad . \quad (7)$$

$$\int_0^{\pi/2} \sin^2 \phi \Delta d\phi = \frac{2k^2 - 1}{3k^2} E + \frac{1 - k^2}{3k^2} F. \quad . \quad (8)$$

The following theorems for transforming the modulus of elliptic functions are of fundamental importance. By their means simple formulæ can be found from which, as Legendre has shown, the elliptic functions can be calculated to any required degree of accuracy with extreme ease. The first formula* was given by Landen.

$$\left. \begin{aligned} F &= (1 + k')F' \quad . \quad . \quad . \\ E &= \frac{2}{1 + k'} E' - (1 - k')F' \end{aligned} \right\}, \quad . \quad . \quad . \quad (9)$$

where the modulus of E' and F' is k' , and

$$k' = (1 - \sqrt{1 - k^2}) / (1 + \sqrt{1 - k^2})$$

and $k = 2\sqrt{k'} / (1 + k')$.

3. Formulæ for the magnetic force at any point due to the current in a circular filament in rectangular, polar, and bipolar coordinates.

In fig. 1, BR is the conducting circular filament and OZ' is its axis. It is obvious from symmetry that the lines of force will all be in planes passing through OZ'. We shall first find the component forces Z and X, parallel to OZ' and OX' respectively, at the point P in the plane Z'OX'. Let the radius of the circular filament be a , let PO be r , and let the angles POX' and X'OR be θ and ϕ respectively. We shall consider the force at P due to the current in an element $ad\phi$

* A proof is given in Todhunter's 'Integral Calculus,' Chapter X.

Hence, writing $\phi = \pi - 2\phi'$, we get by (6) after a little reduction

$$\left. \begin{aligned} Z &= \frac{2i}{r_1} \left(\frac{a^2 - r^2}{r_2^2} E + F \right) \\ X &= \frac{2i \tan \theta}{r_1} \left(\frac{a^2 + r^2}{r_2^2} E - F \right) \end{aligned} \right\}, \quad \dots \quad (12)$$

where

$$r_1^2 = a^2 + r^2 + 2ar \cos \theta,$$

$$r_2^2 = a^2 + r^2 - 2ar \cos \theta,$$

and the modulus k of the elliptic functions is given by $k^2 = 1 - r_2^2/r_1^2$.

If R and T be the component magnetic forces along and perpendicular to OP (fig. 1), we have

$$\left. \begin{aligned} R &= Z \sin \theta + X \cos \theta = \frac{4ia^2 \sin \theta}{r_1 r_2^2} E, \\ \text{and} \\ T &= Z \cos \theta - X \sin \theta = \frac{2i}{r_1 r_2^2} \left[\frac{a^2 \cos 2\theta - r^2}{\cos \theta} E + \frac{r_2^2}{\cos \theta} F \right] \end{aligned} \right\}. \quad (13)$$

In rectangular coordinates, we have

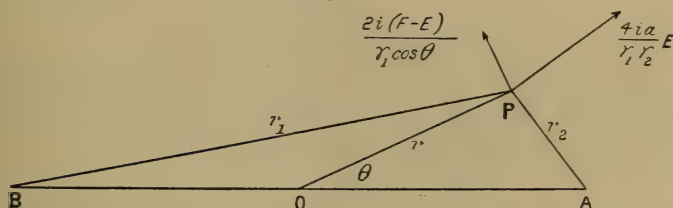
$$\left. \begin{aligned} X &= \frac{2i}{r_1} \left[\frac{2az}{r_2^2} E - \frac{z}{x} (F - E) \right], \\ Z &= \frac{2i}{r_1} \left[\frac{2a(a-x)}{r_2^2} E + (F - E) \right], \end{aligned} \right\} \quad \dots \quad (14)$$

$$\text{where } r_1^2 = (a+x)^2 + z^2,$$

$$r_2^2 = (a-x)^2 + z^2,$$

$$\text{and } k^2 = 1 - r_2^2/r_1^2.$$

Fig. 2.



The magnetic force at P due to the circular current is the resultant of the forces $2i(F-E)/(r_1 \cos \theta)$ and $4iaE/r_1 r_2$ acting at right angles to OP and AP respectively.

If ϕ be the angle APB (fig. 2), $OP = r$, the angle $POA = \theta$, and $OA = OB = a$, we have $AP = r_2$ and $PB = r_1$.

Hence in bipolar coordinates

$$\left. \begin{aligned} Z &= -\frac{2i}{r_2} E \cos \phi + \frac{2i}{r_1} F, \quad . \quad . \quad . \quad . \quad . \quad . \\ X &= \frac{2i}{r_2} \cdot \frac{r_1^2 + r_2^2}{r_1^2 - r_2^2} \cdot E \sin \phi - \frac{2i}{r_1} \cdot \frac{2r_1 r_2}{r_1^2 - r_2^2} \cdot F \sin \phi, \end{aligned} \right\} \quad (15)$$

where $\cos \phi = (r_1^2 + r_2^2 - 4a^2)/(2r_1 r_2)$, and $k^2 = 1 - r_2^2/r_1^2$.

A simple and useful way of considering the force at any point P due to a circular current is to consider that it is the resultant (fig. 2) of two forces at right angles to AP and OP respectively. The component at right angles to AP equals $(4ia/r_1 r_2)E$, and the component at right angles to OP equals $2i(F - E)/(r_1 \cos \theta)$, the modulus of E and F being $(1 - r_2^2/r_1^2)^{1/2}$.

4. *Applications of the Formulæ.*

i. *Force in the plane of a circular current.*

From fig. 2 we see at once that the force at any point in the plane of the circular current and inside the circle is given by

$$\begin{aligned} Z &= \frac{2i}{r_2} E + \frac{2i}{r_1} F \\ &= \frac{2i}{a-r} E + \frac{2i}{a+r} F, \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned} \quad (16)$$

the modulus being $2(ar)^{1/2}/(a+r)$.

Again, at points in the plane of the circular current, but outside the circle, we have

$$\begin{aligned} Z &= -\frac{2i}{r-a} E + \frac{2i}{r+a} F \\ &= \frac{2i}{a-r} E + \frac{2i}{a+r} F. \end{aligned}$$

Hence, whether the point be inside or outside the circle, formula (16) gives the magnetic force. When the point lies within the circle the author at first used the formula *

$$\begin{aligned} Z &= \frac{4ai}{a^2 - r^2} E' \\ &= \frac{2i}{a-r} E' + \frac{2i}{a+r} E', \end{aligned}$$

where the modulus is r/a . By means of (9) we see that the

* 'The Electrician,' vol. xxxi. p. 212 (1893) or 'Alternating Currents,' vol. i. p. 30.

formulæ give identical results. For points outside the circle, however, we find by (16) and (9), or otherwise,

$$Z = -\frac{4\pi i}{r^2 - a^2} E'' + \frac{4i}{r} F'',$$

the modulus now being a/r . It will be seen that it is best to consider (16) as the standard formula, as it gives the force at points in the plane of the filament, both outside and inside the circle.

It is worth noticing that if r be not greater than $0.7a$, the maximum inaccuracy of the formula

$$Z = \frac{\pi i}{2(a^2 - r^2)} \{a + \sqrt{a^2 - r^2} + \sqrt{4a^2 - 2r^2}\}$$

is less than 1 in 10,000.

Let us now find the value of the magnetic force very close to the filament. In this case, as the modulus of E and F in (16) is nearly unity, we can use Legendre's formulæ (1) and (2) to find E and F . Hence

$$\begin{aligned} Z &= \frac{2i}{a-r} \left[1 + \frac{1}{2} \left(\frac{a-r}{a+r} \right)^2 \left\{ \log \frac{4(a+r)}{a-r} - \frac{1}{2} \right\} + \dots \right] \\ &\quad + \frac{2i}{a+r} \left[\log \frac{4(a+r)}{a-r} + \frac{1}{4} \left(\frac{a-r}{a+r} \right)^2 \left\{ \log \frac{4(a+r)}{a-r} - 1 \right\} + \dots \right] \\ &= \frac{2i}{a-r} + \frac{2i}{a+r} \log \frac{4(a+r)}{a-r}, \dots \dots \dots (17) \end{aligned}$$

when $(a-r)^2/(a+r)^2$ is negligibly small compared with unity. The first term on the right-hand side of this equation is very great compared with the second when r is nearly equal to a , and, in this case, we may write

$$Z = \frac{2i}{a-r}, \text{ very approximately.}$$

Similarly, when r is greater than a , we have

$$Z = -\frac{2i}{r-a} + \frac{2i}{r+a} \log \frac{4(r+a)}{r-a}, \dots \dots (18)$$

approximately.

ii. The mutual inductance of two coaxial circular currents.

Let the radii of the circles be a and b respectively, and let c be the distance between their planes. We shall calculate the flux M through the circle whose radius is b due to unit current in the circle whose radius is a . Let Z be the

magnetic force at an element $r'd\theta'dr'$ of the area of the circle whose radius is b , then

$$M = \int_0^b \int_0^{2\pi} Zr'd\theta'dr',$$

and noticing that $r^2 = r'^2 + c^2$, and $r' \cos \theta = c$, we get

$$\frac{M}{4\pi a} = \int_0^b \int_0^\pi \frac{(a - r' \cos \phi) r' d\theta' dr'}{(a^2 + r'^2 + c^2 - 2ar' \cos \phi)^{3/2}}.$$

Changing to rectangular coordinates, we have

$$\begin{aligned} \frac{M}{4\pi a} &= \int_0^b \int_{-\sqrt{b^2-y^2}}^{+\sqrt{b^2-y^2}} \frac{(a-x)dx dy}{\{(a-x)^2 + y^2 + c^2\}^{3/2}} \\ &= \int_0^b \left[\frac{1}{\{(a-x)^2 + y^2 + c^2\}^{1/2}} \right]_{-\sqrt{b^2-y^2}}^{+\sqrt{b^2-y^2}} dy. \end{aligned}$$

Put $y = b \sin \phi$ and simplify. In the second integral also, write $\phi' = \pi - \phi$. We get

$$\frac{M}{4\pi ab} = \int_0^\pi \frac{\cos \phi d\phi}{(a^2 + b^2 + c^2 - 2ab \cos \phi)^{1/2}}.$$

Now putting $\phi = \pi - 2\theta$, we have

$$\frac{M}{4\pi ab} = -\frac{2}{r_1} \int_0^{\pi/2} \frac{\cos 2\theta}{\Delta} d\theta, \quad . \quad . \quad (19)$$

and therefore, by (4)

$$M = 4\pi \sqrt{ab} \{ (2/k - k) F - (2/k) E \}, \quad . \quad . \quad (20)$$

where

$$r_1^2 = (a+b)^2 + c^2, \quad \text{and} \quad k^2 = 4ab/r_1^2.$$

Formula (20) agrees with that given by Helmholtz * for the analogous problem of the vortex ring.

If we transform (20) by the Landen-Legendre formulæ (9), we get

$$M = \frac{8\pi \sqrt{ab}}{\sqrt{k'}} (F' - E'), \quad . \quad . \quad . \quad (21)$$

where

$$k' = (r_1 - r_2)/(r_1 + r_2), \quad r_1^2 = (a+b)^2 + c^2, \quad \text{and} \quad r_2^2 = (a-b)^2 + c^2.$$

This is Maxwell's formula.

iii. *Approximate formula for the self-inductance of a ring.*

We shall suppose that the radius of the ring is a , that the radius of the cross section is r , and that the current density at all points of the cross section is the same. We suppose

Crelle's *Journal*, vol. lv. p. 25 (1858), or *Ges. Abh.* t. 1, p. 101.

also that r/a is small compared with unity. From the construction given in fig. 2 we see that the lines of force near a circular filament are very approximately circles with their centres on the circular axis of the filament. The field is therefore similar to that round a straight cylindrical conducting tube, in which the current flow is parallel to the axis. Now it is well known* that the magnetic force outside a straight cylindrical conducting tube can be calculated as if the current were concentrated along the axis and that the magnetic force inside the tube is zero. Let us suppose that the circular conductor is built up of an infinite number of infinitely thin ring-tubes, every point on any tube having the same minimum distance from the circular axis, and make the assumptions that the current in each ring-tube produces no magnetic field inside it, and that the field outside can be calculated as if the current were concentrated along the axis. From the corresponding electrostatic problem of a charged conducting ring we see, by the method of duality, that the first assumption could be made rigorously true by assuming that the density of the current distribution over the cross section of the infinitely thin ring-tube was proportional to the surface density of the electrostatic charge. In this case, however, the magnetic force at points outside the tube is not the same as if the current were concentrated along the circular axis, and therefore the gain in rigour by making the first assumption accurate is problematical. When r is very small compared with a the surface density is approximately constant over the ring†. In this case, therefore, when the current density is uniform over the circular tube, the magnetic force produced inside is negligible, and therefore our assumptions are legitimate in obtaining an approximate solution.

Let Φ_1 be the flux linked with the whole current when unit current is flowing in the ring, and Φ_2 the flux linked with part of the current only. By (21) we have

$$\Phi_1 = 8\pi a(F - E),$$

where the modulus is $(a-r)/a$.

By (17), the force Z at a point in the plane of the circular axis and at a distance $a-\xi$ from its centre is given by

$$Z = \frac{2i'}{\xi} + \frac{i'}{a} \log \frac{8a}{\xi},$$

approximately, where $i' = \xi^2/r^2$, since the whole current in the ring is unity.

* A. Russell, 'Alternating Currents,' vol. i. p. 32.

† F. W. Dyson, Phil. Trans. vol. clxxxiv. A, p. 67 (1893).

Hence, on our assumptions

$$\Phi_2 = 2\pi \int_0^r (a - \xi) Z(\xi^2/r^2) d\xi.$$

This integral can be easily found. Since, however, we are supposing that r and ξ are very small, we may write $Z = 2i'/\xi$ simply, and thus $\Phi_2 = \pi a$, approximately.

If L , therefore, denote the self-inductance, we have

$$\begin{aligned} L &= \Phi_1 + \Phi_2, \\ &= 8\pi a(F - E) + \pi a. \end{aligned}$$

Substituting for F and E their approximate values $(1/2) \log (8a/r)$ and 1 , we get

$$L = 4\pi a \{ \log (8a/r) - 1.75 \}, \quad . \quad . \quad (22)$$

approximately.

The formula given by Lord Rayleigh* and Sir W. D. Niven is

$$L = 4\pi a \{ \log (8a/r) - 1.75 \} + (\pi r^2/2a) \{ \log (8a/r) + 1/3 \},$$

and by Max Wien†

$$L = 4\pi a \{ \log (8a/r) - 1.75 \} + (\pi r^2/2a) \{ \log (8a/r) - 0.0664 \}.$$

When $r/a = 1/100$ we should expect (22) to give L with considerable accuracy. We see that in this case, the value of L found by it differs from that given by either Rayleigh's or Wien's formula by less than the five-hundredth part of one per cent.

If we suppose that the conductivity of the ring is infinite so that the current flows entirely on the surface, then our method of proof shows that

$$L = \Phi_1 = 8\pi a(F - E) = 4\pi a \{ \log (8a/r) - 2 \},$$

approximately. When $r/a = 1/100$, the value of L given by this formula is more than five per cent. lower than that given by any of the preceding formulæ. It will be seen, therefore, that even with a thin wire the manner in which the current is distributed over the cross section has an appreciable effect on the inductance. It must also be noticed that when r/a is small, $dL/dr (= -4\pi a/r)$ is large, and hence L varies rapidly with r and a small error made in measuring r may lead to a large error in the calculated value of L .

It is interesting to compare the formula for the self-inductance of a ring with the formula for its electrostatic

* 'Scientific Papers,' vol. ii. p. 15.

† Wiedemann's *Annalen*, liii. p. 934 (1894).

capacity. Formulæ for the capacity K have been found by W. M. Hicks* and F. W. Dyson†.

When the radius of the ring is large compared with the radius of the circular section, they both agree in giving the formula as

$$K = \pi a / \{ \log (8a/r) \}.$$

Hence, if the resistivity of the ring be zero, we have

$$KL = (2\pi a)^2 \{ 1 - 2/\log (8a/r) \}. \quad . \quad . \quad (25)$$

When $100r = a$, this equation becomes

$$KL = 0.7(2\pi a)^2, \text{ approximately.}$$

It will be seen, therefore, that we can only use the equation $KL = (2\pi a)^2$ when r/a is an exceedingly small fraction. The corresponding equation $KL = l^2$, for two long parallel cylinders of infinite conductivity is strictly true (see A. Russell, 'Alternating Currents,' vol. i. p. 141).

5. Applications to Hydrodynamics.

Lord Kelvin has shown ('Papers on Electrostatics and Magnetism,' p. 444) that the problem of finding the velocity at any point of an incompressible fluid near a vortex filament is, from the point of view of mathematical analysis, the same as the problem of finding the magnetic force near a current filament. The strength m of the vortex corresponds to the strength i of the current and the fluid velocity to the magnetic force. If r be the radius of the cross section (supposed circular) of a vortex ring and ω the angular velocity of points on the ring round the circular axis, the strength ‡ m of the vortex is generally defined to be $\pi r^2 \omega$. In the electrical problem the magnetic force tangential to the ring is $2i/r$, and in the hydrodynamical problem the corresponding velocity is ωr . Hence to convert the formulæ for the magnetic force at a point into the corresponding formulæ for the fluid velocity at that point we must multiply by $\omega r^2/2i$, that is, by $m/2\pi i$. For instance, we see from the construction given in fig. 2 that the velocity at a point P in the fluid near a vortex ring can be found as follows:—The component velocity at right angles to AP (fig. 2) equals $(2ma/\pi r_1 r_2)E$, and the component at right angles to OP equals

$$m(F - E)/(\pi r_1 \cos \theta),$$

the modulus of E and F being $(1 - r_2^2/r_1^2)^{1/2}$. At points on

* Phil. Trans. vol. clxxii. p. 643 (1881).

† F. W. Dyson, p. 68, *l. c. ante*.

‡ It is sometimes defined as $2\pi r^2 \omega$.

the surface of the vortex ring we have $r_2=r$, and we may write $r_1=2a$, $\cos \theta=1$, $E=1$, and $F=\log (8a/r)$. Hence points on the surface of the ring have a velocity $m/\pi r$, that is, ωr about the circular axis and a linear velocity w upwards, where

$$w = \frac{m}{2\pi a} \left(\log \frac{8a}{r} - 1 \right). \quad . \quad . \quad . \quad . \quad . \quad (a)$$

We see, therefore, that the vortex ring moves bodily upwards with the velocity w . We see also at once that the velocity of the fluid at the centre of the vortex ring is $2\pi i/a \times m/2\pi i$, that is, m/a .

Again, if the components of the velocity of an element of the volume dv of the fluid be u and w , and if ρ be the density of the fluid, we have

$$\begin{aligned} T_1 &= \frac{1}{2} \rho \Sigma (u^2 + w^2) dv = \frac{\rho m^2}{8\pi^2 i^2} \Sigma (X^2 + Z^2) dv \\ &= \frac{\rho m^2}{\pi i^2} \Sigma \left(\frac{R^2}{8\pi} \right) dv, \end{aligned}$$

where T_1 is the kinetic energy of the moving fluid and R =the resultant magnetic force at dv in the electrical problem. But by Kelvin's formula

$$\Sigma (R^2/8\pi) dv = Li^2/2,$$

and thus

$$T_1 = \frac{\rho m^2}{2\pi} L,$$

where $Li^2/2$ is the energy due to the linkages of the lines of force with the whole of the electric current.

Hence, by the preceding section

$$\begin{aligned} T_1 &= \frac{\rho m^2}{2\pi} 4\pi a \left(\log \frac{8a}{r} - 2 \right), \\ &= 2\rho m^2 a \left(\log \frac{8a}{r} - 2 \right). \end{aligned}$$

Let us now suppose that the core of the vortex-ring is solid so that the angular velocity of every point in it round the circular axis is constant. This corresponds to assuming that the current density over the cross section of the ring in the electrical problem is constant. Let us also suppose that the density of the core is the same as that of the fluid. The kinetic energy T_2 of the core is given by

$$\begin{aligned} T_2 &= \frac{1}{2} \rho \pi r^2 2\pi a \{ (r^2/2) \omega^2 + w^2 \}, \\ &= \frac{1}{2} \rho m^2 a + \text{small terms.} \end{aligned}$$

Hence the total kinetic energy of the system $T_1 + T_2$ equals

$$2\rho m^2 a \left(\log \frac{8a}{r} - 1.75 \right).$$

This is the formula given by Lamb*. By means of (22) we could have written it down at once. The formula, however, Lamb gives for the velocity of translation of a vortex ring is

$$w = \frac{m}{2\pi a} \left(\log \frac{8a}{r} - \frac{1}{4} \right). \quad \dots \quad (b) \dagger$$

This formula is due to Lord Kelvin‡. In Basset's 'Hydrodynamics' § both formulæ are given. The author thinks that when r/a is small formula (a) must be the more accurate. If this be not the case, then the electrical analogy must break down at points contiguous to the filament. Chree ||, in his paper 'Vortex Rings in a Compressible Fluid,' gives formula (a). He also points out reasons for the slight divergences in the formulæ for the motion of vortex rings given by various physicists. In another important paper ¶ by the same author, the similarity of the equations of vorticity to those of electrodynamics is shown very clearly. When the compressibility of the fluid is considered, the hydrodynamical problem is much the more difficult.

6. The magnetic force due to a cylindrical current sheet.

We shall suppose that there are n bands of current per unit length. We shall first find the magnetic force at a point P in a plane through the base of the cylinder and at a distance r from the axis. Let the length of the axis of the cylinder be l and its radius a . Then, by (10),

$$Z = 2ia \int_0^\pi \int_0^l \frac{a - r \cos \phi}{(z^2 + a^2 + r^2 - 2ar \cos \phi)^{3/2}} \cdot \frac{ndz}{l} \cdot d\phi,$$

where i is the current in a filament.

$$\begin{aligned} \text{Hence} \quad Z &= \frac{2ni}{l} \int_0^\pi \frac{a(a - r \cos \phi)}{\Delta^2} \left[\frac{z}{(z^2 + \Delta^2)^{1/2}} \right]_0^l d\phi \\ &= 2ni \int_0^\pi \frac{a(a - r \cos \phi)}{\Delta^2 (l^2 + \Delta^2)^{1/2}} d\phi, \end{aligned}$$

where $\Delta^2 = a^2 + r^2 - 2ar \cos \phi$.

* 'Hydrodynamics,' p. 227, 3rd edition.

† 'Hydrodynamics,' p. 227. [Note that $\kappa = 2m$.]

‡ Phil. Mag. [4] vol. xxxiii. Supp. p. 511 (1867).

§ Vol. ii., (5) p. 62, and (86) p. 87.

|| Proc. Edin. Math. Soc. vol. vi. p. 59 (1887).

¶ Proc. Edin. Math. Soc. vol. viii. p. 43 (1889).

The magnetic force, therefore, at a point distant $h+z$ from one end and $h-z$ from the other end of a cylindrical current sheet is given by

$$Z = \frac{N_1 i}{h_1} \int_0^\pi \frac{a(a-r \cos \phi)}{\Delta^2} \left[\frac{h_1+z}{\{(h_1+z)^2 + \Delta^2\}^{1/2}} + \frac{h_1-z}{\{(h_1-z)^2 + \Delta^2\}^{1/2}} \right] d\phi, \quad (26)$$

where $2h_1$ = the axial length of the cylinder, and $N_1 = 2h_1 n$.

Now noticing that

$$\frac{h_1+z}{\{(h_1+z)^2 + \Delta^2\}^{1/2}} = \frac{h_1+z}{d} \left[1 + \frac{1}{2} \cdot \frac{2ar \cos \phi}{d^2} + \frac{1}{2 \cdot 4} \cdot \left(\frac{2ar \cos \phi}{d^2} \right)^2 + \dots \right],$$

where $d^2 = (h_1+z)^2 + a^2 + r^2$, and that

$$\begin{aligned} \int_0^\pi \frac{a(a-r \cos \phi)}{\Delta^2} d\phi &= \pi, \\ \int_0^\pi (2ar \cos \phi) \frac{a(a-r \cos \phi)}{\Delta^2} d\phi &= \pi r^2, \\ \int_0^\pi (2ar \cos \phi)^2 \frac{a(a-r \cos \phi)}{\Delta^2} d\phi &= \pi r^2(r^2 + 2a^2), \\ &\dots \dots \dots \end{aligned}$$

we see that,

$$\begin{aligned} Z = \frac{\pi N_1 i}{h_1} \left[\frac{h_1+z}{d} \left\{ 1 + \frac{1}{2} \cdot \frac{r^2}{d^2} + \frac{1}{2 \cdot 4} \cdot \frac{r^2(r^2 + 2a^2)}{d^4} \right. \right. \\ + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{r^4(r^2 + 3a^2)}{d^6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{r^4(r^4 + 4r^2a^2 + 6a^4)}{d^8} \\ + \frac{1 \dots 9}{2 \dots 10} \cdot \frac{r^6(r^4 + 5r^2a^2 + 10a^4)}{d^{10}} + \dots \left. \right\} \\ + \frac{h_1-z}{d'} \left\{ 1 + \frac{1}{2} \cdot \frac{r^2}{d'^2} + \dots \right\} \right], \quad (27) \end{aligned}$$

where $d'^2 = (h_1-z)^2 + a^2 + r^2$.

The terms in the bracket multiplying $(h_1-z)/d'$ can be found at once from the corresponding multiplier for $(h_1+z)/d$, by writing d' for d in the terms in the latter bracket. As a rule the series converges rapidly.

Particular cases of the formula are interesting. For instance, when h_1 is infinite we get

$$Z = 4\pi \left(\frac{N_1}{2h_1} \right) i,$$

and when r is zero,

$$Z = \frac{\pi N_1 i}{h_1} \left[\frac{h_1 + z}{d} + \frac{h_1 - z}{d'} \right].$$

Both of these results are well known. It is not difficult to show that the latter result is true for a helical current i , the radius of the helix being a , the axial length $2h_1$, and the number of turns N_1 .

When z is zero, that is at points on the plane bisecting the axis of the cylindrical current sheet at right angles, we have

$$Z = \frac{2\pi N_1 i}{\{h_1^2 + a^2 + r^2\}^{1/2}} \left[1 + \frac{1}{2} \cdot \frac{r^2}{h_1^2 + a^2 + r^2} + \frac{3}{8} \cdot \frac{r^2(r^2 + 2a^2)}{(h_1^2 + a^2 + r^2)^2} + \frac{5}{16} \cdot \frac{r^4(r^2 + 3a^2)}{(h_1^2 + a^2 + r^2)^3} + \dots \right]. \quad (28)$$

When r^2 is small compared with $h_1^2 + a^2 + r^2$, the series converges with great rapidity. This result, however, is not true for a helical current.

7. *Formulae for the mutual inductance between two coaxial cylindrical-current sheets.*

Let us suppose that there is a cylinder of radius b and axial length $2h_2$, coaxial and concentric with the cylinder considered above. We shall suppose that b is less than a and that we have N_2 filaments equally spaced on the inner cylinder so that $N_2/(2h_2)$ gives the turns per unit length. We have now to calculate the linkages M of the flux, due to unit current in the outer cylinder, with the filaments of the inner cylinder.

If dM denote the linkages due to the flux through the element of area $r d\theta dr$ from $z = h_2$ to $z = -h_2$, we have

$$dM = \int_{-h_2}^{+h_2} Z r d\theta dr N_2 (dz/2h_2),$$

and hence, by (26),

$$dM = \frac{N_1 N_2}{h_1 h_2} \int_0^b \int_0^\pi \frac{a r d\theta (a - r \cos \phi)}{\Delta^2} \left[\{ (h_1 + h_2)^2 + \Delta^2 \} - \{ (h_1 - h_2)^2 + \Delta^2 \}^{1/2} \right] d\phi dr.$$

Therefore

$$M = \frac{2\pi N_1 N_2}{h_1 h_2} \int_0^b \int_0^\pi \frac{ar(a-r \cos \phi)}{\Delta^2} [\{(h_1+h_2)^2 + \Delta^2\}^{1/2} - \{(h_1-h_2)^2 + \Delta^2\}^{1/2}] d\phi dr. \quad (29)$$

It can be proved mathematically that this formula is equivalent to the following:

$$M = 4\pi abc^2 \frac{N_1 N_2}{h_1 h_2} \int_0^{\pi/2} \frac{\sin^2 \theta \cos^2 \theta}{1 - c^2 \sin^2 \theta} [R_1(1 - k_1^2 \sin^2 \theta)^{1/2} - R_2(1 - k_2^2 \sin^2 \theta)^{1/2}] d\theta, \quad (30)$$

where $c^2 = 4ab/(a+b)^2$; $R_1^2 = (a+b)^2 + (h_1+h_2)^2$; $R_2^2 = (a+b)^2 + (h_1-h_2)^2$; $k_1^2 = 4ab/R_1^2$ and $k_2^2 = 4ab/R_2^2$. As we give a proof of the identity of (29) and (30) in the next section, it is unnecessary to give a mathematical proof here.

It is not difficult to find a series for M by expanding the radicals in (29) by the binomial theorem and then integrating the terms. The author has found several series formulæ for M . The following two approximate formulæ deduced from them are simple, and when $h_1^2 + h_2^2 + a^2$ is greater than $b^2 + 2ab$ they give accurate results. The first formula given (31) is much the more accurate, especially when h_1 and h_2 are not very different.

$$M = \frac{2\pi N_1 N_2 (\pi b^2)}{d} \left\{ \frac{2d}{(d^2 + 2h_1 h_2)^{1/2} + (d^2 - 2h_1 h_2)^{1/2}} + \frac{3a^2 b^2}{8d^4} - \frac{5a^2 b^4}{16d^6} \right\}, \quad (31)$$

and

$$M = \frac{2\pi N_1 N_2 (\pi b^2)}{d} \left\{ 1 + \frac{3a^2 b^2 + 4h_1^2 h_2^2}{8d^4} \right\}, \quad (32)$$

where $d^2 = h_1^2 + h_2^2 + a^2$.

Let us suppose, for instance, that $N_1 = 300$, $N_2 = 200$, $a = 5$, $b = 4$, $h_1 = 15$, and $h_2 = 2.5$, the lengths being measured in centimetres. Formula (31) makes $M = 0.0011995$ henry, and (32) makes $M = 0.0011992$. By the complete formula given in the next section $M = 0.0011995$. It will be seen that the accuracy of the approximate formulæ in this case is quite satisfactory.

From considerations in connexion with the magnetic potential of a cylindrical current sheet, G. F. C. Searle* has

* G. F. C. Searle and J. R. Airey, 'The Electrician,' Dec. 8th, 1905.

deduced the formula

$$M = \frac{2\pi^2 N_1 N_2 b^2}{d_1} \left(1 - \frac{a^2}{2d_1^4} \cdot \frac{4h_2^2 - 3b^2}{4} - \frac{a^2(4h_1^2 - 3a^2)}{8d_1^8} \cdot \frac{8h_2^4 - 20h_2^2 b^2 + 5b^4}{8} - \dots \right),$$

where $d_1^2 = h_1^2 + a^2$. The value of M found by this formula in the case considered is 0.0011999 henry.

We shall find a formula in the next section by means of which M can always be evaluated to any specified degree of accuracy.

8. Formulæ for the mutual inductance between a helix and a coaxial cylindrical current sheet.

We shall first find Jones's formula* for the mutual inductance between a circle and a coaxial helix.

From (19) we see that the mutual inductance M between two coaxial circular currents is given by

$$M = -8\pi ab \int_0^{\pi/2} \frac{\cos 2\theta \, d\theta}{\{(a+b)^2 + z^2 - 4ab \sin^2 \theta\}^{1/2}},$$

where z is the distance between their planes, and a and b are their radii. Hence the mutual inductance dM between a sector of the circle whose radius is b and the circle whose radius is a is given by

$$dM = -4ab \, d\psi \int_0^{\pi/2} \frac{\cos 2\theta \, d\theta}{\{(a+b)^2 + z^2 - 4ab \sin^2 \theta\}^{1/2}},$$

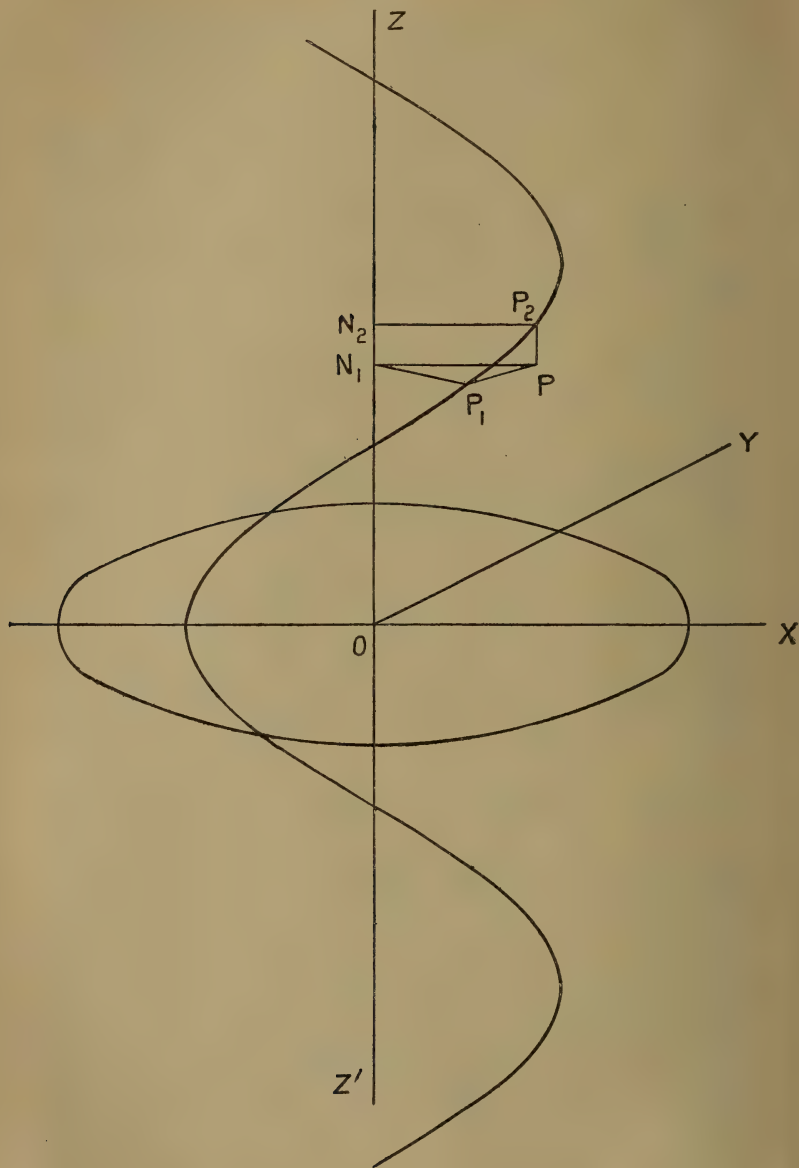
where $d\psi$ is the angle of the sector.

Let p be the pitch of the helix, and let the coordinates of points on it be given by $x = b \cos \psi$, $y = b \sin \psi$, and $z = p\psi$.

We shall first find the mutual inductance between this helix and a coaxial circle in the plane XY (fig. 3). Let us find the linkages of the helix with the flux due to unit current in the circular filament. Consider two contiguous points P_1 and P_2 on the helix and draw P_1N_1 and P_2N_2 perpendicular to OZ . Also draw P_2P parallel to OZ to meet the plane passing through P_1N_1 and perpendicular to OZ at P . Join PN_1 and PP_1 . Now every line of force due to the circular current must lie in a plane which passes through OZ . Hence, since P_1P_2 are infinitely close together all the

* Proc. Roy. Soc. vol. lxii. p. 247 (1897).

Fig. 3.



Helix and Circle. ZOZ' is the axis of the helix and XOY is the plane of the circle. The axial length of the helix is $2h_1$, its radius b and the number of turns N_1 . The radius of the circle is a .

lines of force linked with the circuit $P_1P_2N_2N_1$ pass through the sector N_1P_1P . Hence if the helix have an integral number of turns and its ends be joined by a conductor lying in a plane containing the axis of the helix, we get

$$M = -4ab \int_0^{\pi/2} \int_{\psi_1}^{\psi_2} \frac{\cos 2\theta \, d\psi \, d\theta}{\{(a+b)^2 - 4ab \sin^2 \theta + p^2 \psi^2\}^{1/2}},$$

where ψ_1 and ψ_2 are the values of ψ at the ends of the helix.

Hence, integrating, we get

$$\frac{pM}{4ab} = \int_0^{\pi/2} \cos 2\theta [\log (p\psi_1 + \sqrt{p^2\psi_1^2 + \Delta^2}) - \log (p\psi_2 + \sqrt{p^2\psi_2^2 + \Delta^2})] d\theta,$$

where $\Delta^2 = (a+b)^2 - 4ab \sin^2 \theta$.

Integrating by parts and noticing that $\sin 2\theta$ vanishes at both limits, we find that

$$\begin{aligned} \frac{pM}{4ab} &= 4ab \int_0^{\pi/2} \frac{\sin^2 \theta \cos^2 \theta \, d\theta}{(p^2\psi_1^2 + \Delta^2)^{1/2} (p\psi_1 + \sqrt{p^2\psi_1^2 + \Delta^2})} - \dots \\ &= 4ab \int_0^{\pi/2} \frac{\sin^2 \theta \cos^2 \theta}{\Delta^2 (p^2\psi_1^2 + \Delta^2)^{1/2}} \left(\sqrt{p^2\psi_1^2 + \Delta^2} - p\psi_1 \right) d\theta - \dots \\ &= 4ab \int_0^{\pi/2} \frac{\sin^2 \theta \cos^2 \theta}{\Delta^2} \left\{ \frac{p\psi_2}{\sqrt{p^2\psi_2^2 + \Delta^2}} - \frac{p\psi_1}{\sqrt{p^2\psi_1^2 + \Delta^2}} \right\} d\theta. \end{aligned}$$

Therefore

$$\frac{M}{4ab} = c^2 \int_0^{\pi/2} \frac{\sin^2 \theta \cos^2 \theta}{1 - c^2 \sin^2 \theta} \left\{ \frac{\psi_2}{R_2 \Delta_2} - \frac{\psi_1}{R_1 \Delta_1} \right\} d\theta, \quad (33)$$

where $c^2 = 4ab/(a+b)^2$; $R_1^2 = (a+b)^2 + p^2\psi_1^2$;

$R_2^2 = (a+b)^2 + p^2\psi_2^2$; $k_1^2 = 4ab/R_1^2$; $k_2^2 = 4ab/R_2^2$;

$\Delta_1^2 = 1 - k_1^2 \sin^2 \theta$, and $\Delta_2^2 = 1 - k_2^2 \sin^2 \theta$.

If $-\psi_1 = \psi_2 = \psi$, $R_1 = R_2 = R$, and $k^2 = 4ab/R^2$, we have

$$\frac{M}{4ab} = \frac{c^2 \psi}{R} \int_0^{\pi/2} \frac{\sin^2 \theta \cos^2 \theta \, d\theta}{(1 - c^2 \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}}. \quad (34)$$

Noticing that

$$c^2 \sin^2 \theta \cos^2 \theta = -\frac{1 - c^2}{c^2} + (1 - c^2 \sin^2 \theta) \left(\sin^2 \theta + \frac{1 - c^2}{c^2} \right),$$

and using (3), we get

$$M = 2\psi(a+b)ck \left[\frac{1}{k^2} (F-E) + \frac{1-c^2}{c^2} (F-\Pi) \right], \quad (35)$$

where

$$\Pi = \int_0^{\pi/2} \frac{d\theta}{(1-c^2 \sin^2 \theta)(1-k^2 \sin^2 \theta)^{1/2}},$$

and is consequently the complete elliptic integral of the third kind in which $n = -c^2$. This is the final form in which Jones gave his formula.

If we put $\psi = \pi$, and $p = 0$, so that $c = k$, then, by (6), we get

$$M = 4\pi \sqrt{ab} \left\{ (2/k - k) F - (2/k) E \right\},$$

which is formula (20).

M may be found from (35) by a formula, due to Legendre, given on p. 138 of Vol. I. of his *Traité des Fonctions Elliptiques*, and by the tables given in Vol. II.

Legendre's formula may be written in the form

$$F - \Pi = \frac{c}{k'^2 \sin \alpha \cos \alpha} [E \cdot F(k', \alpha) + F \cdot E(k', \alpha) - F \cdot F(k', \alpha) - \pi/2]. \quad (36)^*$$

In this formula $k'^2 = 1 - k^2$, and $F(k', \alpha)$, $E(k', \alpha)$ represent elliptic integrals of modulus k' and amplitude α , where $\sin \alpha = (1 - c^2)^{1/2}/k'$.

We shall now find the mutual inductance between a helix and a coaxial concentric cylindrical current sheet. Let N_1 be the number of turns of the helix, $2h_1$ its axial length, and a its radius. If p be the pitch of the helix, $h_1 = pN_1\pi$. Let also N_2 , $2h_2$, and b be the number of turns, axial length and radius of the cylindrical sheet. Let us consider a filament of the cylinder at a height z above the median plane. Then its distances from the ends of the axis of the helix will be $h_1 + z$ and $h_1 - z$ respectively. Hence if dM be the linkages of the flux due to unit current in the helix embraced by the circular filament $N_2(dz/2h_2)$, we get by (33)

$$dM = \frac{4abc^2}{p} \int_0^{\pi/2} \frac{\sin^2 \theta \cos^2 \theta}{1 - c^2 \sin^2 \theta} \left[\frac{h_1 + z}{\{(h_1 + z)^2 + \Delta^2\}^{1/2}} + \frac{h_1 - z}{\{(h_1 - z)^2 + \Delta^2\}^{1/2}} \right] d\theta \frac{N_2 dz}{2h_2},$$

* See also Cayley's 'Elliptic Functions,' Chapter V. p. 139, 2nd edition.

and therefore, integrating from $-h_2$ to $+h_2$ we get

$$M = 4\pi abc^2 \frac{N_1 N_2}{h_1 h_2} \int_0^{\pi/2} \frac{\sin^2 \theta \cos^2 \theta}{1 - c^2 \sin^2 \theta} [R_1 (1 - k_1^2 \sin^2 \theta)^{1/2} - R_2 (1 - k_2^2 \sin^2 \theta)^{1/2}] d\theta. \quad (37)$$

Now

$$\begin{aligned} & c^2 \sin^2 \theta \cos^2 \theta \sqrt{1 - k_1^2 \sin^2 \theta} \\ &= \frac{(c^2 - k_1^2) \sin^2 \theta \cos^2 \theta + k_1^2 \sin^2 \theta \cos^2 \theta (1 - c^2 \sin^2 \theta)}{\sqrt{1 - k_1^2 \sin^2 \theta}}, \end{aligned}$$

and thus, by (5)

$$\begin{aligned} M &= 4\pi ab \frac{N_1 N_2}{h_1 h_2} \left[R_1 \left(1 - \frac{k_1^2}{c^2} \right) \left\{ \frac{1 - c^2}{c^2} (F_1 - \Pi_1) + \frac{1}{k_1^2} (F_1 - E_1) \right\} \right. \\ &\quad \left. + R_1 k_1^2 \left\{ \frac{2 - k_1^2}{3k_1^2} E_1 - \frac{2 - 2k_1^2}{3k_1^2} F_1 \right\} \right. \\ &\quad \left. - R_2 \left(1 - \frac{k_2^2}{c^2} \right) \left\{ \frac{1 - c^2}{c^2} (F_2 - \Pi_2) + \frac{1}{k_2^2} (F_2 - E_2) \right\} \right. \\ &\quad \left. - R_2 k_2^2 \left\{ \frac{2 - k_2^2}{3k_2^2} E_2 - \frac{2 - 2k_2^2}{3k_2^2} F_2 \right\} \right] \\ &= 4\pi ab \frac{N_1 N_2}{h_1 h_2} \left[R_1 \left(1 - \frac{k_1^2}{c^2} \right) \left(\frac{1 - c^2}{c^2} \right) (F_1 - \Pi_1) \right. \\ &\quad \left. + R_1 \left(\frac{1}{k_1^2} - \frac{1}{c^2} + \frac{k_1^2}{3} - \frac{2}{3} \right) (F_1 - E_1) + R_1 \frac{k_1^2}{3} F_1 \right. \\ &\quad \left. - R_2 \left(1 - \frac{k_2^2}{c^2} \right) \left(\frac{1 - c^2}{c^2} \right) (F_2 - \Pi_2) \right. \\ &\quad \left. - R_2 \left(\frac{1}{k_2^2} - \frac{1}{c^2} + \frac{k_2^2}{3} - \frac{2}{3} \right) (F_2 - E_2) - R_2 \frac{k_2^2}{3} F_2 \right] \quad (38) \end{aligned}$$

By the aid of (36), therefore, and the tables given in Legendre's treatise, M may be calculated by this formula to any desired accuracy. If a four-figure accuracy suffice, the tables given on pp. 68-75 of Dale's 'Mathematical Tables' will be found ample.

It is to be noticed that provided a , b , h_1 and h_2 remain the same, the formula for the mutual inductance between a helical current and a cylindrical current sheet may be written in the form mN_1N_2 where m is a constant. By considering the

case, therefore, of N_1 infinitely great, that is, when the helix is practically identical with a cylindrical current sheet of the same axial length and diameter, we see that the formula for two cylindrical current sheets must also be mN_1N_2 . We thus see that the formulæ (29) and (38) must be identical.

It has to be remembered that the above formula needs to be modified when both the coils are helical. If the pitch of one of the helices be a very small quantity the correcting factor will be small. It is advisable, however, when actually measuring the mutual inductance to investigate whether rotating* one of the helices round the axis through various angles alters the mutual inductance.

Even when Legendre's tables are accessible, the evaluation of (38) is laborious. We shall, therefore, give a series formula for M which is convenient in many cases. By the binomial theorem, we always have

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin^2 \theta \cos^2 \theta}{1 - c^2 \sin^2 \theta} (1 - k_1^2 \sin^2 \theta)^{1/2} d\theta \\ = P_1 - \frac{1}{2} P_2 k_1^2 - \frac{1 \cdot 1}{2 \cdot 4} P_3 k_1^4 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} P_4 k_1^6 - \dots, \end{aligned}$$

where

$$\begin{aligned} P_n &= \int_0^{\pi/2} \frac{\sin^{2n} \theta \cos^2 \theta}{1 - c^2 \sin^2 \theta} d\theta \\ &= \frac{P_{n-1}}{c^2} - \frac{1}{c^2} \int_0^{\pi/2} \sin^{2n-2} \theta \cos^2 \theta d\theta \\ &= \frac{P_{n-1}}{c^2} - \frac{1}{2n} \cdot \frac{1 \cdot 3 \cdot 5 \dots 2n-3}{2 \cdot 4 \cdot 6 \dots 2n-2} \cdot \frac{\pi}{2} \cdot \frac{1}{c^2} \dots \quad (39) \end{aligned}$$

Now it is easy to show that

$$P_1 = \frac{\pi}{4(1 + \sqrt{1 - c^2})^2} = \frac{\pi}{16} \cdot \frac{(a+b)}{a^2},$$

and if we denote P_n/P_1 by q_n we get

$$q_n = \frac{(a+b)^2}{4ab} q_{n-1} - \frac{1}{n} \cdot \frac{1 \cdot 3 \cdot 5 \dots 2n-3}{2 \cdot 4 \cdot 6 \dots 2n-2} \cdot \frac{a}{b} \dots \quad (40)$$

Thus the successive values of q_n can be readily computed.

Since P_n is always less than P_{n-1} , and k_1 is always less than unity, it follows that the series given above is always convergent. The formula for M can, therefore, be written

* Lord Rayleigh, British Association Report (1899), p. 241.

in the form

$$M = \pi^2 b^2 \frac{N_1 N_2}{h_1 h_2} \left[R_1 \left\{ 1 - \frac{1}{2} q_2 k_1^2 - \frac{1 \cdot 1}{2 \cdot 4} q_3 k_1^4 - \dots \right\} \right. \\ \left. - R_2 \left\{ 1 - \frac{1}{2} q_2 k_2^2 - \frac{1 \cdot 1}{2 \cdot 4} q_3 k_2^4 - \dots \right\} \right], \quad (41)$$

where $R_1^2 = (a+b)^2 + (h_1+h_2)^2$; $k_1^2 = 4ab/R_1^2$,

and $R_2^2 = (a+b)^2 + (h_1-h_2)^2$; $k_2^2 = 4ab/R_2^2$.

This formula gives a complete practical solution of the problem.

Let us take the case of the two cylindrical coils for which $a=5$, $b=4$, $h_1=15$, $h_2=2.5$, $N_1=300$, and $N_2=200$, mentioned in the preceding section. By (39) we easily obtain

$$q_2=0.7, \quad q_3=0.5525, \quad q_4=0.4618, \quad q_5=0.3992,$$

$$q_6=0.3527, \quad q_7=0.3170, \quad \&c.$$

$$\text{Also} \quad R_1^2 = (a+b)^2 + (h_1+h_2)^2 = 387.25; \quad R_1 = 19.6787; \\ R_2^2 = (a+b)^2 + (h_1-h_2)^2 = 237.25; \quad R_2 = 15.4029; \\ k_1^2 = 80/387.25; \quad k_2^2 = 80/237.25.$$

Thus, substituting in (41), we have

$$M = \frac{(4\pi)^2 6 \cdot 10^4}{37.5} [19.6787(1-0.07230-0.00295 \\ -0.00026-0.00003-\dots) \\ -15.4029(1-0.11802-0.00785 \\ -0.00100-0.00020 \\ -0.00004-0.00001-\dots)] 10^{-9} \\ = 0.0011995 \text{ henry,}$$

which agrees with the answer found from (29).

When $h_1=h_2=h$,

$$k_1^2 = 4ab/\{(a+b)^2 + 4h^2\} \quad \text{and} \quad k_2^2 = 4ab/(a+b)^2 = c^2.$$

In this case the series of terms with the suffix 2 in (41) may converge slowly. It is better, therefore, to express the second half of the formula on the right-hand side of (41) in terms of elliptic integrals. The formula * now becomes:

$$M = \pi^2 b^2 \frac{N_1 N_2}{h^2} \left[R_1 \left\{ 1 - \frac{1}{2} q_2 k_1^2 - \frac{1 \cdot 1}{2 \cdot 4} q_3 k_1^4 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} q_4 k_1^6 - \dots \right\} \right] \\ + \frac{8\pi ab}{3(a+b)} \cdot \frac{N_1 N_2}{h^2} [(a^2+b^2)(F-E) - 2abF], \quad (42)$$

where the modulus of the elliptic functions is $2(ab)^{1/2}/(a+b)$.

* For another formula in terms of incomplete elliptic integrals, see J. G. Coffin, p. 124, *l. c. ante*.

For the calculation of the mutual inductance standards required for calibrating the galvanometers used in iron-testing, this formula is suitable. In order that the coils be as compact as possible, it is necessary to make $h=a$ and b large, but even in this case the series can be easily summed.

Let us now consider the case when the cylinder has the same radius as the helix. We may suppose, for instance, that both coils are wound on the same insulating cylinder. The formula (37) now becomes

$$M = 4\pi a^2 \frac{N_1 N_2}{h_1 h_2} \int_0^{\pi/2} \sin^2 \theta [R_1(1 - k_1^2 \sin^2 \theta)^{1/2} - R_2(1 - k_2^2 \sin^2 \theta)^{1/2}] d\theta.$$

Hence, from (8),

$$M = 4\pi a^2 \frac{N_1 N_2}{h_1 h_2} \left[R_1 \frac{2k_1^2 - 1}{3k_1^2} E_1 + R_1 \frac{1 - k_1^2}{3k_1^2} F_1 - R_2 \frac{2k_2^2 - 1}{3k_2^2} E_2 - R_2 \frac{1 - k_2^2}{3k_2^2} F_2 \right], \quad (43)$$

where

$$R_1^2 = 4a^2 + (h_1 + h_2)^2, \quad R_2^2 = 4a^2 + (h_1 - h_2)^2, \\ k_1^2 = 4a^2/R_1^2, \quad \text{and} \quad k_2^2 = 4a^2/R_2^2.$$

The value of M can be readily found from this formula with the help of Dale's Mathematical Tables or by the series for E and F given in § 2.

If we have, in addition, $h_1 = h_2 = h$, then

$$R_1 = 2(a^2 + h^2)^{1/2}, \quad R_2 = 2a, \quad k_1^2 = a^2/(a^2 + h^2), \quad \text{and} \quad k_2^2 = 1.$$

Hence

$$M = \frac{8\pi a^2}{3h} N_1 N_2 \left[\left(1 + \frac{a^2}{h^2} \right)^{1/2} \left\{ E_1 + \frac{h^2}{a^2} (F_1 - E_1) \right\} - \frac{a}{h} \right]. \quad (44)$$

If the height $2h$ equal the diameter $2a$, so that the coils are as compact as possible, we may write

$$M = 13.589 N_1 N_2 a. \quad . \quad . \quad . \quad . \quad (45)$$

9. Formulae for the self-inductance of a helical current when the pitch of the helix is small.

If we have a wire closely wound on a smooth insulating cylinder, the diameter of the wire being small compared with that of the cylinder, then the mutual inductance between it and a cylindrical current sheet of the same radius as the helix formed by the axis of the wire and having the same

axial length, will be approximately equal to the self-inductance L of the coil. Hence, by (44), we find that

$$L = \frac{8\pi a^2}{3h} N^2 \left[\left(1 + \frac{a^2}{h^2}\right)^{1,2} \left\{ E_1 + \frac{h^2}{a^2} (F_1 - E_1) \right\} - \frac{a}{h} \right], \quad (46)$$

where N is the number of turns of the helix and $a/(\alpha^2 + h^2)^{1,2}$ is the modulus of the elliptic functions.

From a paper by Rosa*, the author has recently discovered that this formula has been previously given by Lorenz†. Before finding this out, however, he had given the formula to his students, and several of them had constructed standards of self-inductance by simply turning cylinders of well-seasoned teak in a lathe and winding cotton-covered wire round them. The agreement between measurement and calculation was found quite satisfactory.

When h is less than a , we get by the Binomial Theorem and (1) and (2),

$$L = 4\pi a N^2 \left[\log \frac{4a}{h} - \frac{1}{2} + \frac{h^2}{8a^2} \left(\log \frac{4a}{h} + \frac{1}{4} \right) \right]. \quad (47)^\ddagger$$

This is in exact agreement with a formula previously given by Lord Rayleigh ('Scientific Papers,' vol. ii. p. 15).

When h equals $(3/4)a$, the number found by this formula is too small by about the quarter of one per cent. For smaller values of h the numbers found by (46) and (47) are in practical agreement. When h is small, a small percentage error in determining it introduces a large percentage error into the calculated value of L . Hence, in making standards, it is advisable to have h not less than $2a$. In this case we can use the following remarkably simple formula :

$$L = \frac{2\pi^2 a^2}{h} N^2 \left[1 - \frac{4a}{3\pi h} + \frac{a^2}{8h^2} - \frac{a^4}{64h^4} \right]. \quad (48)$$

When $h=2a$, the inaccuracy of the formula (48) as compared with the more accurate elliptic integral formula is less than 1 in 8000. If h be greater than $2a$, therefore, it is sufficiently accurate for all practical purposes.

It is convenient to write

$$L = \beta(a^2/h) N^2, \quad (49)$$

and tabulate the values of β for various values of h/a :—

* Bull. Bureau of Standards, p. 162, Aug. 1906.

† Wied. Ann. vii. p. 161 (1879).

‡ The much more elaborate formula given by Coffin (*l. c. ante*, p. 113) may be deduced in the same way from (46).

h/a	1/100.	1/10.	1/5.	1/4.	1/2.	3/4.	1.	2.
β	0.5159	4.059	6.299	7.209	10.38	12.34	13.59	16.15
h/a	3.	4.	5.	6.	7.	8.	9.	10.
β	17.22	17.80	18.16	18.41	18.59	18.73	18.84	18.91

When h is greater than $40a$ we may write

$$L = 19.74(1 - 4a/3\pi h)(a^2/h)N^2; \quad . \quad . \quad (50)$$

and when h is greater than $400a$, we get the well-known formula

$$L = 19.74(a^2/h)N^2.$$

XXXIV. *Simultaneous Vibration-Curves of String and Bridge photographically obtained from a Monochord.* By E. H. BARTON, D.Sc., F.R.S.E., Professor of Experimental Physics, and J. PENZER, Research Scholar, University College, Nottingham*.

[Plates XI. & XII.]

PREVIOUS papers in this series have dealt with (i.) the belly† and string, and (ii.) the air‡ and string; the present communication gives the results experimentally obtained for the bridge and string of the same monochord. The motions of the bridge are recorded vertically and horizontally, the string being excited both by bowing and plucking. Examples of beats are also obtained by using in addition to the vibrations of the string those of a tuning-fork placed upon the instrument. In the later figures curves are shown for the bridge and string when the latter is weighted in the middle so as to make its partial tones inharmonic.

Experimental Arrangements.—The string's motion is obtained and recorded as in the previous papers. The bridge's vertical motion is detected and photographed by the rocking motion of an optical lever, as in the case of the motion of the belly in the first paper. These effects are shown in figs. 1–6 & figs. 14–26 (Plate XI.), in which the string's

* Communicated by the Authors.

† Barton & Garrett, Phil. Mag. July 1905.

‡ Barton & Penzer, Phil. Mag. Dec. 1906.

diameter was 0.92 mm., and it was illuminated at 5 or 6 cms. from the centre. In the original plates the string's motion was magnified $1\frac{1}{3}$ and the bridge's vertical motion 1425 times. Thus on the reduced prints, though both magnifications are changed, their ratio 1069 remains constant and gives the magnification of the bridge's motion relative to that of the string.

Fig. 1.—Plan of Optical Arrangements.

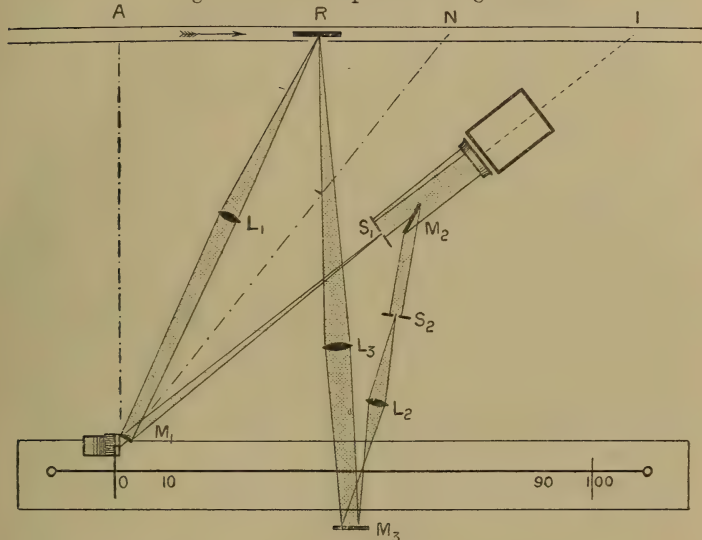
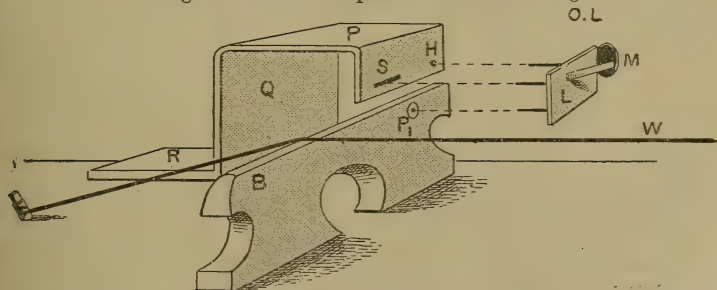


Fig. 2.—Detail of Optical Lever and Bridge.



For the horizontal motion of the bridge in the direction parallel to the length of the string a special form of optical lever was adopted. A plan of the entire optical arrangements is shown in fig. 1, and an enlarged detail of the optical lever and bridge in fig. 2. Consider first the light which falls on the optical lever. It starts from the arc

lantern and passes through the pin-hole S_1 (fig. 1) in a tin-foil screen, falls upon the mirror M_1 of the optical lever, and is thence reflected to the lens L_1 which focusses it upon the photographic plate at R which is on rails in a dark room. The normal to the mirror is $M_1 N$, and $M_1 A$ shows the axis about which it rotates if the bridge moves lengthwise of the string.

In the ordinary use of an optical lever as previously made, the plane of rotation coincides with that of incidence and the reflected beam describes an angle exactly double that described by the lever. Or, if these angles are respectively called δ and μ , then $\delta/\mu=2$.

But in the special arrangement and use of the optical lever here adopted, not only are the planes of rotation and incidence inclined, but also the normal to the mirror is inclined to the axis of rotation of the lever. Hence, as the lever turns, the reflected beam traces out a conical surface instead of a plane. And the angular magnification δ/μ varies according to the circumstances of the case between its limiting values unity and infinity! Thus when the lever turns through the small angle μ about its axis $M_1 A$, let the point N on the mirror's normal rise to N' , and let R in consequence rise to R' . Then since I , N' and R' are in the plane of incidence and reflexion, $IN'R'$ is a straight line. Hence

$$RR'/NN'=RI/NI. \quad \text{Also } \mu=NN'/AN \text{ and } \delta=RR'/AR.$$

If we now write α for the angle AM_1N between the axis of rotation of the lever and the normal to the mirror, and θ for the angle of incidence IM_1N , it can easily be shown that the angular magnification is given by

$$\delta/\mu = \frac{2 \tan \alpha}{\tan \alpha - \tan \theta}.$$

Thus, if the incident beam is coincident with the axis of rotation, $\theta=-\alpha$ and $\delta/\mu=1$. Again, if the incident beam is along the mirror's normal, $\theta=0$ and $\delta/\mu=2$ as in the ordinary form and use of the optical lever. But if the angle of incidence is nearly equal to α , the reflected beam is almost coincident with the axis of rotation and the angular magnification is very great. Thus writing $\theta=\alpha$ we have $\delta/\mu=\infty$!

In dealing with the actual magnification of the arrangement used, it is convenient to first omit the lens L_1 and find as above how R would move to R' by the action of the optical lever simply. Next trace this reflected beam back through the mirror M_1 to the virtual image of the pin-hole S_1

and find the consequent motion of this virtual image. Finally, omit the mirror M_1 but regard the lens L_1 as present, and find the motion of the bright spot on the photographic plate at R as due to the lens L_1 which focusses upon it the virtual image of S_1 in the mirror M_1 . A calculation on this principle gave the linear magnification of the bridge's motion lengthwise of the string to be 108 on the original negative.

By inserting a piece of very thin blown glass under the leg of the optical lever, see P_1 (fig. 2), and noting the consequent movement of the bright spot on a card, the linear magnification was estimated as 93 times. Thus we may say that the magnification near enough for our purpose is of the order 100.

The exact action of the optical lever is best seen from fig. 2, in which it is shown detached from its place at the bridge, where when in use it is secured by indiarubber bands so as to keep its three legs in contact with the hole, slot, and plane shown by H and S on the bracket P Q R and by P_1 on the bridge B. The plane P_1 consisted of a piece of glass cemented on to the head of a pin driven home into the bridge. On the optical lever O L is seen a little tie-piece or strut from the mirror M to the plate L. Before that piece was inserted the mirror yielded a little in the middle of its rotation, and so displaced the spot *horizontally* in the photographic plate. And though this yield was of the order of only two minutes of arc, it produced a horizontal motion of about 2 mm. on the plate and rendered it worthless.

In conjunction with the above special optical lever the optical arrangements for recording the motion of the string, now of 0.88 mm. diameter, are seen in fig. 1. In this M_2 represents a mirror of black glass which diverted part of the beam from the lantern to the tin-foil screen having a vertical slit at S_2 , whence the light passed to the lens L_2 which focussed a bright image of the slit on the string at a point about 5 cm. from its centre. Thence the light fell upon the mirror M_3 , this reflected it to the lens L_3 which focussed it upon the plate at R. The string's motion is accordingly shown on the prints by a black line on a light ground due to its shadow crossing the bright image of the slit. The motion of the string at this point is hereby magnified $1\frac{1}{2}$ on the original plates. So on those and the reductions accompanying this paper, the relative magnification of the bridge's motion lengthwise of the string is $100/1\frac{1}{2} = 67$ nearly.

In all the prints and their reproductions the left side corresponds to the beginning of the time and the string's

motion is inverted. The bridge's motion vertically in figs. 1-6 and 14-26 is also shown inverted. In figs. 7-13 and 27-47 showing the motion of the bridge horizontally the high positions of the spot denote a yielding of the bridge towards the centre of the string.

Results.—The curves obtained by these experiments are represented in the 47 photographs on Plates XI. and XII., each showing the simultaneous vibrations of string and bridge. These fall naturally into five groups which are taken in order below.

Group I. includes figs. 1-6 in column 1 of Plate XI. These show the vertical movements of the bridge, the string being excited by bowing and plucking at the places shown in the margin of the plate.

Group II. includes figs. 7-13 in column 1 of Plate XI. These show horizontal motions of the bridge parallel to the length of the string, the string being excited by bowing and plucking as noted against each figure. Two examples of beats are represented in figs. 9 and 10. In the former the string had a frequency of 138 per second and was plucked at 10 cm. from the bridge whose motion is shown, while a fork of 128 per second was excited by striking and then held vertically with its stem on the belly of the sound-box at 18 cm. from the bridge. The photographic plate was also shot slower than usual in the hope of showing a couple of maxima or minima of the resultant vibrations. This state of things was not obtained, but as the left end of the print indicates the beginning of the time we see there is evidence of increasing amplitude, thus proving that the bridge could take up vibrations from a fork on the belly although the string was passing over it and vibrating at a different frequency. This plate paved the way for the next, fig. 10, in which beats were obtained by the device of making them 18 per second instead of 10 and shooting the plate still slower than before. In this case the frequency of the string was raised to 146 per second, and the 128 fork was held upright with its stem on the bridge near the point B in fig. 2. The string as before was plucked at 10 cm. from this bridge. The upper half of the print shows clearly two maxima of the resultant vibrations of the bridge, while the lower part shows that the string's vibrations were dying away.

Group III. consists of figs. 14-26 contained in column 2 of Plate XI. Here bowing at a tenth of the string's length is adhered to throughout, the pitches forming the tempered chromatic scale of frequencies 92 to 184 per second. All the figures in this group show the vertical motions of the bridge.

Group IV. consists of figs. 27-39 contained in column 3 of Plate XI. These all show the vibrations of the bridge parallel to the length of the string, which is throughout bowed at 90 cm. from the bridge as in the previous group. Thus columns 2 and 3 of Plate XI. allow comparison of the two motions of the bridge under given conditions of pitch and excitation. They also allow of comparison with the corresponding columns in the previous papers*, where the motions of belly and air were shown.

Group V. comprises figs. 40-47 on Plate XII. Here the string is loaded in the middle by having a wire coiled round it and cemented where necessary to preclude all jarring. In the first column of this plate the photographic plates were shot fast; in the second slow, to obtain more vibrations on a plate and reveal the progressive changes in type due to the inharmonic relation between their partials. In all these figs. the white curves show the bridge's motion lengthwise of the string. The loads used had masses which were respectively 0.002, 0.004, and 0.008 of the mass of the string between the bridges. After fig. 42 the attempt to bow the string was abandoned. A moment's reflection will show its difficulty. When the string carries a load at the middle each partial having a node there is unaffected in pitch, *i. e.* all the evenly-numbered partials. Whereas all the other partials having an anti-node at the middle are flattened by the presence of that load. Hence, when the string is tightened to restore the fundamental tone to its original pitch, all the even partials are sharpened. But the compound vibration excited by the forcing action of the bow must be strictly periodic, and hence by Fourier's theorem all its partials commensurate. Thus when the departure from this state of things is slight the bow can force the partials into tune, but when the departure is great the bow refuses to "bite" and it becomes difficult to coax the string to speak with any tolerable quality of tone.

Accordingly for the figs. 44-47 constituting the second column of Plate XII., plucking only was adopted. And this had the advantage of leaving each partial free to vibrate at its own natural period although inharmonic with the others. The inharmonic character of the partials, of course, causes a slight change in the resultant vibration as their relative phases vary. And this progressive change of type is well shown in these four figures, where the plates were shot slow so as to obtain a number of complete vibrations on each. These curves give a graphic representation of the disturbance

* Phil. Mag. July 1905 and Dec. 1906.

caused by any irregularity in a musical string on a violin or harp, and illustrate the impurity of tone which must follow. The curves show also that these irregularities are all taken up by the bridge, and so no doubt extend to the sound-box and the adjacent air.

Looking now over the results as a whole for the bridge's motion, it is noteworthy that the fundamental tone of the string is also revealed in the vibration of the bridge. Whereas it might have been thought that whether the string were up or down the bridge would be equally deflected inwards towards the string's centre*. But if this were the case the fundamental would be lost by the bridge, its slowest vibration being the octave. This, however, is not the case, as all experience shows and as is confirmed by the curves.

As to the general character of the vibrations of the bridge, they seem to be intermediate in type between those of the belly dealt with in the first paper and those of the air dealt with in the second.

University College, Nottingham,
Dec. 21, 1906.

XXXV. *On the Appreciation of Difference of Phase of Sound-Waves.* By L. T. MORE, Ph.D., Professor of Physics, and H. S. FRY, Ph.D., Instructor in Chemistry, The University of Cincinnati†.

THE experiments described in this paper, in which an attempt has been made to show that we can locate sounds by an appreciation of the difference of phase of the waves coming to the ears, were carried out during the winter of 1902. Certain psychologists, to whom the results were submitted, considered them to be inconclusive, and their publication was postponed until additional work could be done. But this winter Lord Rayleigh, in his Sidgwick lecture‡, described some experiments he had performed which convinced him that we do locate sounds, partially at least, by this means. As his method differs considerably from ours, it seems best to publish these in spite of some incompleteness.

There have been several attempts made to prove that the ear is capable of detecting phase-differences of sound-waves as well as their amplitude and complexity. Apparently the attempts have not had much success, and König is perhaps the only one who maintained that one could distinguish two

* Compare the longitudinal form of Melde's experiment, see Rayleigh's 'Theory of Sound,' 2nd ed. vol. i. p. 81.

† Communicated by the Authors.

‡ See Phil. Mag. [6] xiii, p. 214 (1907).

complex tones similar in all respects, save the phase relations of their harmonics. His experiments with his rotating "phase-disks" have been repeated in many laboratories, and most observers agree with Helmholtz that there was no certainty in distinguishing the tones. In fact the theory of audition elaborated by Helmholtz apparently excludes the recognition of the phase of the wave by the auditory nerves. While this may be true of the sensation in a single ear, the experiments described in this paper make it probable that phase relations of sounds do influence binaural audition and enable us to locate sounds.

If we accept Helmholtz's theory of sound perception, we are, in explaining our method of locating sounds, compelled to rely on the ability of the mind to judge the intensity and complexity of the waves which enter the ears. That is, sounds coming from the right or left are louder or more complex in one ear than in the other, and by comparing the relative intensities or complexities of the two, we determine with considerable accuracy the direction of a sound.

The effect of the complexity of the wave we know little about from direct experiment and it is probably of minor importance in the general problem, as one can locate pure tones quite accurately. However, it seems natural to suppose the sound heard by the averted ear is weaker than the impression made on the other, and that the mind is capable by comparison to judge the location of the sound. But the work of Lord Rayleigh has raised very grave objections to this as an adequate method of locating sounds in all cases. We can dismiss at once the fact that one ear is a little further from the sound source than the other, because we can as readily determine the direction of sounds coming from a great distance as of those from near by. But the intensity of sounds is affected by obstacles in their path, and it is true that the head is generally more in the path of the wave to one ear than to the other. Although the head may be thus interposed, the calculations of Lord Rayleigh, 'Theory of Sound,' vol. ii. p. 442, show that it is too small an obstacle to affect appreciably the intensity of the long waves given out by grave tones. And this can be verified experimentally. If these two functions of sound-waves are inadequate, we are compelled to fall back on the only other property of a wave, its phase.

In one case, at least, we do know that the mind takes notice of the phase relations. Professor S. P. Thompson proved this by applying the receivers of two telephones, each to an ear. When the telephone disks vibrated so as to give

waves in the same phase, the impression on the mind was quite different from that when the disks vibrated oppositely. But no experiments had then been published which showed that our power of locating sounds is largely due to this phase interpretation*.

There are three reasons which support this theory: first, experiments made on animals; second, the fact that we can locate complex tones made in the medial plane between the ears; third, the systematic experiments given below.

It is a significant fact that the ears (certainly at least the outer parts) of many animals which have need to locate sounds accurately, are placed on top of the head and close together. This position is such that the head can make absolutely no sound shadow unless the source of the sound is directly below them. A sound coming from a considerable distance to a horse must affect each of his ears with the same intensity. Yet a horse, or any other of the animals tried, had no difficulty in locating the direction of a noise. This can be readily proved. If a person stations himself behind and somewhat to the *right* of an animal and then whistles, the animal invariably moves his head or ears to the *right* and continues the motion until pointing directly to the person, although he may be hidden. Of course the same holds true for experiments made on the left side. The results are quite certain, that such animals locate sounds accurately and yet have no means of doing this by comparing the intensity of the sounds in the two ears.

It is well known that a pure tone made directly in front of a blind-folded person cannot be distinguished from a similar tone sounded directly behind. But one has not much difficulty in distinguishing complex tones under like circumstances, and especially is this true when the tones are made by the human voice. As neither a difference in intensity nor in phase exists, we were led to the belief that a complex tone produced a different sensation according to the direction the waves entered the outside ear. Certainly the shape of the outer ear is well adapted to modify the short waves of the higher harmonics, and thus to change the character of complex tones.

Lastly, we devised a set of experiments to see if the apparent direction of a simple tone could not be changed by

* The preceding remarks are, of course, no longer correct, as Lord Rayleigh, in his Sidgwick lecture of last fall and in his paper in the February issue of this journal, states his unqualified conviction that phase-differences do affect the hearing. But, as they give the views held at the time our experiments were made and in fact were the cause of their being made, we decided not to alter them.

producing a difference in phase of the waves which entered the two ears. It was easy to produce such an effect, but an exact relation between phase-difference and direction was more difficult to obtain, although, allowing for personal variations, something has been accomplished.

A listener seated himself at the centre of a large circle marked on the floor of a room. The circle was divided by radii every $22\frac{1}{2}^\circ$, so that 0° was directly behind him and 180° in front, $+90^\circ$ being to his right and -90° to his left. A glass funnel, 13.5 cm. in diameter, was mounted horizontally on a table about seven feet behind him. Heavy rubber tubing, 1.2 cm. internal diameter, connected the funnel to a glass Y tube, on the other two branches of which, rubber tubing of the same size was fitted. These branch tubes ended in glass tubes bent so as to fit into the ears of the listener. Each of the branch tubes was cut in two in the middle and, by inserting pieces of glass tubing, their lengths were readily altered without the listener being at all aware of it. Fourteen different students of the college assisted, and to be sure they might not be biassed in their judgments, they remained unacquainted with the purpose of the experiment. In experimenting, a tuning-fork or the voice singing an open vowel at the same pitch was sounded at the mouth of the funnel. The listener was then asked to locate the direction from which the sound appeared to come; this direction was compared with the angles of the circle, and the relative lengths of the branch tubes were measured.

The results, in general terms, prove that when the branch tubes were of exactly equal length, the sound seemed to come from behind, but if one were longer than the other, by say two centimetres, the direction of the sound was shifted to the right or left; when the tube to the left ear was the longer the sound came from the right, and *vice versa*. However the different observers might vary as to the exact angle, they were unanimous so far. Some objections to the experiments are obvious, but they can be discussed better after the results are displayed in detail.

In the following two tables of results, the capital letters stand for the different listeners. $\delta\phi$ is the difference in phase of the two waves of sound when they reach the ears, expressed in eighths of the wave-length of the tone used. The angles give the apparent direction of the sound for various phase-differences noted by each listener. Since no mistakes were made in respect to the general directions, right and left, it is not necessary to distinguish the angles with plus and minus signs, if we remember only that 0° signifies

behind and 180° in front. When the experiments were made, the order of changing the lengths of the tubes was entirely arbitrary, that is for a given observer the two tubes might be equal for the first observation, then the right tube might be lengthened a half wave-length, or the left tube be lengthened a quarter of a wave-length and so on.

TABLE I.

Pitch of fork = 512.

Wave-length (λ) = 64 cm.

	$\delta\phi = 0.$	$\delta\phi = \frac{1}{8}\lambda.$	$\frac{1}{4}\lambda.$	$\frac{3}{8}\lambda.$	$\frac{1}{2}\lambda.$	$\frac{5}{8}\lambda.$	$\frac{3}{4}\lambda.$	$\frac{7}{8}\lambda.$	$\lambda.$
A ...	0	22.5	67.5	67.5	67.5	90	90	67.5	67.5
B ...	0	22.5	45	67.5	90	67.5	90	67.5	45
C ...	0	0	22.5	45	45	67.5	90	67.5	67.5
D ...	0	22.5	22.5	67.5	67.5	90	45	45	67.5
E ...	0	22.5	67.5	67.5	45	67.5	45	67.5	67.5
F ...	0	22.5	45	67.5	90	67.5	67.5	67.5	90
G ...	0	22.5	22.5	45	67.5	90	45	22.5	0
H ...	0	22.5	45	67.5	67.5	22.5	90	90	90
I ...	0	22.5	45	67.5	90	67.5	90	90	112.5
K ...	0	22.5	67.5	90	67.5	45	67.5	45	90

TABLE II.

Pitch of fork = 320.

Wave-length (λ) = 104 cm.

	$\delta\phi = 0.$	$\delta\phi = \frac{1}{8}\lambda.$	$\frac{1}{4}\lambda.$	$\frac{3}{8}\lambda.$	$\frac{1}{2}\lambda.$	$\frac{5}{8}\lambda.$	$\frac{3}{4}\lambda.$	$\frac{7}{8}\lambda.$	$\lambda.$
L ...	0	22.5	45	45	90	90	90	90	90
M ...	0	22.5	45	67.5	67.5	67.5	90	90	90
N ...	0	22.5	90	90	90	90	90	67.5	90
O ...	0	22.5	67.5	45	22.5	67.5	90	90	90
C ...	0	22.5	45	67.5	90	90	90	90	90
F ...	0	45	67.5	67.5	67.5	67.5	67.5	67.5	90
I ...	0	22.5	45	67.5	67.5	90	90	90	90

In spite of a certain indefiniteness in the results and considerable variations in the exact direction for any given phase-difference, the observations of these fourteen students, who, without an exception, correctly noted changes of direction, go to show that the hearing is affected by phase variation in the waves reaching the ears. Examining the data more carefully, we can divide them into two parts. Up to a

certain increase in phase-difference, the change of direction increases regularly and uniformly. But when the phase-difference becomes greater than $\frac{3}{8}\lambda$, for $\lambda = 64$ cm., and $\frac{1}{4}\lambda$ for $\lambda = 104$ cm., decision is more difficult, the apparent direction remaining unchanged to some and to others the angle is diminished. This may be shown by the following summary:—

When $\lambda = 64$ cm.,

if $\delta\phi$ is changed from 0	to $\frac{1}{8}\lambda$,	—all but one noted a change in direction.
" "	" $\frac{1}{8}\lambda$ "	" $\frac{1}{4}\lambda$, — " two " further increase in angle.
" "	" $\frac{1}{4}\lambda$ "	" $\frac{3}{8}\lambda$, — { eight noted a further increase in angle.
" "	" "	" two " no " " " "
" "	" "	" four " a " " " "
" "	" $\frac{3}{8}\lambda$ "	" $\frac{1}{2}\lambda$, — { four " no " " " "
" "	" "	" two " a decrease " "
" "	" $\frac{1}{2}\lambda$ "	" $\frac{5}{8}\lambda$, — { five " a further increase " "
" "	" "	" five " a decrease " "

When $\lambda = 104$ cm.,

if $\delta\phi$ is changed from 0	to $\frac{1}{8}\lambda$,	—all noted a change in angle.
" "	" $\frac{1}{8}\lambda$ "	" $\frac{1}{4}\lambda$, — " " a further increase in angle.
" "	" $\frac{1}{4}\lambda$ "	" $\frac{3}{8}\lambda$, — { three noted a further increase in angle
" "	" "	" three " no " " " "
" "	" "	" one " a decrease " "
" "	" $\frac{3}{8}\lambda$ "	" $\frac{1}{2}\lambda$, — { two a further increase " "
" "	" "	" four " no " " " "
" "	" "	" one a decrease " "

We may then say, for the wave-lengths used, a phase-difference up to about a quarter of a wave-length determines partially or wholly the apparent direction of a sound. From the properties of summation wave forms, we should expect a difference of phase of a quarter wave-length to be the greatest which would alter the form of the wave made by this addition. The agreement between our observations and this property of waves cannot, however, be said to be established except for these wave-lengths, and experiments to test this point should be made with sounds presenting great differences in pitch.

The experiments were continued up to a phase-difference of a whole wave-length, and roughly for greater differences, but the latter were not recorded as the angle continued, with most observers, to be about 90° .

Analytically, of course, the complex waves are identical when the phase-differences of the component waves are zero or a whole wave-length; and to explain the fact that most observers noticed a difference in direction for these two cases, we are led to believe that when one tube is so much longer than the other, the intensity of the sound in the longer tube then becomes appreciably less than in the other. This supposition is supported by the fact that the majority found

a maximum change in direction before the phase-difference of a wave-length was reached for the fork of shorter wave-length, but did not find such a maximum for the graver tone. And also we should remember that in the problem of locating sounds of these pitches in the free air, such large differences of phase would not occur. For example, the average distance between the ears of a man is about 15.5 centimetres, and that is less than a quarter of these wave-lengths.

If it be granted that we locate sounds entirely by their phase relations up to a phase-difference determined by the distance between the ears, we still find this insufficient as an explanation for locating sounds which come from the extreme right or left. For these large angles, difference in intensity may be our principal aid, especially as the head is best fitted then to cast a sound shadow. It is also a matter of observation that it is not so easy to locate either very grave tones or very shrill ones, and this may possibly be so because the difference in path to produce a phase-difference of a quarter of a wave-length is either much greater or much less than the width of the head.

During the experiments, attention was paid to the question whether difference in phase produced a change in the intensity of the combined tones. Analytically two sounds interfere when the phase-difference is a half wave-length. But the mind evidently does not combine the separate sensations in the ears in so simple a manner, as not one of the observers could detect any alteration in the loudness of the sounds when the relative phases were changed. Neither were they heard as two tones, nor was the sound louder in one ear than in the other. The effect was as if the fork had been sounded in the free air at different points on the circumference of a circle.

These experiments are perhaps open to the objection that, to produce a difference in phase, one branch tube must be made longer than the other. And so a doubt may arise whether the intensity of the sound going through the longer tube is not diminished more than that through the other tube. But the observations show that this objection is not valid. If it were a question of change of intensity, the change in direction would increase continually and not reach an angle where further increase in length of the tube produces either a doubtful increase in angle or even at times a decrease. We tested the least change in the relative lengths of the tubes which would shift the angle appreciably and found, for some listeners, two centimetres to be sufficient. The

diminution in intensity for a length of two centimetres of large tubing would be quite unnoticeable. Also it should be remembered that, to explain the location of a sound by differences in intensity, the idea of a sound shadow produced by the head was introduced, as it was evident that the difference in length of path of the sound to the two ears was not sufficient a cause.

The writers of this paper are not well enough acquainted with physiology to offer any explanation of the way the mind takes cognisance of phase-differences of tones. The theory of the mechanism of hearing advanced by Helmholtz seems entirely unsuitable, for a simple rod-like filament of the auditory nerve, vibrating sympathetically to waves of sound, apparently has no apparatus for recognizing phase-differences. So far as we know there has been but little adverse criticism of Helmholtz's work on the ear except in one instance. Dr. Howard Ayers, after a careful and laborious examination and measurement, maintains that his theory of audition is incorrect. Unfortunately the work of Dr. Ayers is not at hand, so a comparison of his theory with our experiments is not possible at present.

Before the laws of this effect can be stated completely a large number of experiments must be made, and it is hoped that those planned to be done this year will prove adequate.

Cambridge, England,
January 1907.

XXXVI. *On Convective Equilibrium of a Spherical Mass of Gas subject only to the Mutual Gravitation of its parts.* By J. PRESCOTT, *Lecturer in Mathematics at the Manchester School of Technology**.

BEFORE attacking the question from a mathematical standpoint, let us get some idea from physical considerations, how the radius of the sphere of zero density depends on the mass of the gas.

Suppose that, when a mass of gas acted on by no forces except its own gravitation has taken up its equilibrium state, it were possible suddenly to double its density at every point by placing in every portion of space an exactly equal and similar mass of gas to that which already occupies the space. The pressure would be doubled everywhere, and therefore the resultant of the pressures acting on the gas enclosed within any given space would be doubled. But the mass within the space would be doubled, and the attracting mass

* Communicated by the Author.

likewise doubled; hence the resultant attraction would be quadrupled. And since, before the alteration in density, the pressures and attraction balanced, it follows that the attraction under the new conditions would be preponderant everywhere and each element would move nearer the centre of the sphere. Thus the larger mass, starting from the same initial conditions as the smaller mass, will occupy the smaller volume. In the preceding argument there is no assumption concerning the variation of temperature throughout the mass; and therefore the conclusion is true however the temperature may vary from point to point.

The consideration that the smaller the mass the larger is the sphere occupied by a gas led me to suspect that there might be a critical mass of gas which, starting with a given amount of internal energy per unit mass, would just diffuse to infinity. In the case of convective equilibrium, however, I find that this surmise is not correct. Although the bounding sphere may become very great for small masses yet it never becomes infinite. But practically a small mass of gas could not be held together by its own gravitation in the neighbourhood of other bodies; for the attraction on the outer portions of the mass would be so small that the slightest attraction, even from distant external bodies, would suck some of it away, and this process would go on until nothing was left.

Now I propose to obtain the relation between the density ρ and the distance from the centre r in a mass of gas in convective equilibrium.

The equation expressing the condition of equilibrium is

$$\frac{dp}{dr} = -\rho F, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where p is the pressure, and F is the attraction on unit mass at a distance r . Thus

$$F = \frac{K}{r^2} \int_0^r \rho 4\pi r'^2 dr', \quad . \quad . \quad . \quad . \quad . \quad (2)$$

K being the constant of gravitation.

Hence

$$\frac{dp}{dr} = -\frac{K\rho}{r^2} \int_0^r 4\pi \rho r'^2 dr'. \quad . \quad . \quad . \quad . \quad (3)$$

Assuming convective equilibrium we have

$$p = c\rho^\gamma.$$

The equation of equilibrium now becomes

$$c\gamma\rho^{\gamma-1}\frac{d\rho}{dr} = -\frac{K\rho}{r^2}\int_0^r 4\pi\rho r^2 dr, \quad . \quad . \quad . \quad (4)$$

or

$$c\gamma\frac{r^2}{\gamma-1}\cdot\frac{d(\rho^{\gamma-1})}{dr} = -4\pi K\int_0^r \rho r^2 dr, \quad . \quad . \quad . \quad (5)$$

By means of the substitutions

$$\begin{aligned}\rho^{\gamma-1} &= y \\ \frac{1}{r} &= z,\end{aligned}$$

this last equation reduces to

$$\frac{dy}{dz} = -\frac{4\pi K(\gamma-1)}{c\gamma}\int_{\infty}^z y^{\frac{1}{\gamma-1}}\frac{1}{z^4}dz. \quad . \quad . \quad . \quad (6)$$

Now putting, for convenience,

$$x = z \cdot \sqrt{\frac{c\gamma}{4\pi K(\gamma-1)}},$$

and differentiating with respect to the upper limit, we get

$$\frac{d^2y}{dx^2} = -\frac{y^{\frac{1}{\gamma-1}}}{x^4} \cdot . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Being unable to find the general solution of this equation I have concentrated my attention on a single solution of the type which applies to the physical problem, and I shall shortly show that this single solution can be adapted to any mass of gas.

Let us take $\gamma=1.4$. If we start with $y=0$ when $\frac{dy}{dx}=m$ and $x=a$, m and a being both positive, then so long as the curve for x and y does not cross the axis of x again we know that $y < mx$; consequently

$$-\frac{d^2y}{dx^2} < \frac{m^{2.5}x^{2.5}}{x^4};$$

therefore

$$m - \frac{dy}{dx} < \int_a^x \frac{m^{2.5}}{x^{1.5}} dx.$$

The right hand side of this inequality is finite when the upper limit is infinite. Hence, if we start with suitable

values of m and a , the value of $\frac{dy}{dx}$ will approach some constant as x becomes infinite; *i. e.*, the curve will have an asymptote. The only types of curve which will apply to the physical problem are those to which this asymptote is parallel to the axis of x and at a finite distance from it; for the value of y when x is infinite is the value of $\rho^{\gamma-1}$ at the centre of the sphere. It is easier to start with x infinite and $\frac{dy}{dx}$ zero and work towards the origin. I therefore began with $y=100$, $\frac{dy}{dx}=0$ when $x=\infty$, and proceeded by successive steps until y became zero. The values obtained are only approximate ones, but I do not think the errors in the values of x and $\frac{dy}{dx}$ when y is zero are greater than 5 or 6 per cent. The following table gives the calculated values of $\frac{dy}{dx}$ and y .

x .	$\frac{dy}{dx}$	y .	x .	$\frac{dy}{dx}$	y .
∞	0	100	21	2.347	72.07
100	.0333	98.33	20	2.588	69.60
80	.0525	97.47	19	2.856	66.88
60	.0993	95.95	18	3.154	63.87
50	.1689	94.61	17	3.483	60.55
40	.3045	92.25	16	3.844	56.89
36	.5321	90.78	15	4.232	52.85
32	.8796	88.28	14	4.649	48.40
30	1.051	86.38	13	5.085	43.54
29	1.141	85.29	12	5.529	38.23
28	1.241	84.10	11	5.960	32.49
27	1.352	82.81	10	6.352	26.34
26	1.475	81.41	9	6.670	19.83
25	1.613	79.87	8	6.877	13.06
24	1.767	78.18	7	6.957	6.14
23	1.939	76.33	6.10	6.98	0
22	2.131	74.30			

The accompanying curve shows the relation between y and x from the point where $y=0$ to $x=40$. The numbers in the table can easily be converted into distance from the centre and density by means of the relations

$$r = \frac{1}{x} \cdot \sqrt{\frac{c\gamma}{4\pi K(\gamma-1)}},$$

and

$$\rho = \frac{1}{y^{\gamma-1}} = y^{2.5}.$$

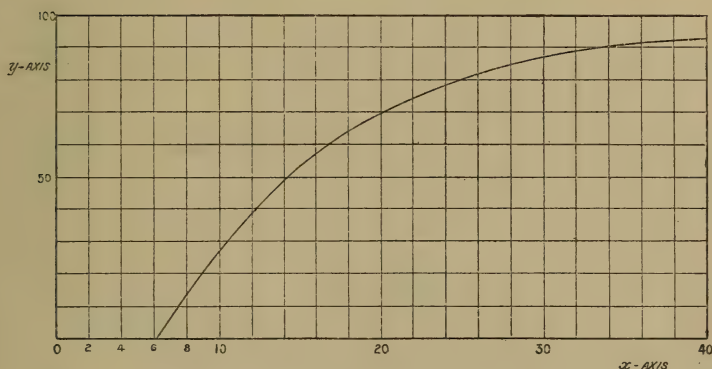
Taking $c = 15 \times 144 \times 13^{1.4} \times 32.2,$

which is the value, in foot pound units, of $\frac{p}{\rho^{1.4}}$ in our atmosphere at the surface of the earth; and taking also

$$K = \frac{1}{9.3 \times 10^8},$$

we get that

$$\begin{aligned} r &= \frac{1}{x} \times 25.5 \times 10^6 \text{ feet,} \\ &= \frac{1}{x} \times 4820 \text{ miles.} \end{aligned}$$



When ρ is zero $r=790$ miles, and when r is zero $\rho=100,000$. This, however, does not correspond to any physical conditions, for the density of a gas cannot have such a large value. I shall now show, however, how the solution can be adapted to give any arbitrary value of ρ at the centre.

If we put

$$\rho^{\gamma-1} = py,$$

and

$$r = q \frac{1}{x} \sqrt{\frac{c\gamma}{4\pi K(\gamma-1)}}$$

in equation (5), p and q being constants, we get

$$pq^2 \frac{d^2 y}{dx^2} = -p^{2.5} q^4 \frac{y^{2.5}}{x^4};$$

and if we make $p^{1.5} q^2=1$, this reduces to equation (7) whose

solution has been obtained. Denoting by M the total mass of the sphere, we know that

$$M = 4\pi \int_0^{r_0} \rho r^2 dr,$$

the subscript $(_0)$ indicating values of quantities when $\rho = 0$. Substituting for ρ and r we get

$$\begin{aligned} M &= 4\pi \left\{ \frac{c\gamma}{4\pi K(\gamma-1)} \right\}^{\frac{3}{2}} \int_{x_0}^{\infty} p^{2.5} q^3 \frac{y^2}{x^4} dx \\ &= 1.20 \times 10^{11} \times q^{-\frac{1}{3}} \int_{x_0}^{\infty} \frac{y^{2.5}}{x^4} dx \\ &= 1.20 \times 10^{11} \times q^{-\frac{1}{3}} \int_{\infty}^{x_0} \frac{d^2 y}{dx^2} dx \\ &= 1.20 \times 10^{11} \times q^{-\frac{1}{3}} \left(\frac{dy}{dx} \right)_0 \\ &= 1.20 \times 10^{11} \times 6.98 \times q^{-\frac{1}{3}}. \end{aligned}$$

Therefore $q^{\frac{1}{3}} = 8.4 \times 10^{11} \times M^{-1},$

and $p^{\frac{1}{3}} = q^{-\frac{1}{3}}$

$$= \frac{M}{8.4 \times 10^{11}}.$$

Hence $r = \left(\frac{8.4 \times 10^{11}}{M} \right)^{\frac{3}{2}} \frac{1}{x} \sqrt{\frac{c\gamma}{4\pi K(\gamma-1)}}$

and $\rho = (py)^{2.5}$

$$= \left(\frac{M}{8.4 \times 10^{11}} \right)^{10} y^{2.5}.$$

Thus, by means of the table given, the density at any point can be obtained when the mass is given.

Let us find the radius of the sphere whose density at the centre is 10 lbs. per cubic foot. At the centre

$$\rho = 10 = \left(\frac{M}{8.4 \times 10^{11}} \right)^{10} 10^5.$$

Therefore $\frac{8.4 \times 10^{11}}{M} = 10^{\frac{4}{10}}.$

And at the surface of zero density

$$\begin{aligned} r &= 10^{\frac{12}{10}} \times \frac{1}{6.1} \times 25.5 \times 10^6 \text{ feet} \\ &= 12600 \text{ miles about.} \end{aligned}$$

Also if the density at the centre is 1 lb. per cubic foot the radius of the containing sphere is about 25100 miles.

I will now investigate what effect the contraction of the gas has on the temperature.

Let us first transform the equation of equilibrium so as to make θ , the temperature, the dependent variable, instead of ρ .

We have assumed

$$\frac{p}{\rho^\gamma} = c.$$

But
$$\frac{p}{\rho} = R\theta.$$

Hence
$$c\rho^{\gamma-1} = R\theta.$$

Thus, since
$$c \frac{d(\rho^{\gamma-1})}{dr} = R \frac{d\theta}{dr},$$

the equation of equilibrium becomes

$$\begin{aligned} R \frac{\gamma}{\gamma-1} \frac{d\theta}{dr} &= - \frac{K}{r^2} \int_0^r 4\pi \rho r^2 dr \\ &= - \frac{K}{r^2} \int_0^r 4\pi \left(\frac{R\theta}{c} \right)^{\frac{1}{\gamma-1}} r^2 dr. \quad (8) \end{aligned}$$

Now if the gas loses heat c will diminish. Let us find the changes in r and θ for the same element of gas—or rather, for the gas occupying the same relative position in the mass, corresponding to a diminution in c .

Suppose the new value of c is c_1 such that

$$c_1 = nc,$$

n being less than unity; and let n_1, θ_1 be the new values of r, θ for the corresponding element of gas. Then it can easily be shown that

$$r_1 = n^\alpha r$$

$$\theta_1 = n^\beta \theta$$

satisfy all the conditions of the problem provided α and β be properly chosen. For, the new equation of equilibrium is

$$R \frac{\gamma}{\gamma-1} r_1^2 \frac{d\theta_1}{dr_1} = -K \int_0^{r_1} 4\pi \left(\frac{R\theta_1}{nc} \right)^{\frac{1}{\gamma-1}} r_1^2 dr_1. \quad (9)$$

Now putting $n^{\alpha}r$ for r_1 and $n^{\beta}\theta$ for θ_1 we find

$$R \frac{\gamma}{\gamma-1} r_1^2 \frac{d\theta_1}{dr_1} = R \frac{\gamma}{\gamma-1} r^2 \frac{d\theta}{dr} n^{\alpha+\beta}; \quad \dots \quad (10)$$

and

$$\begin{aligned} \int_0^{r_1} \left(\frac{\theta_1}{nc} \right)^{\frac{1}{\gamma-1}} r_1^2 dr_1 &= \int_0^r \left(\frac{n^{\beta}\theta}{nc} \right)^{\frac{1}{\gamma-1}} r^2 n^{\beta} dr \\ &= n^{\frac{\beta-1}{\gamma-1} + 3\alpha} \int_0^r \left(\frac{\theta}{c} \right)^{\frac{1}{\gamma-1}} r^2 dr. \end{aligned} \quad (11)$$

Hence we see that with these values for r_1, θ_1 , equation (9) is reduced to equation (8) provided

$$n^{\alpha+\beta} = n^{\frac{\beta-1}{\gamma-1} + 3\alpha};$$

i. e., provided

$$\alpha + \beta = \frac{\beta-1}{\gamma-1} + 3\alpha. \quad \dots \quad (12)$$

This means that, if equation (12) holds, $n^{\alpha}r$ and $n^{\beta}\theta$ are the distance from the centre and the temperature respectively of an element of gas in convective equilibrium with the value nc for $\frac{p}{\rho^{\gamma}}$. Now to ensure that r_1, θ_1 shall refer to the same element of gas as r, θ referred to in the old state, we must make the mass within the sphere r_1 the same as that which formerly occupied the sphere r . But the expression on the left-hand side of equation (11) is proportional to this mass in the new state. It will be the same mass as that originally in the sphere r if

$$n^{\frac{\beta-1}{\gamma-1} + 3\alpha} = 1,$$

or

$$\frac{\beta-1}{\gamma-1} + 3\alpha = 0. \quad \dots \quad (13)$$

Equations (12) and (13) give the following values for α and β ,

$$\alpha = \frac{1}{3\gamma-4},$$

$$\beta = -\frac{1}{3\gamma-4}.$$

Hence we see that

$$r_1 = n^{\frac{1}{3\gamma-4}} r$$

$$\theta_1 = n^{-\frac{1}{3\gamma-4}} \theta.$$

Now there are two cases to consider.

Case 1. $3\gamma > 4$.

Then, n being less than unity,

$$r_1 < r;$$

that is, the whole sphere contracts uniformly in consequence of loss of heat.

Case 2. $3\gamma < 4$.

Now $r_1 > r$;

that is, the sphere expands uniformly in consequence of loss of heat.

In both cases

$$r_1 \theta_1 = r \theta.$$

In the first case, therefore, as the gas loses heat the sphere contracts and the temperature rises; whereas, in the second case, with loss of heat the sphere expands and the temperature falls.

It is clear that, since a gas for which γ (the ratio of the specific heats) is less than $1\frac{1}{3}$, loses energy as it expands, it must be supplied with energy from outside in order to contract. Also it follows that in any given state of convective equilibrium such a gas has more internal energy than it could have obtained by contraction from a state of infinite diffusion. This conclusion agrees with Perry's theorem as stated by Lord Kelvin on page 368 of 'Nature' for February 14, 1907.

If, however, the gas has been supplied by some means with the heat necessary for a state of convective equilibrium, the gradual inevitable loss of heat will only drive the particles further and further apart until the outer portions come within the sphere of attraction of some other body, after which this body will gradually suck in the whole of the gas.

But a gas for which γ is greater than $1\frac{1}{3}$ will condense as it loses heat until a liquid nucleus is formed at its centre. After that stage is reached the results obtained for a wholly gaseous mass will no longer apply.

XXXVII. *On Transformer Indicator Diagrams.* By
THOMAS R. LYLE, M.A., Sc.D., *Professor of Natural
Philosophy in the University of Melbourne*.*.

[Plate XIII.]

1. **T**HE term "transformer indicator diagram" has been applied by Professor Fleming to any series of periodic curves which gives the forms, relative phase-positions, and magnitudes of the waves of current and E.M.F. on both the primary and secondary sides of a transformer when working. Such diagrams have been obtained by many investigators in different ways, but by none of the methods hitherto used has it been possible to determine directly and independently either the wave of magnetic flux F in the core, or the wave of magnetizing-current turns usually represented by the vector sum $n_1C_1 + n_2C_2$.

Both these quantities are of fundamental importance in the theory of the transformer. When they are known for any given load, all the other quantities (currents and E.M.F.s) can be determined for the same load when the primary and secondary turns, resistances, and leakage coefficients are known †.

In addition, since, as will be shown later, the integral

$$\int (n_1C_1 + n_2C_2) dF$$

for one cycle is equal to the total iron loss per cycle, the advantage of being able to determine both $n_1C_1 + n_2C_2$ and F directly and accurately is apparent.

Theoretically, $n_1C_1 + n_2C_2$ can be obtained by the vector addition of n_1C_1 and n_2C_2 , but as the latter quantities are, when the transformer carries a load, approximately equal in magnitude and opposite in phase, their vector sum is a small quantity compared with either of them. Hence small errors in the magnitudes of n_1C_1 and n_2C_2 may cause a large percentage error in the magnitude of $n_1C_1 + n_2C_2$, while very small errors in the magnitudes and phase-difference of n_1C_1 and n_2C_2 may render the phase of $n_1C_1 + n_2C_2$ calculated from them utterly unreliable.

By means of the wave-tracer ‡ designed by the author not only can the E.M.F. and current waves be accurately determined, but also the wave of magnetic flux pulsating in the core of the transformer, and in addition, as will be shown in

* Communicated by the Physical Society: read February 22, 1907.

† See Lyle: "The Alternate Current Transformer," *Proc. Roy. Soc. Victoria*, vol. xviii. pt. 1.

‡ Lyle: "Wave-Tracer and Analyser," *Phil. Mag.* Nov. 1903.

the sequel, we can obtain by its means the magnetizing-current wave, $n_1C_1 + n_2C_2$, with the same accuracy as any of the other quantities.

Incidentally will be given new methods of comparing mutual inductances, and of measuring both mutual and self inductances in terms of a resistance and a time.

2. In the paper just quoted I have shown that if a periodic current C flows in the primary of a pair of coils whose mutual inductance is M , and if the secondary be joined through a suitably arranged commutator (running synchronously with the generator of the periodic current), which commutes twice per period, to a large resistance r and thence to a galvanometer, there will be a steady deflexion γ in the latter which is connected with the instantaneous value C of the periodic current at the instant of commutation by the relation

$$MC = \frac{\lambda r T}{4} \gamma,$$

where λ is the reducing factor of the galvanometer and T the period.

By arranging so that the commutating brushes can be rotated on a divided circle round the drum of the commutator, commutation can be effected at any desired instant of the period, and the corresponding galvanometer reading when multiplied by the factor given in the above equation gives the ordinate of the current wave at that instant.

Take now a triad (see fig. 1, T) of coils of which p_1 and p_2 are to serve as primaries and the remaining one s placed between p_1 and p_2 to serve as common secondary. Let s be connected as before through commutator and resistance r to the galvanometer, and let M_1 be the mutual inductance of p_1 and s , and M_2 that of p_2 and s . Then, when a current C_1 circulates in p_1 and none in p_2 , the galvanometer deflexion γ_1 produced is connected with the ordinate of C_1 corresponding to the instant in the period at which commutation takes place by the equation

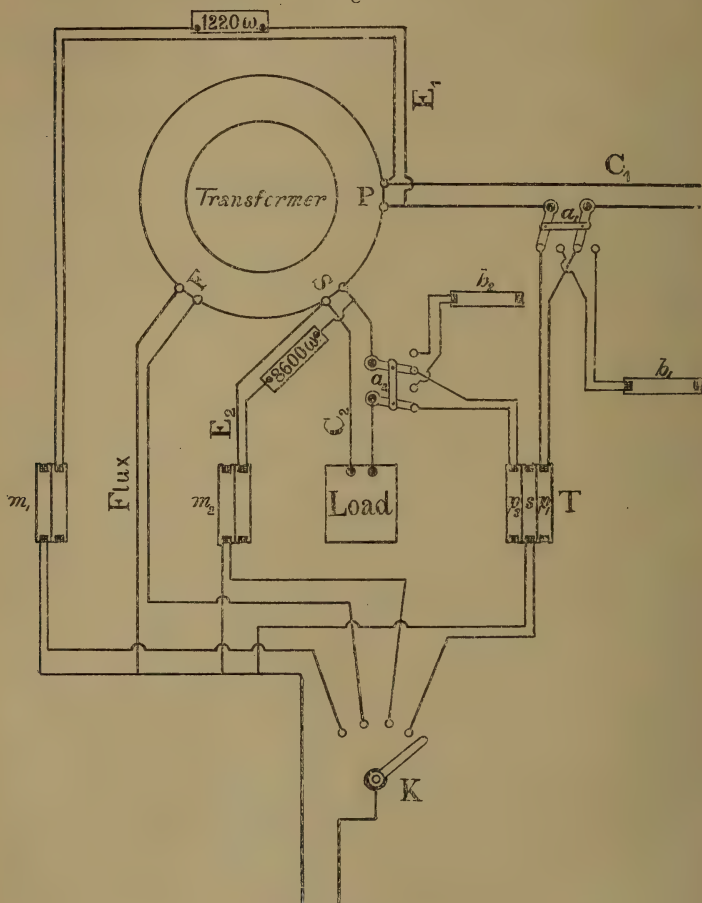
$$M_1C_1 = \frac{\lambda r T}{4} \gamma_1,$$

and when a current C_2 , equiperiodic with C_1 , circulates in p_2 and none in p_1 , a galvanometer deflexion γ_2 is produced which is similarly connected with the corresponding ordinate of C_2 by the equation

$$M_2C_2 = \frac{\lambda r T}{4} \gamma_2.$$

When both C_1 and C_2 are flowing in their respective primaries at the same time, the galvanometer deflexion γ will

Fig. 1.



to Commutator.

be the sum of γ_1 and γ_2 produced by C_1 and C_2 separately;
hence

$$M_1 C_1 + M_2 C_2 = \frac{\lambda r T}{4} \gamma.$$

If now n_1 and n_2 are numbers such that

$$\frac{M_1}{n_1} = \frac{M_2}{n_2} = M,$$

then
$$n_1 C_1 + n_2 C_2 = \frac{\lambda r T}{4 M} \gamma;$$

so that if γ be the galvanometer deflexion for a given position of the commutating brushes when C_1 is flowing in p_1 and C_2 , correctly directed, in p_2 , the ordinate of the wave which is the vector sum of $n_1 C_1$ and $n_2 C_2$ at the instant of commutation is the product of γ by $\lambda r T / 4 M$.

The same principle can obviously be extended to the determination of the vector sum of any number of equi-periodic currents each affected by an independent numerical multiplier.

In the paper already quoted it is also shown that if a coil of v turns of wire is looped on a magnetic circuit in which a periodic magnetic flux of period T is pulsating, and if the ends of the coil are connected through the synchronous commutator that changes twice per period to a high resistance r , and thence to a galvanometer as before, the ordinate F of the flux-wave and the corresponding galvanometer deflexion β for any position of the commutating brushes are connected by the relation

$$F = \frac{\lambda r T}{4v} \beta.$$

Hence any desired set of ordinates of a flux-wave can be obtained.

3. From what precedes it is obvious that in order to obtain practically the vector sum of $n_1 C_1$ and $n_2 C_2$, where n_1 and n_2 are numbers and C_1 and C_2 currents of equal period, it is necessary to have a triad p_1 , p_2 , and s of coils such that the mutual inductance of p_1 and s is to that of p_2 and s as n_1 is to n_2 . These can be wound in three grooves turned out of a circular disk of seasoned wood that has been well baked and then soaked in melted paraffin. A rough approximation to the required ratio of M_1 to M_2 can be obtained by attention to the relative numbers of turns, and a preliminary adjustment can most quickly be made by the wave-tracer as follows:—Send a periodic current of suitable value through the common secondary s . Join together one end from each of p_1 and p_2 and connect the junction to one fixed brush of the commutator. Bring the other ends of two terminals of a two-way key, from whose moving tongue a connexion is made, to the other fixed brush. Join the two movable brushes of the commutator through a suitable resistance r to the galvanometer. Thus, when a current C is flowing in s , either p_1 or p_2 can be put in circuit with the galvanometer, and the deflexions γ_1 and γ_2 due to p_1 and p_2 respectively for every position of the moving brushes are in the ratio of M_1 to M_2 , since

$$C = \frac{\lambda r T}{4 M_1} \gamma_1 = \frac{\lambda r T}{4 M_2} \gamma_2.$$

Obviously in making the comparison it is desirable to place the moving brushes so that the deflexions obtained are near the maximum ordinate of the current wave.

Otherwise we might put a resistance r_1 in the galvanometer circuit when p_1 is in, and r_2 when p_2 is in, so as to obtain equal deflexions; then we would have that

$$\frac{M_1}{M_2} = \frac{r_1}{r_2}.$$

Hence the coils s and p_1 may be finished and their terminals permanently fixed, and, using the latter (equal deflexion) method a resistance r_1 found to give a large deflexion when p_1 is in circuit and the brushes placed so as to give approximately the maximum ordinate of the current wave. Now substitute r_2 obtained by the above relation, allowing for galvanometer or other appreciable resistance that may be in circuit, and switch p_2 on to the commutator and galvanometer. An assistant can increase or reduce the turns (or fractions) until the same deflexion is obtained as was given by p_1 with r_1 in circuit. Spare wire should be left in case the further treatment of the coils causes any change in their M ratio.

As it is necessary, if high accuracy with the wave-tracer be desired, that the M coils used should have high insulation, the triad should now be thoroughly dried in an oven to drive off moisture from the cotton or silk covering and placed in an iron dish that can be fitted with an air-tight lid that has an exhausting tube through it. A layer of solid paraffin is in the bottom of the dish, and on this the coil is placed and weighted with a piece of metal. The lid is luted on with melted paraffin, the air pumped out, and the dish then heated so as to melt the paraffin it contains. The coil now sinks in the melted paraffin, and as the air has been removed it becomes thoroughly impregnated. We can thus obtain coils of high insulation and permanent mutual inductance if, in addition, proper attention is paid to the insulation of the terminals.

A final and careful adjustment of the ratio of M_1 to M_2 should now be made with the wave-tracer.

The absolute values of M_1 and M_2 must also be known, and they can easily be obtained by either of the methods just described in terms of a known standard of mutual inductance. An alternating current is sent through the primary of the standard, and either p_1 or p_2 of the triad (the one whose M is nearest in value to that of the standard) and the

secondary of the standard and s of the triad arranged so that either can be switched on to the commutator and galvanometer. Then for any position of the commutating brushes the two deflexions are proportional to the mutual inductances when the resistance in circuit is fixed, or if equal deflexions be obtained the inductances will be proportional to the resistances. Obviously it will be desirable to make a number of independent comparisons by varying the position of the commutating brushes about the place at which the maximum ordinate of the current is obtained.

The absolute value of a mutual inductance can be determined by the wave-tracer in terms of a resistance and a time, as follows. An alternating current is sent through a Kelvin balance, and at the same time a number (usually 30) of equispaced ordinates (γ) of it embracing one full wave is obtained by the wave-tracer using the mutual inductance M to be measured; as

$$C_t = \frac{\lambda r T}{4M} \gamma$$

the square root of the mean squares of C_t , which is the Kelvin balance reading (B say), is $\lambda r T / 4M$ times the square root of mean squares ($\bar{\gamma}$ say) of the galvanometer readings, hence

$$M = \frac{\lambda r T}{4} \frac{\bar{\gamma}}{B};$$

T is given by the chronograph attached to the wave-tracer, and the ratio of λ , the reducing factor of the galvanometer, to B can be obtained as follows. Send a continuous current $= B$ through the balance and through a standard resistance ρ (usually .1 ohm). From the terminals of ρ lead a shunt circuit through a resistance R to the galvanometer and let the deflexion of the latter be d .

Then

$$\lambda d = \rho \frac{B}{R},$$

and hence

$$M = \rho T \frac{r \gamma}{4 R d}.$$

4. The transformer, some of whose indicator diagrams will be given, was a small experimental one of the ring type of about one-half kilowatt capacity. It was used as a step-up one of ratio 1 to 6 transforming from about 40 volts (virtual) to 240 at about 50 periods per second.

Its details are as follows :—

Core :—144 annular laminæ annealed and paper insulated.

Laminæ :—Internal diameter..... 15.25 cm.

External diameter 27.75 cm.

Thickness 047 cm.

Primary coil :—No. of turns (n_1) = 100.

Resistance (warm) = 0.0676 ohm.

Secondary coil :—No. of turns (n_2) = 600.

Resistance (warm) = 2.034 ohms.

Before either coil was wound a single turn of well-insulated wire was looped on the core to serve as a search-coil for the determination, by means of the wave-tracer, of the magnetic flux pulsating in the core. The secondary coil was wound next the core.

The primary current was drawn from the alternating side of a small rotary converter that was supplied with direct current from storage-cells. The spindle of the commutator and that of the converter were in line and rapidly connected so that perfectly synchronous commutation was obtained. A chronograph took a continuous record of the period, recording once every 200 alternations.

The arrangement of the transformer and the mutual inductances by means of which the different waves were determined is shown in fig. 1. The primary current C_1 from the converter enters by the two leads marked C_1 . In one of these leads is placed a two-pole switch a_1 , by which C_1 can be sent through the primary p_1 of the triad T, or deflected through an equal compensating coil b_1 , leaving p_1 completely disconnected from the live circuit in case the triad is being used for the determination of C_2 .

E_1 is determined by obtaining the trace of the current wave it sends through a non-inductive resistance of 1220 ohms. The mutual inductance m_1 of .00061 henry is used for obtaining this trace. E_2 is similarly determined, the non-inductive resistance in circuit being 8660 ohms and the mutual inductance m_2 .003535 henry.

The secondary current C_2 may, by means of the switch a_2 , be directed through the primary p_2 of the triad T or through the equal compensating coil b_2 . When $n_1C_1 + n_2C_2$ is being determined, both C_1 and C_2 flow through their respective primaries p_1, p_2 of the triad in the proper relative directions, the common secondary s being joined as shown to the commutator and thence through a resistance to the galvanometer. When C_1 alone is being determined, C_2 is deflected through its compensating coil b_2 ; similarly when C_2 alone is being

determined, C_1 is deflected through b_1 . The mutual inductance of p_1 and s was $\cdot 0000485$ henry, while that of p_2 and s was $\cdot 000291$ henry, which bear the same ratio to each other as n_1 to n_2 , that is as 1 to 6.

The points marked F in fig. 1 represent the terminals of the single loop of insulated wire wound round and close to the transformer core. When the flux-wave is being obtained these are connected direct to the commutator, and thence through a suitable resistance to the galvanometer.

By means of the key K either the flux leads or those of any of the secondaries of the different mutual inductances can be connected with the commutator so as to obtain the corresponding wave-trace.

A glance at fig. 1 will show that the true primary current is less than the measured C_1 by the small current by which E_1 is determined. To correct for this (if correction is necessary) we subtract from each galvanometer C_1 reading an easily determinable fraction of the equi-phase E_1 galvanometer-reading. A similar correction has to be applied to the C_2 readings; but in this case the corrections have to be added. The galvanometer-readings for $n_1C_1 + n_2C_2$ have also to be similarly corrected, and in this case the corrections are of great importance as the C_1 correction is affected by the factor n_1 and the C_2 correction by the factor n_2 , and both corrections are approximately in the same phase.

5. For the present paper three separate sets of connected transformer quantities were determined, and they will be given below by their harmonic expressions and also represented by curves or indicator-diagrams.

These are for the transformer :—

- (1) At no load.
- (2) At (q.p.) full non-inductive load.
- (3) At (q.p.) full inductive load.

The method by which the wave-tracer galvanometer-readings are reduced to absolute measure and the harmonic expression for the periodic quantity deduced from them has been fully explained in former papers*.

In the present experiments, for each of the periodic quantities determined, 30 wave-tracer galvanometer-readings were taken, each differing from the next in order by 12° . The full wave was thus covered. Corresponding deflexions in each half of the wave were added, that is the 1st and 16th, the

* "Wave-Tracer and Analyser," Phil. Mag. Nov. 1903. "Variation of Magnetic Hysteresis with Frequency," Phil. Mag. Jan. 1905. "Expeditious Practical Method of Harmonic Analysis," Phil. Mag. Jan. 1906.

2nd and 17th, and so on, and 15 equispaced ordinates per half-wave obtained. These were subjected to harmonic analysis, and the amplitudes of the different harmonics were affected by their proper factors to reduce them to flux, current, or E.M.F., as the case might be, in absolute measure.

The results obtained are as follows:—

(1) *For the transformer at no load.*

$$\text{Period} = .02015 \text{ sec. ; } \omega = 311.8.$$

$$\frac{E_1}{10^8} = 57.71 \sin(\omega t - 6.98) - 4.34 \sin 3(\omega t - 15.7) + 0.35 \sin 5(\omega t - 29).$$

$$C_1 = .1304 \sin(\omega t - 44.23) + .0224 \sin 3(\omega t - 33) + .0057 \sin 5(\omega t - 27).$$

$$F = 184700 \sin(\omega t - 97.11) + 4300 \sin 3(\omega t - 105.9) - 300 \sin 5(\omega t - 84).$$

And as E_2 at no load $= -n_2 \frac{dF}{dt}$, we find that

$$\frac{E_2}{10^8} = 345.5 \sin(\omega t - 187.11) - 24.1 \sin 3(\omega t - 195.9) + 2.8 \sin 5(\omega t - 210).$$

(The different quantities are in absolute units and the phase angles in degrees.)

(2) *For the transformer at (q.p.) full non-inductive load.*

$$\text{Period} = .02053 \text{ sec. ; } \omega = 306.$$

The load was a manganin non-inductive resistance of 104.7 ohms.

$$\frac{E_1}{10^8} = 56 \sin(\omega t - 2.59) + 4.18 \sin 3(\omega t - 51.9) + .78 \sin 5(\omega t - 61.2).$$

$$C_1 = 1.945 \sin(\omega t - 5.71) + .145 \sin 3(\omega t - 50.1) + .034 \sin 5(\omega t - 33.7).$$

$$n_1 C_1 + n_2 C_2 = 12.6 \sin(\omega t - 45.21) + 2.67 \sin 3(\omega t - 33.4) + .75 \sin 5(\omega t - 30)$$

$$F = 178900 \sin(\omega t - 93.22) + 3900 \sin 3(\omega t - 83.2) - 400 \sin 5(\omega t - 93)$$

$$C_2 = .309 \sin(\omega t - 183.16) + .022 \sin 3(\omega t - 233.5) + .005 \sin 5(\omega t - 216)$$

And as $E_2 = R_2 C_2 = 104.7 \times 10^9 \cdot C_2$, we find that

$$\frac{E_2}{10^8} = 323.5 \sin(\omega t - 183.16) + 23 \sin 3(\omega t - 233.5) + 5.2 \sin 5(\omega t - 216).$$

(3) *For the transformer at (q.p.) full inductive load. Power factor = .73.*

$$\text{Period} = .01835 \text{ sec. ; } \omega = 342.4.$$

The load was a manganin non-inductive resistance in series with a copper wire inductance-coil. The total external

resistance was 76.52 ohms, and the inductance, measured independently, was .212 henry.

$$\frac{E_1}{10^8} = 54.13 \sin(\omega t - 10.5) - 5.36 \sin 3(\omega t - 12.5) + 1.37 \sin 5(\omega t - 13.2).$$

$$C_1 = 1.907 \sin(\omega t - 52.1) - .065 \sin 3(\omega t - 34.7) - .017 \sin 5(\omega t - 63.1).$$

$$n_1 C_1 + n_2 C_2 = 11.18 \sin(\omega t - 48.37) + 1.82 \sin 3(\omega t - 42.14) + .31 \sin 5(\omega t - 35).$$

$$F = 154900 \sin(\omega t - 99.9) + 5200 \sin 3(\omega t - 101.7) + 800 \sin 5(\omega t - 98).$$

$$C_2 = .299 \sin(\omega t - 232.38) - .015 \sin 3(\omega t - 215.4) - .002 \sin 5(\omega t - 235).$$

$$E_2 = 312.9 \sin(\omega t - 189.08) - 31.8 \sin 3(\omega t - 189.5) - 8.3 \sin 5(\omega t - 223).$$

6. The same wave-tracer deflexions were individually multiplied by their proper factors to reduce them to absolute measure, and the products plotted as wave ordinates against the corresponding wave-tracer divided-circle readings (*i. e.* against ωt where $\omega = 2\pi/\text{period}$) as abscissæ. Fig. 2 (Pl. XIII.) represents correctly in amplitude and relative phase the different periodic quantities for the transformer at no load; fig. 3 (Pl. XIII.) for the transformer at (q.p.) full non-inductive load, and fig. 4 (Pl. XIII.) for the transformer at (q.p.) full inductive load.

Obviously in figs. 2, 3, and 4 the same periodic quantities are graphically represented as are analytically expressed in series 1, 2, and 3 respectively of the preceding paragraph.

In addition to the sets of related waves, there is, in each of the three diagrams, an area very similar to the well-known hysteresis indicator-diagram. In the present case these closed curves were obtained by plotting the flux as ordinate against the corresponding value of the magnetizing-current turns (*i. e.*, $n_1 C_1 + n_2 C_2$) as abscissa for a complete period.

Thus the area enclosed (A say) is

$$\int_t^{t+T} (n_1 C_1 + n_2 C_2) dF,$$

where T is the period, and this integral can be shown to be equal to the total core loss per cycle due to both hysteresis and eddy currents as follows:—

Neglecting re^2 losses, the energy entering the transformer on the primary side in any element of time dt is $eC_1 dt$, where e is the back E.M.F. due to variation of the flux; and as

$$e = n_1 \frac{dF}{dt}$$

this energy is equal to

$$n_1 C_1 \frac{dF}{dt} dt.$$

Similarly the energy leaving the transformer on the secondary side in the same element of time dt is equal to

$$-n_2C_2 \frac{dF}{dt} dt;$$

hence in the time dt the transformer absorbs energy to the amount

$$(n_1C_1 + n_2C_2) \frac{dF}{dt} dt,$$

so that in one cycle, of duration T , the energy absorbed is equal to

$$\int_t^{t+T} (n_1C_1 + n_2C_2) \frac{dF}{dt} dt = \text{area } A.$$

7. It is easy to show that when $n_1C_1 + n_2C_2$ and F are expressed in the forms

$$n_1C_1 + n_2C_2 = m_1 \sin(\omega t - \mu_1) + m_3 \sin 3(\omega t - \mu_3) + m_5 \sin 5(\omega t - \mu_5) + \&c.,$$

$$F = f_1 \sin(\omega t - \phi_1) + f_3 \sin 3(\omega t - \phi_3) + f_5 \sin 5(\omega t - \phi_5) + \&c.,$$

the integral or area A and therefore the total core loss, in ergs, per cycle is equal to

$$\pi \{ m_1 f_1 \sin(\phi_1 - \mu_1) + 3m_3 f_3 \sin 3(\phi_3 - \mu_3) + 5m_5 f_5 \sin 5(\phi_5 - \mu_5) + \&c. \},$$

which when divided by $10^7 T$, where T is the period, gives the total power lost in the core in watts.

This form of expression has the advantage of giving separately the power absorbed by the iron by means of the harmonics of different orders, and from the analytic expressions for E.M.F.s and currents we can also obtain for the different orders of harmonics the input, output, and copper losses. Hence we can draw up a debit and credit account for the individual harmonics which will afford a good test of the accuracy of the wave-tracer, as the account for each order should balance.

This has been done for the first and third harmonics of the three series given in this paper and the results shown in the following tables. The quantities for the fifth harmonics are negligible. The figures represent watts.

TABLE I.—No Load.

	1st Harmonic.	3rd Harmonic.	Total.
Output	0	0	0
Iron loss	29.93	— .28	29.65
Copper loss, I. ...	.06	.002	.06
Copper loss, II....	0	0	0
Sum	29.99	— .28	29.71
Input	29.95	— .30	29.65

TABLE II.—Non-inductive Load.

	1st Harmonic.	3rd Harmonic.	Total.
Output	499.80	2.53	502.33
Iron loss	25.63	.24	25.87
Copper loss, I.	12.79	.07	12.86
Copper loss, II....	9.71	.05	9.76
Sum	547.93	2.89	550.82
Input.....	543.8	3.02	546.82

TABLE III.—Inductive Load.

	1st Harmonic.	3rd Harmonic.	Total.
Output	340.5	.51	341.01
Iron loss	23.18	.01	23.19
Copper loss, I.	12.29	.01	12.30
Copper loss, II....	9.13	.02	9.15
Sum	385.10	.55	385.65
Input.....	386.0	.69	386.69

The output in Table III. is the mean value of E_2C_2 . A second determination of it, obtained from the mean value of RC_2^2 , where R is the external resistance (76.52 ohms), gives 342.86 watts as against 341.

8. Other tests can be applied to the harmonic expressions given in § 5, and though their results may not be so good as those in § 7, yet they will prove fairly satisfactory when the number of independent determinations involved in obtaining each series is considered.

Thus the pressure drop from E_1 to $E_2/6$ is too small in both series 2 and 3, and this is probably due to a common error or errors. Moreover the speed never remained quite constant throughout a complete series. Again in series 3, knowing the external secondary resistance, and assuming as correct the value given for the external inductance (.212 henry), C_2 can be calculated from E_2 . When this is done we find that

$$C_2 = .2966 \sin(\omega t - 232.58) - .014 \sin 3(\omega t - 213) \\ - .002 \sin 5(\omega t - 239),$$

which agrees well with the value of C_2 obtained with the wave-tracer.

Conversely from E_2 and C_2 obtained with the wave-tracer,

both the external resistance R and the external inductance L can be calculated, or if R is known two determinations of L can be obtained from each order of harmonic.

If, however, as in the present case $\omega L/R$ is considerable, the first order is the only one suitable for this purpose; for if ψ_1 , ψ_3 , and ψ_5 be the difference of phase of the first, third, and fifth harmonics respectively of E_2 and C_2 , then, since

$$\tan \psi_1 = \frac{\omega L}{R}, \quad \tan \psi_3 = \frac{3\omega L}{R}, \quad \tan \psi_5 = \frac{5\omega L}{R},$$

and $\omega L/R$ is large, a small error in the observed values of ψ_3 or ψ_5 will cause a large error in the value of L deduced from them.

For the first harmonics, as $\psi_1 = 43^\circ.3$, $R = 76.52$, $\omega = 342.4$, we find from

$$\tan \psi_1 = \frac{\omega L}{R}$$

that $L = .2106$ henry; and from

$$C = \frac{E}{\sqrt{R^2 + \omega^2 L^2}}$$

that $L = .2084$ henry.

These give a mean value for L of .2095 henry, which is as likely to be correct as the value .212 already given and which was obtained by the Wheatstone's bridge method. As ω or $2\pi/T$ was determined by the chronograph attached to the wave-tracer, the above constitutes a new method of measuring an inductance in terms of a resistance and a time.

9. In Table I. § 7, it will be noticed that the iron loss for the 3rd harmonic is negative ($-.28$ watt). This means that all the watts put down as iron loss for the 1st harmonic are not dissipated as heat, but that some of them are transformed by the iron to 3rd harmonic power; a portion of these are dissipated as heat, while the remainder .28 watt are given out as electrical power by the iron to the copper circuits. This phenomenon has already been drawn attention to in a former paper * by the author.

10. In the diagrams, figs. 2, 3, and 4 (Pl. XIII.), it is worth drawing attention to the approximate permanence of form of the magnetizing current wave, and to the permanence of the relative phase relations of E_1 , $n_1 C_1 + n_2 C_2$, and F , for the three essentially different conditions of working for which the diagrams were obtained.

* "Variation of Magnetic Hysteresis with Frequency," *Phil. Mag.* Jan. 1905.

XXXVIII. *Improvements in the Hüfner Type of Spectrophotometer.* By F. TWYMAN*.

THE following is a description of a Spectrophotometer of the Hüfner type to which an addition has been made which converts it into a Polarimeter, by means of which optical rotations can be measured for light of any wavelength in the visible part of the spectrum. Incidentally an improvement in design is described whereby greater accuracy is attained.

Fig. 1.

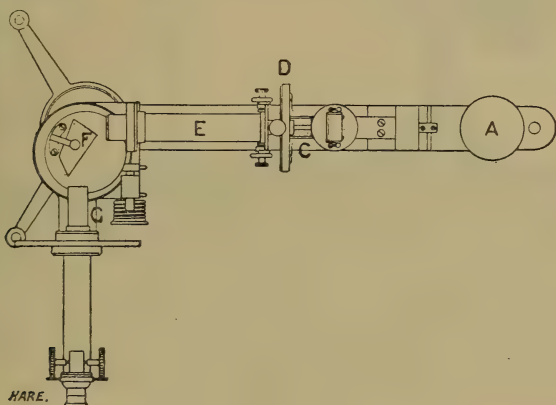
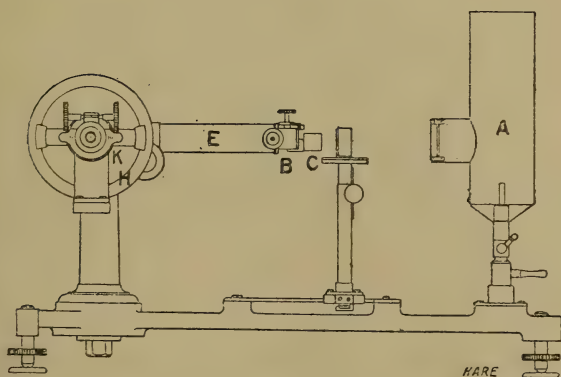


Fig. 2.

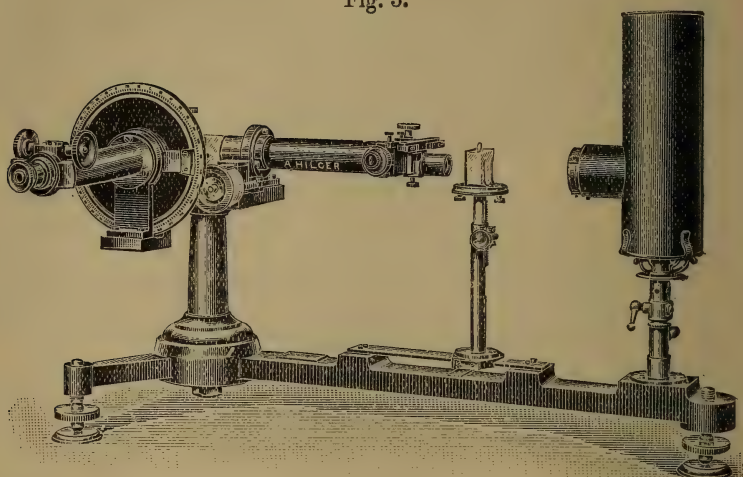


The Spectrophotometer part of the apparatus (shown in plan and elevation in figs. 1 and 2 and in fig 3) was designed and constructed in the workshop of Messrs. Adam Hilger Ltd.,

* Communicated by the Physical Society : read January 25, 1907.
Phil. Mag. S. 6. Vol. 13. No. 76. April 1907. 2 L

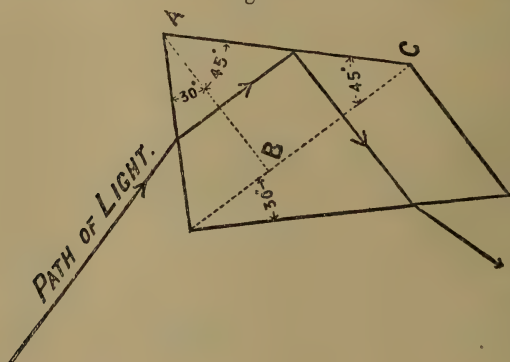
for Dr. C. E. Kenneth Mees in 1904, and embodies many suggestions of his and of Mr. S. E. Sheppard's. The Optical Train is the same as that of the Hufner Spectrophotometer

Fig. 3.



as made by Albrecht; with the exception that instead of the dispersion prism being the usual prism of 60° , the now fairly well-known constant-deviation prism is employed.

Fig. 4.



The prism is shown in fig. 4. It may be considered as built up of two 30° prisms and one right-angled prism from the hypotenuse of which light is internally reflected as shown in the figure. The prism is, however, made in one piece. This construction of prism has the very important mechanical advantage that the telescope and collimator are both fixed, being rigidly attached to a strong cast-iron base. The passing

from one part of the spectrum to another is effected by rotation of the prism by a micrometer screw.

The light source, A, is fixed to the cast-iron bed of the instrument. The light passes partly above and partly through a polarizing prism C. The upper beam passes first through

Fig. 5.



the substance whose absorption is to be measured and then through a thin wedge of neutral-tint glass, by the lateral translation of which the two beams may be made equal in intensity. Both beams then pass

through the rhomb of glass B (fig. 5) by means of which the two beams are brought into close juxtaposition. Immediately behind this rhomb is the slit of the spectroscope, the edge of the rhomb and the jaws of the slit both being in focus simultaneously when one observes them in the telescope.

The light after collimation in the usual way then undergoes dispersion by the constant-deviation prism, and passes on to the second polarizing prism G, and so on to the telescope. The second polarizing prism is mounted on a circle H divided into degrees and reading by a vernier to minutes, and by rotation of this the polarized beam of light can be reduced in intensity in the usual way till the intensity of the top and bottom beams is equal; and the density of the absorbing substance under test is then determined by means of the $\cos^2$ law (the intensity of the light passing through crossed nicols being proportional to the square of the cosine of the angle between their planes of polarization).

The instrument as used as a spectrophotometer will be found very completely described (together with an admirable bibliography on the subject of Spectrophotometry) in a paper by Messrs. C. E. Kenneth Mees and S. E. Sheppard printed in the 'Photographic Journal' for July 1904.

I might, however, again mention here the fact that the wave-length of the light under observation is read off direct on the helical drum of the micrometer-screw, see fig. 1. This is an arrangement which is found of great convenience in practice. The wave-lengths can be relied on to an average accuracy of well within 10 Angström units.

The present paper is principally concerned

(A) with the mode of correction of a serious error to which polarization spectrophotometers are liable;

(B) with the use of the instrument as a polarimeter.

(A) The error referred to is that due to the polarization of the light on transmission through the dispersing prism,

whether it be an ordinary prism of 60° or other angle, or a constant-deviation prism as in the present instrument.

Considering the case where a beam of light, collimated but initially unpolarized, is passed through the prism. Such a beam is the upper beam in the present instrument, which passes over the first nicol. The beam when it emerges is partially polarized, owing to the unequal reflexion of vibrations in and at right angles to the plane of incidence.

(In the case of a beam initially polarized in a plane neither in nor at right angles to the plane of incidence, matters are further complicated by the reflexion within the prism, which causes elliptical polarization; a state of things which has to be avoided by very carefully setting up the first nicol, until it is found that satisfactory extinction of the polarized beam can be obtained by rotation of the second nicol.)

Considering the appearance of the field of view as seen in the eyepiece, in the zero position of the second nicol: the position, namely, in which its plane of polarization is parallel to that of the first nicol. One sees a slit of light wider or narrower according to the width to which one may think it advisable to open the shutters in the eyepiece which limit the field of view. This slit of light is bisected by a fine horizontal line, the edge of the rhomb in front of the slit, which is almost invisible when the intensities above and below are equalized by the adjustment of the wedge of neutral-tint glass. The light-absorbing medium, whose density it is required to measure, is introduced into the top beam, and the second nicol is then rotated to obtain equality of intensity. The lower half varies in intensity according to the $\cos^2$ law; but owing to the polarization due to the dispersion prism the top beam also varies in intensity, and the absorption if determined simply by the application of the $\cos^2$ law will consequently be incorrect. The amount of the error can be determined by applying Fresnel's equations to any particular case. We will consider the actual state of things in the original instruments made, *i. e.*, a prism of refractive index 1.658 for the light under observation.

Let θ and θ_1 represent respectively the angles of incidence and refraction at the surface of the dispersion prism.

According to Fresnel's equations, if unpolarized light of intensity a^2 fall on a reflecting surface at an angle of incidence θ , and if θ_1 represent the angle of refraction, the intensity of that part of the reflected light which is polarized

in the plane of incidence is (see Schuster's 'Theory of Optics,' first edition, p. 49)

$$\frac{1}{2}a^2 \frac{\sin^2(\theta_1 - \theta)}{\sin^2(\theta_1 + \theta)}.$$

The intensity of the transmitted light if that light be polarized in the plane of incidence will then be

$$\frac{1}{2}a^2 \left\{ 1 - \frac{\sin^2(\theta_1 - \theta)}{\sin^2(\theta_1 + \theta)} \right\}.$$

As in a prism at the angle of minimum deviation the angle of incidence on entering the prism is equal to that of refraction on leaving it, the intensity of the transmitted light if that light be polarized in the plane of incidence, is, after passing right through the prism,

$$\frac{1}{2}a^2 \left\{ 1 - \frac{\sin^2(\theta_1 - \theta)}{\sin^2(\theta_1 + \theta)} \right\}^2.$$

For the case under consideration this works out to $\frac{a^2}{2} \times .651$.

The polarization due to the prism in the plane at right angles to this will be found, when evaluated, to be negligible.

When, then, a spectrum produced by such a prism is observed through a polarizing prism, and the latter is rotated, the beam varies in intensity from 1 to .651 (for the particular wave-length for which the μ of the prism = 1.658).

It will be seen how extremely serious is this polarizing effect of the dispersion prism. Want of leisure has prevented me from reading up fully the bibliography of the subject, but although this error has been previously mentioned and seems fairly well known, there certainly seems to have been insufficient attention brought to bear on the subject, as there are still polarization photometers made which suffer from the full amount of this error.

The Hufner instrument even in its original form is to a very great extent protected from this failing by the presence of the so-called Hufner rhomb in front of the slit. It will be seen at once that this produces a partial polarization of the upper beam of light in a plane perpendicular to that of the partial polarization produced by the dispersion prism. By applying Fresnel's equations as in the case of the dispersion prism above, we can find the amount of this partial polarization. The rhombs in the first instrument made were of Jena borosilicate crown of $\mu_D = 1.517$, the angle of the front and back

edges of the rhomb being 70° (which makes the angle of incidence of the beam upon the face of the rhomb 55°). With these values we find that the intensity of the light transmitted by the rhomb, if that light be polarized after transmission in the plane of incidence, would vary from 1 to 0.732: the partial polarization due to the rhomb in a plane at right angles to this being negligible.

Thus if the beam after passage through both rhomb and dispersion prism be polarized, it will vary from 0.732 to 0.651 as the analyzing nicol is turned.

This variation in the intensity of the top beam produces errors of the following amounts:—

Density of the absorbing medium under test (i. e. $\log_{10} \frac{I_1}{I_0}$, I_1 being the intensity of the incident beam and I_0 that of the transmitted beam).	Percentage error in the density, to be added or subtracted according as the first nicol is set with its plane of polarization vertical or horizontal.
0.054	11.5
0.231	9.4
0.602	6.4
1.521	3.3

The errors of reading are considerably less than the errors in the above table, so that it is very important to get rid of the latter. This is done in the more recent instruments by the simple and obvious plan of making the rhomb of the same glass as the prism, and of such angles that the angles of incidence of the beam on its surfaces equal those on the surfaces of the dispersion prism.

The following table will show the effect of this in several actual instances. A piece of neutral-tint glass was retained as a standard, and its density measured on various instruments, some with the old form of rhomb and some with the new. The plane of polarization of the front nicol was sometimes vertical, sometimes horizontal.

It was in the testing of Nos. 2 and 3 that we first became fully aware of the magnitude of the error.

Having calculated the amount on the hypothesis that the error was caused in the manner I have described, the front nicol was rotated through 90° , and an altered reading of the expected amount was obtained. On replacing the Hufner rhomb by a fresh one of the same glass as the dispersion-prism and of corresponding angles, a reading was obtained exactly the mean between the two former ones. (See No. 3 below.)

Several instruments were made before readings on a test density were taken, and instrument No. 4 was one of these

sent back for repair. The opportunity was taken of replacing the rhomb, after which it gave the correct reading, although I have no actual record of the final test reading.

Spectrophotometer number.	Density of standard glass at W.L. 5550 as given by the spectrophotometer.	Error due to polarization by the Hüfner rhomb and dispersion prisms. (calculated).	Corrected reading.
1	1.81	+0.05	1.86
2	1.92	-0.05	1.87
3	1.92	-0.05	1.87
3 (nicol rotated through 90°)	} 1.84	+0.05	1.89
3 (new rhomb)			
4	} 1.88	0	1.88
	1.83	+0.05	1.88
5	1.86	0	1.86
6	1.86	0	1.86

One of the great sources of error in polarization photometers is the difficulty of obtaining pure plane-polarized light, and to keep it plane-polarized so that when it arrives at the second nicol it can be entirely extinguished. The extinction of the last two instruments was so good, that by rotation of the second nicol from the zero position to the 90° position, the intensity varied from about 1 to 1/10,000. Imperfect annealing of the optical train and strain by clamping or cementing, or flaws in the first nicol are the chief difficulties met with—they give considerable but not insurmountable difficulties, and are readily detected in the instrument. They all tend to make densities appear too high.

(B) The use of the instrument as a polarimeter was first suggested by me, I believe, to Dr. K. C. Browning in conversation. Some while after that he commissioned Messrs. Hilger to undertake the construction of one of these instruments with the suggested polarimeter addition, which consists merely in the extension of the telescope support sufficiently far to enable one to interpose between the dispersion prism and the second nicol the medium of which one desires to measure the optical rotation.

The readings are taken by putting into the top beam a

suitable density. This density is matched by rotation of the second nicol before and after the interposition of the substance under test. The difference of readings gives, of course, the optical rotation for the part of the spectrum under test.

In the instrument exhibited room is provided for the interposition of a 100 mm. tube for liquids, but there is of course no difficulty in extending this to any desired amount. It will be seen that one measures by reducing one half of the field to the intensity of the other half which is already fairly dark. Thus one would expect the sensitiveness to be somewhat less than that obtained by the use of polarizers, in which one half of the field decreases while the other half increases in intensity. But it is an exceedingly useful addition to the spectrophotometer and produces a polarimeter free from some serious defects of most instruments excepting those with the Lippisch and similar polarizers. Moreover it is, of course, suitable either for use by monochromatic light or for the measurement of optical rotations for different wave-lengths.

I may add that a photographic negative is the best neutral-tint medium with which I am acquainted, and is what is employed when using the instrument as a spectro-polarimeter.

XXXIX. *The Density of the Æther.*

By SIR OLIVER LODGE *.

PART I.

1. **T**HE ætherial constants, μ and κ ,—the magnetic permeability and the electric inductivity of free space, have hitherto been of completely unknown value, both as to kind of quality and as to numerical magnitude; though the reciprocal of their geometric mean is well known to be a velocity, the velocity with which the æther transmits transverse waves of all lengths and intensities.

2. For many reasons I have been accustomed to think of $4\pi\mu$ (the ungeometrical part of a coefficient of self-inductance) as of something which by legitimate analogy could be spoken of as ætherial density, and of $4\pi/\kappa$ as of something which by similar material analogy must be expressible as ætherial elasticity or rigidity: this kind of rigidity being probably due to the æther's own intrinsic energy of constitution, and being explicable on a recondite hydrodynamical basis, as treated in various forms by Lord Kelvin.

3. I have also adduced reasons (in 'Modern Views of Electricity') for regarding the æther as exceedingly, perhaps infinitely, incompressible; this view being based in the first

* Communicated by the Author.

instance on the "superficial-charge" experiment of Cavendish (carefully repeated by Clerk Maxwell and the present Principal of Glasgow University with modern appliances)*; and in the second instance on the phenomenon of gravitation.

4. Furthermore, of late years, the whole world has become accustomed to think of minute electric charges—configurations probably of æther of some kind—moving about freely, and endowed with a definite mass and bulk: and, as is well known, the hypothesis has been mooted that matter itself may consist of bodies of this sort of character.

But if so, it is apparent how very porous, and so to speak rarefied, a structure matter must exhibit, to everything except electric disturbance; for the interstices must be enormous compared with the regions of definite structure. So, just as the ratio of mass to volume is small in the case of a solar system or a nebula or a cobweb, I have been driven to think that the observed mechanical density of matter is probably an excessively small fraction of the total density of the substance, or æther, contained in the space which it thus partially occupies—the substance of which it may hypothetically be held to be composed.

5. Thus, for instance, consider a mass of platinum, and assume that its atoms are composed of electrons, or of some structures not wholly dissimilar: the space which these bodies actually fill, as compared with the whole space which in a sense they 'occupy,' is comparable to 10^{-10} th of the whole, even inside each atom; and the fraction is still smaller if it refers to the visible mass. So that a kind of minimum estimate of ætherial density, on this basis, would be something like ten-thousand-million times that of platinum.

This, standing alone, would be a wild guess; but other physicists have probably of late years been led, on various grounds, to suspect that the ætherial density may be large; though probably it has not occurred to them to surmise that it is as great as that. Nevertheless if more detailed consideration should urge us to contemplate a density of enormous magnitude for the æther, even greater than the above, there is no need to be shocked or surprised.

6. With this introduction I will state the argument as briefly as possible; referring for fuller or more explanatory details to the concluding chapter of a new edition of 'Modern Views of Electricity':—

* Maxwell, 'Cavendish Researches,' p. 104, and Note 19, p. 417. See also Proc. Camb. Phil. Soc. vol. iii. part iii., and Maxwell's Collected Works, p. 612.

It is admitted that each negative electron has a certain intrinsic inertia or massiveness, dependent upon the amount of electricity which constitutes it, and on its size when pictorially represented as like a charged sphere; the mass being constant for all ordinary ranges of speed.

The simplest way of illustrating such a mass is to treat it as if it were a sphere of radius a moving through a simple irrotational perfect liquid, of density ρ , which it merely displaces.

The æther being supposed incompressible, as stated above, the density inside and outside the sphere must, on the simplest hypothesis, be assumed the same. And accordingly the total effective mass of the moving body is half as much again as the mass of the sphere itself, viz. $2\pi\rho a^3$; since in such a case the body is known, from ordinary hydrodynamics, to behave as if it carried half its bulk of fluid with it, while all the rest was annihilated.

That is the simplest mode of regarding the matter.

7. But it is known that the apparent mass of an electric charge really resides in the space outside it; being due to the circular magnetism which surrounds its path. If its speed is u , the intensity of magnetic field at any place $r\theta$, due to the motion, is, in every system of units,

$$H = \frac{eu \sin \theta}{r^2}.$$

[Parenthetically we may here observe, without any hypothesis, how enormous is the magnetic intensity immediately surrounding the equator of an electron, moving along an axis at say $\frac{1}{30}$ th the speed of light. It is no less than 10^{15} c.g.s. units. At a third the speed of light, even in the completer theory of which the above is the dominant term at ordinary speeds, it is still not appreciably more than 10^{16} ; but it begins to run up suddenly towards infinity when the velocity of light is closely approached. The reason probably being that it is then no longer a small fraction of the intrinsic rotational energy of the æther itself.]

8. Now mathematical speculation has suggested that there is an ætherial flow or circulation along lines of magnetic induction, which are well known to be always closed curves; and it is natural to assume that the energy of the field may be represented by the energy of this motion. Denote the speed of this hypothetical æther-drift by w , then the energy per unit volume is expressible in two equivalent ways, namely,

$$\text{as } \frac{1}{2}\rho w^2, \text{ and as } \frac{\mu H^2}{8\pi}.$$

Wherefore, in the case when the field is excited by a single spherical moving charge, e ,

$$\frac{w}{u} = \frac{e \sin \theta}{r^2} \sqrt{\left(\frac{\mu}{4\pi\rho}\right)}.$$

At the equator of the sphere, where $r=a$ and $\theta=90^\circ$, w is a maximum, say w_0 , such that

$$\frac{w_0}{u} = \frac{e}{a^2} \sqrt{\left(\frac{\mu}{4\pi\rho}\right)};$$

and so
$$\frac{w}{w_0} = \frac{a^2}{r^2} \sin \theta.$$

Thus the magnetic circulation round an electron, though certainly intense in its immediate neighbourhood, decreases as the square of the distance; and at any perceptible microscopic interval of space is utterly insignificant.

9. Certain imperfect cog-wheel, or rack-and-pinion, analogies, elsewhere developed—with a screw motion on the axes of the pinions by reason of the gyrostatic structure of the medium—suggest that w_0 and u are velocities of the same order of magnitude,—suggest even that they may be equal, or connected by the ratio π .

Now although this is uncertain and hypothetical, yet it is difficult to suppose w_0 *greater* than u , since it is excited by it; moreover u is already often so enormous, sometimes nearly equal to the velocity of light. While if w_0 be regarded as decidedly *less* than u , the density of the æther, shortly to be estimated on this basis, will be still further increased, in proportion to the square of the ratio $u:w_0$, to an amount which becomes even more immense. As a working hypothesis, therefore, I propose to assume that w_0 and u are of the same order of magnitude and practically equal.

10. An immediate consequence of this is that, in every system of units,

$$e = 4\pi a^2 \sqrt{\left(\frac{4\pi\mu}{\rho}\right)};$$

and as there are reasons (indicated in 'Modern Views') for supposing that the quantity under the square root is unity, the electronic charge comes out as equal, both in magnitude and dimensions, to the superficies of its supposed spherical bulk. The ordinary electrostatic unit of charge, moreover, happens to equal the superficies of the conventional atom: and the dimensions of all electrostatic and magnetic quantities are as given in Appendix *p* of 'Modern Views of Electricity' (1889).

11. But this is a minor matter; let us proceed to estimate

the massiveness of the hypothetical magnetic circulation, without any assumption of that kind.

Its energy will be

$$\int_0^\pi \int_a^\infty \frac{1}{2} \rho w^2 \cdot 2\pi r \sin \theta \cdot r d\theta \cdot dr.$$

Substituting for w its value just found in §§ 8, 9,¹

$$w = \frac{a^2}{r^2} \cdot u \sin \theta,$$

the term to be integrated becomes

$$\pi \rho a^4 u^2 \cdot \frac{dr}{r^2} \sin^3 \theta d\theta;$$

and so the energy comes out

$$\frac{4\pi \rho a^4 u^2}{3a}.$$

Expressing this as the kinetic energy of a certain mass m moving with speed u , this gives

$$m = \frac{8}{3} \pi \rho a^3;$$

or twice the mass of the content of the sphere, considered as composed of the surrounding magnetically circulating medium of density ρ . In so far as w_0 and u are not equal, this expression for m becomes multiplied by the factor

$\left(\frac{w_0}{u}\right)^2$, which justifies the remark of § 9.

12. All the estimates of inertia agree, therefore, as regards order of magnitude :—

- (1) The simple sphere composed of the coagulated liquid;
- (2) The sphere moving through effectively displaced simple incompressible fluid;
- (3) The more elaborate and truer consideration of the magnetic circulation outside the sphere.

For they are to one another merely in the ratio

$$1 : 1\frac{1}{2} : 2;$$

and they all give the effective mass of the electron as due to a disturbance, moving and identified with itself, affecting a region comparable with its own bulk. That region therefore must have a certain density.

What is that density?

We all admit now that the mass of an electron is comparable

with 10^{-27} gramme, while the volume is comparable with 10^{-39} c.c.; so that the density of the medium must be of the order 10^{12} grammes per c.c.;

and the absolute value of the magnetic constant μ for free space will be about 10^{11} of the same units.

The electrostatic rigidity of the medium is therefore 10^{33} dynes per sq. cm.;

or the absolute value of the dielectric constant κ for free space is 10^{-32} cubic centimetre per erg.

The intrinsic constitutional kinetic energy of the æther, which confers upon it its properties and enables it to transmit waves, is thus comparable with 10^{33} ergs per c.c.;

or say 100 foot-lbs. per atomic volume.

This is equivalent to saying that 3×10^{17} kilowatt-hours, or the total output of a million-kilowatt power station for thirty million years, exists permanently, and at present inaccessible, in every cubic millimetre of space.

13. Just as the ætherial rigidity is of a purely electric character, and is not felt mechanically—since mechanically it is perfectly fluid,—so its density is likewise of an electromagnetic character, and again is not felt mechanically, because it cannot be moved by mechanical means. It is by far the most stationary body in existence; though it is endowed with high intrinsic energy of local movement, analogous to turbulence, conferring on it gyrostatic properties.

Optically, its rigidity and density are both felt, since optical disturbances are essentially electromotive. Matter loads the æther optically, in accordance with the recognized fraction $\frac{n^2 - 1}{n^2}$; and this loading, being part and parcel of the matter, of course travels with it. But it is not a mechanical loading.

The mechanical density of matter is a very small proportion of the ætherial density; but the optical or electrical density of matter—being really that of æther affected by the intrinsic or constitutional electricity of matter—is not so small. The relative optical virtual density of the æther inside and outside matter is measured by n^2 ; but it may be really a defect of elasticity, at least in non-magnetic materials.

Electrical and optical effects may be said to depend upon e . Mechanical or inertia effects may be said to depend upon e^2 .

Detached and inelastic charges can load the æther optically, quite appreciably; but as regards mechanical loading, the densest matter known is trivial and gossamer-like compared with the unmodified æther in the same space.

Some consequences of this estimate of Density.

14. The amplitude of a wave of light, in a place where it is most intense, namely near the sun where its energy amounts to 2 ergs per c.c., comes out only about 10^{-17} of the wavelength. The maximum tangential stress called out by such strain is of the order 10^{11} atmospheres.

The hypothetical luminous circulation-velocity, conferring momentum on a wave-front, in accordance with Poynting's investigation, comes out 10^{-22} cm. per sec. The arithmetic of these calculations is given in the new concluding chapter of 'Modern Views' already referred to, § 6, which is likely also to appear in 'Nature.'

The supposed magnetic ætherial drift, along the axis of a solenoid or other magnetic field, is comparable to '003 centim. per sec., or 4 inches an hour, for a field of intensity 12,000 c.g.s.

But it is not to be supposed that this hypothetical velocity is slow everywhere. Close to an electron the speed of magnetic drift is comparable to the locomotion-velocity of the electron itself, and may therefore rise to something near the speed of light; say $\frac{1}{30}$ th of that speed: but in spite of that, at a distance of only 1 millimetre away, it is reduced to practical stagnation, being less than a millimicron per century.

In any solenoid, the ampere-turns per linear inch furnish a measure of the speed of the supposed magnetic circulation along the axis—no matter what the material of the core may be—in millimicrons per sec.

[1 micron = 10^{-6} metre, 1 millimicron is 10^{-9} metre = 10^{-7} centimetre.]

To get up an ætherial speed of 1 centimetre per second—such as might be detected experimentally by refined optical appliances, through its effect in accelerating or retarding the speed of light sent along the lines of magnetic force,—would need a solenoid of great length, round every centimetre of which 1000 amperes circulated 3000 times. That is to say, a long field of four million c.g.s. units of intensity.

15. The magnetic circulation being thus extremely difficult if not hopeless to test experimentally, it may be thought that by sending light down a torrent of cathode rays in a vacuum-tube, a better attempt to observe *some* accelerating or retarding effect could be made; because here a small portion of the æther is carried forward with enormous speed. But the excessive tenuity of the moving portion is against the possibility of detection. It is not the *mechanical* density of the residual gas, as compared with that of the stagnant æther, that need be considered however;—that would indeed be small,

being $10^{-6} \times 10^{-3} \times 10^{-12} = 10^{-21}$, at the exhaustion of a millionth of an atmosphere, even supposing the whole contents of the tube were electrons,—but it is the *optical* density; and that would appear to be something more like $10^{-6}(n^2-1)/n^2$, where n is the refractive index for common air. The optical fraction being $\frac{1}{20000}$, this amounts to $\frac{1}{2} \cdot 10^{-8}$; and at a speed of 10^9 cm. per sec. it would seem to give something almost appreciable in the way of drift. But the torrent of electrons must be only a small fraction of the whole residual gas; and some positive charges are sailing up in the other direction, in numbers probably sufficient to compensate for their smaller speed. So the outlook for experiment on this kind of influence of electricity on light is not encouraging,—whether the question be examined from the point of view of magnetic flux, or from that of electric current.

The estimates of this paper are:—That the *æther* has a magnetic inertia-coefficient which can be represented by likening it to a mechanical medium with a density of 10^{12} grammes per cubic centimetre; that the intrinsic *ætherial* kinetic energy is 10^{33} ergs per cubic centimetre; and that any streaming along magnetic lines of force is, in centimetres per second, one four-millionth of the number of c.g.s. units of intensity in the magnetic field.

PART II.

Experimental Attempt to observe the Hypothetical Magnetic Æther Drift.

16. During the years 1892–93, being urged thereto by certain mathematical theories, which are referred to and partly quoted in an Appendix in inverted order of date, I made a careful examination of the question of an *æther* flow along magnetic lines of force. This experiment has not been published, and in view of the above estimate its details ought to be recorded. As the result was negative I shall make the record as brief as possible. In July 1892 I constructed four bobbins or frames of brass, each 44 centimetres long, with a core of elliptical section of great eccentricity, so that the major axis of its aperture was about ten centimetres and the minor axis about one centimetre. Each bobbin was provided with carefully glazed ends of optically worked plate-glass, that could be clamped on in a water-tight fashion, and with a means for supplying any desired liquid into the interior of the core. The surrounding brass-sheet walls of the core were double, so that between the core and the winding there was an annular space for water-jacketing, in which a current of cold water could be

kept flowing, so as to screen the core from elevation of temperature due to the current in the exciting coil. The coil of wire wound on each bobbin had a resistance of 31 ohms and consisted of seven thousand turns of silk-covered No. 18 wire (.048 inch diameter), wound in 22 layers, with 7.2 turns to the centimetre in each layer; thus giving a "gauss-gradient," or magnetic intensity, along the axis, of 200 times the amperes flowing in the wire.

With 110 volts applied, the current was $3\frac{1}{3}$ amperes; with 230 volts applied, which could be done by putting the town mains (at that time 110 volts) in series with a large storage-battery, the current was 7 amperes, measured by an ampere-meter in both instances.

The corresponding intensities of field are

$$H = \begin{cases} 670 \\ 1400 \end{cases} \text{ c.g.s. lines to the square centimetre respectively.}$$

17. To check this estimate, the four bobbins were filled with bisulphide of carbon, placed all in a row, and supplied each with a current from the 110-volt mains, the coils being electrically in parallel. Polarized sodium light was then passed through the set, and the plane of polarization was found to be rotated by the current 83° ; of which 5° was due to the glass, and 78° to the bisulphide of carbon.

Hence the drop of magnetic potential between the ends of *each* coil comes out 29,000 gauss; according to Lord Rayleigh's estimate of the Verdet's constant for CS_2 at 18° , viz., .042 minute of arc for every c.g.s. unit of magnetic potential. The other estimate, from current and number of turns, under the same conditions, made the drop of magnetic potential $44 \times 670 = 29,500$ c.g.s.; which therefore agrees quite sufficiently well for the purpose.

18. All this is preliminary. In the actual experiment for trying the accelerating effect of a magnetic field on light, the bobbins were not placed in a row, but were arranged round the sides of a square inscribed at 45° to the sides of another square or optical frame (about 1 metre in the side) of which the details are given in the Phil. Trans. 1893, pp. 757 and 761, or more exactly in the plate 31 at the end of that volume (vol. 184) of the Phil. Trans.; since it was the same optical frame as was used for my æther convection experiments therein described. (See also Preston's 'Light,' § 136, and Proc. Roy. Inst. April 1, 1892.)

It is sufficient to say here that a parallel beam of light, bifurcated by semitransparent silvered glass at 45° , could be sent, by carefully adjusted mirrors, several times round the

inscribed square, the two half beams going in opposite directions; they could then be re-united by the same semi-transparent glass, and the interference bands observed in a telescope with a micrometer eyepiece.

During February 1893 a large number of experiments were made in this way. Sometimes the light was sent five or six times round, when only air was in the core, so that no cover-glasses were required; sometimes water was inserted in the core, and then the light could be got three times round.

19. Sometimes, in the early trials, with a liquid in the cores, a darkening was seen when the current was put on, but never a shift of the bands; the darkening was capricious and could not always be got: it was suspected to be occasionally due to a slight shift of the bobbins through mutual magnetic attraction. Polarized light too was often employed; it emerged still fairly plane-polarized, if the plane of polarization was parallel or perpendicular to the plane of reflexion at the mirrors; otherwise of course it became elliptical. But still there was no shift of the bands.

20. In these early trials, however, I was not sure of seeing more than one-tenth of a band shift, and with bisulphide of carbon not more than one-fourth of a band.

The bisulphide was not often employed, since water was so much easier: water is not nearly so sensitive to changes of temperature. Moreover the strong refractive index of bisulphide of carbon made it difficult to keep the beams to their path, and prevent the bobbins from acting like prisms. But the chief difficulty lay in the extreme sensitiveness to changes of temperature, so that now the water-jacketing failed to protect the bisulphide of carbon from some incipient elevation of temperature, and the slightest change of that sort began to contort the bands and render them undecided; while, if the current was left on any considerable fraction of a minute, they became unrecognizable. The experiment with the bisulphide, therefore, could not be pushed to the same degree of completeness as in the case of air or water; but while the observations lasted the effect was negative, and after each experiment several hours had to be allowed for the liquid to attain a uniform temperature again.

There was no special theoretical reason for supposing that the magnetic drift would be any stronger in bisulphide than in air, so this experiment was not persisted in.

If a transparent *magnetic* liquid could have been used, persistence would have been called for, but no such liquid is known; and though several attempts were made to use a solution of some salt of iron, the minute increase of

permeability, and therefore of etherial density, to be expected, was too insignificant to be of appreciable use. Besides, the oxidation of the sulphate, and the comparative opacity of the perchloride, made these liquids very difficult.

21. It is the experiments with air on which I place most reliance; and of these the best and final determinations were made six months later, immediately after receiving the following two letters from Dr. Larmor, which I am permitted to quote. The first letter seems to initiate the idea that I should try the effect of a longitudinal field on light; but I had already been trying it within the previous year—as just described—among many other things in the course of my “aberration” work.

Extracts from Letters from Dr. Larmor.

“28 Oct., 1893.

“Your kindness in sending me a copy of your Aberration paper encourages me to ask you a question which now interests me very keenly, namely, whether transmission along lines of magnetic force in air appreciably alters the velocity of light. I cannot discover that anyone has tried, but according to an ætherial theory which I have nearly finished writing out, and which covers, I fondly imagine, a very extensive ground, there ought to be such an effect. Perhaps you know, or could easily try with your machinery.”

I suppose I replied that I had tried, and found nothing, but that I would repeat the trial more carefully if he would be satisfied with an experiment in air; for his next letter says:—

“1 Nov., 1893.

“I can correlate most things in one scheme if I am allowed that magnetic force is velocity of the æther. . . . A magnetic field should carry the light along with it,—with what kind of velocity I have not yet formed an opinion; but it has nothing to do with rotary effect, and should be of the same order in vacuum as in a mass.”

22. In the experiment now to be described, as recorded in my laboratory note-book for November 1893, the light was sent four times round, in opposite directions, through the square of bobbins above described; the fact of four times being carefully tested, and sharp broad interference-bands being obtained. A fine micrometer cross-wire was set on the middle band, and watched, while 110 volts were applied to each coil, put on, off, reverse, on, off, &c. No shift whatever was seen, either of the middle or any other band—no effect at all. Repeated carefully several times, and watched for so little as the $\frac{1}{10}$ th of a band shift, which previous experience

had made me sanguine of seeing if it existed (see page 776, Phil. Trans. vol. 184). Could certainly have detected $\frac{1}{50}$ th of a band, probably $\frac{1}{100}$ th; but nothing was seen.

23. The experiment was so good that it was decided to try a higher voltage; so, putting the town mains in series with a storage-battery, 230 volts were applied to each coil, giving a current of 28 amperes on the meter, or 7 amperes to each coil. The bands were again watched while the current was being put on and off and reversed, sometimes slowly and sometimes quickly, but I never saw the slightest motion of any kind. The current could not be left on very long, because of possible heat disturbance penetrating from the windings of wire: for the coils themselves naturally became hot, since 9 horsepower was being expended upon them, but the water-jacketing succeeded in keeping the cores quite cool.

24. Hence I conclude, with security, that if there is any æther-flow at all along magnetic lines of force, it bears to the velocity of light a ratio less than the ratio of $\frac{1}{50}$ th of a wave-length to 32 times the length of a bobbin. The number 32 being justified by the fact that there are four bobbins in optical series, and that the light is sent four times through each, in opposite directions simultaneously; so that the whole virtual length of path along which the slightest acceleration or retardation would have caused a shift of the bands, is $32 \times 44 = 1408$ centimetres. The ratio of a fiftieth of a wave-length to this distance is less than 10^{-9} , so I conclude that any hypothetical æther drift, along a field of intensity 1400 c.g.s., is less than 30 centimetres, or one foot, per second; in other words, the æther flow is not as much as $\frac{1}{9}$ th of a millimetre per second, or say 16 inches an hour, for each c.g.s. unit of field intensity.

25. The total drop of magnetic potential applied was equivalent to $32 \times 60,000 =$ nearly two million c.g.s., and this certainly did not hurry the light by so much as $\frac{1}{50}$ th of a wave-length. As a matter of fact $\frac{1}{100}$ th shift could certainly have been seen, under the conditions; but I estimate $\frac{1}{50}$ th, to keep on the safe side.

So the deduction from this experiment was, either that the theory about longitudinal magnetic drift is erroneous, or else that the æther of space is at least eight times as dense as platinum.

26. That was the experimental conclusion thirteen years ago. I have now brought forward theoretical arguments which suggest that there is a sense in which what we call the

density of æther may turn out to be fifty thousand million times the ordinary material density of platinum. In which case every cubic millimetre of space contains the equivalent of a thousand tons, and every part of it must be internally squirming with the velocity of light.

APPENDIX A.

QUOTATIONS FROM EARLIER WRITERS ON THE SUBJECT OF A HYPOTHETICAL MAGNETIC FLOW.

The experiments above described are referred to by Professor Larmor in his paper on "A Dynamical Theory of the Electric and Luminiferous Medium," in the *Phil. Trans.* for 1894, vol. 185, p. 778. A fairly high value of density for æther is there contemplated with equanimity; though it is pointed out that such density would require the æther to be absolutely stationary, except in a magnetic field, in order to avoid hydrodynamical systems of force between moving bodies. But since everything tends to indicate that the æther is the most stationary substance in existence, at least as regards locomotion—probably the absolutely stationary thing, in so far as the phrase means anything—this necessity involves no difficulty whatever.

27. To illustrate the theoretical bearing and expectation of a magnetic drift, it may be useful to make a few extracts:—

First I will quote from § 73 and 74 from the above 1894 paper of Prof. Larmor's (*Phil. Trans.* vol. 185, p. 773, &c.).

"On the present theory, magnetic force or rather magnetic induction consists in a permeation or flow of the primordial medium through the vortex-aggregate which constitutes the matter; apparently it has not been tried (see however § 81) whether light-waves are carried on by this motion of the medium and their effective velocity is thereby altered, as we would be led to expect. It has been shown, however, by Wilberforce that the velocity of light is not sensibly altered by motion along a field of electric displacement, so far negating any theory that would connect electric displacement with considerable bodily velocity of the æther; and it has also been verified, by Lord Rayleigh, that the transfer of an electric current across an electrolyte does not affect the velocity of light in it. . . .

"As motion of the æther represents magnetic force, the fact that the magnetic permeability is almost the same in all sensibly non-magnetic bodies as in a vacuum, must be taken to indicate that the æther flows with practically its full velocity in all such media, so that there is very little obstruction interposed by the matter; it follows that, in the motion of a body through the æther, the outside æther remains at rest instead of flowing round its sides. . . .

"The notion of illustrating magnetic induction by the permeation of a fluid through a porous medium containing obstacles to its motion has been shown by Lord Kelvin to lead to a complete formal representation of the facts of diamagnetism."

Also on pages 779 and 805:—

“The residual force between bodies at rest in a field of ætherial motion, due to very slight defect of permeability, has already been shown, after Lord Kelvin’s illustration, to simulate diamagnetism; and the fact that there exist no powerfully diamagnetic substances is so far a confirmation of the present hypothesis. The view that the magnetic field of a current involves only slight circulation of the fluid æther is also in keeping with the account which has been given (§ 46) of the genesis of such a field.”

“The idea of representing magnetic force in the equations of electro-dynamics by the velocity of the electric medium has been tried already, for example by Heaviside and by Sommerfeld, not to mention Euler. The objection however has been taken by Boltzmann and also by von Helmholtz, that it would be impossible on such a theory for a body to acquire a charge of electricity. A cardinal feature in the electrical development of the present theory is on the other hand the conception of intrinsic rotational strain constituting electric charge, which can be associated with an atom or with an electric conductor, and which cannot be discharged without rupture of the continuity of the medium.”

28. G. F. FitzGerald also recognized the possibility or probability that the æther may have a high density; as is shown by the following quotation from his Helmholtz Memorial Lecture before the Chemical Society of London in 1896 (page 347 of his *Collected Papers*) while discussing the vortex theory of matter. After explaining that a liquid with intense internal movement, or kinetic structure, could propagate disturbances like light, provided there were no diffusion of the motion, he goes on to point out that “as vortices move of themselves freely and independently of the density of the liquid, there is no necessity for supposing that the density of the liquid is small. In fact, the simple theory would point to the conclusion that its density may be greater than that of platinum. This would require the energy per cubic centimetre of the turbulent motion to which the propagation of light is due to be very large indeed, approaching a million of millions of kilojoules, *i. e.* it would be equivalent to a million horse-power for 10 hours a day for nearly a month.”

This guess at intrinsic ætherial energy makes it 10^{22} c.g.s. My paper has now estimated it as several thousand million times greater still.

29. At an earlier date, Mr. Heaviside’s ideas on a magnetic analogue or model, based on fluid flow, may be typified by the following extract from a paper of his in the ‘*Electrician*’ for 1883. See ‘*Electrical Papers*,’ vol. i. pp. 275, 277.

“But by far the most interesting analogue is that presented by the motion of a perfect incompressible liquid. Let it fill all space (since the magnetic force of our rectilinear current extends without

limit), and replace lines of force by lines of motion of the liquid, its velocity to everywhere correspond to the magnetic force."

"The liquid within the cylindrical space corresponding to the wire is rotating, every particle of it, with the same angular velocity. The motion of the liquid outside the vortex, however, although the lines of motion are circles about the vortex, is differentially irrotational. . . .

"Current direction corresponds to the axis of a vortex, current-density to twice the angular velocity; vortex lines and tubes, analogous to current lines and tubes, are always re-entrant; the line-integral of magnetic force about a current, measuring the amount of current, corresponds to the circulation of liquid about a vortex, being proportional to the angular velocity of the latter, etc.

"But, not to go into details, we will merely note, further, the analogue of the identity

$$\Sigma \frac{1}{2} B_1^2 = \Sigma \frac{1}{2} A_1 C_1;$$

[p. 244 *ante*]. Here, B_1 being the magnetic force of current C_1 , whose potential is A_1 , we have two expressions for the quantity that there is reason to believe is the kinetic energy of the system of magnetic force. Now, let B_1 be the velocity of a perfect liquid of unit density, then $\Sigma \frac{1}{2} B_1^2$ is really the kinetic energy of the whole motion (sum of mass of each elementary portion $\times \frac{1}{2}$ square of its velocity). We have then the remarkable result that the kinetic energy of the moving liquid can be expressed in terms of its angular velocity in those places where the motion is rotational, and of the corresponding vector-potential."

"Further, since if $C_1=0$ everywhere, making $\Sigma A_1 C_1=0$, it follows that $\Sigma B_1^2=0$, which (being the sum of squares) involves $B_1=0$ everywhere. That is to say, if there is no rotation there can be no motion of the liquid [except a uniform motion not vanishing at infinity]. Hence, if the liquid fills all space, its motion, whatever it may be, cannot be everywhere irrotational." . . .

" $V^2 \pi a^3$ is in fact the whole energy in the case of the moving liquid, including that of the sphere of same density."

30. But going still further back, to a time 35 years ago, Lord Kelvin's representation of a magnetic field as a fluid-flow may be illustrated by the following quotations from a paper of his published in 1872; which is where we got the name 'permeability' from. See his 'Papers on Electrostatics and Magnetism,' pp. 581 and 568.

"It is convenient to attribute to the supposed liquid such a density that 4π times the kinetic energy of liquid permeating a solid of unit permeability, reckoned per unit volume of the whole space occupied by porous solid and liquid, shall be equal to half the square of the 'flux'; the word flux being borrowed from Fourier's theory of the conduction of heat and adapted to the use we have to make of it."

"The whole strength of current in each circuit, reckoned in absolute electro-magnetic measure, is to be equal to the circulation

of the fluid through that aperture divided by 4π . The resultant electro-magnetic force at any point will be numerically equal to the resultant fluid velocity at the corresponding point in the hydrokinetic system."

APPENDIX B.

CONSIDERATIONS RESPECTING A ROTATIONALLY ELASTIC *ÆTHER*.

On showing proof-sheets of the above to Prof. Larmor, he favoured me, just before leaving for America, with a communication concerning the outstanding difficulties which had been felt by some mathematicians concerning the dynamical theory of the *æther* considered as a rotationally elastic medium; mentioning especially a critical remark of Lord Rayleigh, quoted by him in Phil. Trans. 1897 A. p. 230, concerning what would happen if a set of electrons were to circulate continually round a ring fixed in space, so as always to affect and keep in rotation the same identical portion of *æther*, instead of constantly operating on fresh and fresh portions. I quote his letter:—

"If we imagine a permanent magnet *fixed in æther* (through which it is to be observed that the earth is moving at something like 20 miles per second), then the circulation of the *æther* round it which represents its magnetic field, though it is at each instant differentially irrotational, would ultimately bring each element of *æther* round again to its original position, yet with a finite resultant rotation acquired by it. This would mean that the circulation in the *æther*, considered as an elastic medium with simple rotational reaction stress, would in such a case ultimately choke itself by becoming gradually held up against an accumulated rotational reaction. It would seem, therefore, that a magnet *fixed in æther* could not exist in absolute permanence. It is pointed out (*loc. cit.*) that this ultimate elastic choking can be delayed in time indefinitely by taking the inertia of the *æther* very great, a hypothesis which there was no reason for excluding and which was accordingly then adopted.

"If the inertia of the *æther* were no greater than your own (itself immense) estimate, then the statement of the fourth section of your § 14 would seem at first sight to hasten up this time of a complete circulation of the *æther*, close around the moving electron there considered, and on the theory of a purely rotational *æther* the region very close around the electron would not be amenable to the ordinary circuital relations of electrodynamic theory, but would be governed by modified relations of form which might perhaps be worked out. There does not seem to be anything definitely known that would absolutely exclude this way out of the difficulty; for, needless to say, the representation of an electron as a small conducting sphere surrounded by free *æther* and with a charge on it, deformable or not, is merely an illustration; and the interesting calculations that can be made about it must not obscure the fundamental fact that it implies a reversal of the order of ideas,

which was to explain electricity by electrons and not to explain electrons by electricity. The only thing experimentally known in this domain seems to be Kauffmann's result, which requires that an electron is some structure expressible analytically as formally *equivalent* to a spherical aggregation of 'electric charge' with invariable volume relations, but with 'density of charge' that may vary, according to any law whatever, in terms of the distance from the centre. For other electrodynamic purposes this structure can be treated simply as if it were an aggregate charge concentrated in a point.

"The case has here been put at its worst, and this aspect of it, along with unusual features in the phenomena of mechanical reaction, has in fact led M. Poincaré* and others following to ascribe insuperable difficulties to a rotationally elastic representation of the æther, notwithstanding the direct and natural representation which it gives of the field of an electron, as a self-locked stress-system keying on to it.

"But as I have been in the habit of explaining in academic lectures for some years, though I have not found time to explain the point in print, the case is far from being so hopeless. It may or may not be true that a permanent magnet (so-called), if absolutely at rest in æther, would through lapse of time have its magnetic field gradually choked: there is no direct means of knowing, so long as we must rush through space along with the earth at 20 miles per second. Take, however, your own supposition of an electron rushing through space and having a circulating movement of æther *relative to itself* which is of the same order of velocity close up to the electron as its velocity of translation. If this were steady actual circulation of the æther, instead of merely apparent circulation relative to the moving electron, such an idea of an electron would be blankly impossible on a rotationally elastic theory. But on consideration it will appear there is in fact no cumulative circulation at all. The electron is constantly shedding off and leaving behind it, as a sort of wake in the stationary æther, these beginnings of circulatory motion which constitute its magnetic field, and thus the choking process never gets a start: in fact in the nature of the case it *can never* get a start (if the inertia of æther is sufficiently great), for the very essence of a magnetic field is that it is a disturbance which arises from the electrons moving on, and thus of necessity leaving it to collapse behind them.

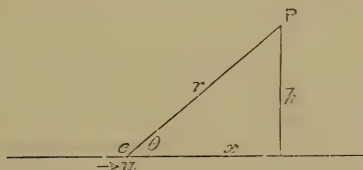
"Problems in hydrodynamics of the general character suggested by these remarks have been dealt with long ago by Clerk Maxwell ("On the Displacement in a Case of Fluid Motion," Proc. London Math. Soc. iii., or Scientific Papers, vol. ii. pp. 208-214). He there considers the special case of a circular cylinder progressing with uniform velocity through quiescent frictionless fluid: the

* See the discussion at the end of the second edition of his *Electricité et Optique*, where the theory was by misunderstanding, doubtless due to its mode of presentation, contrasted with that of Lorentz, the two being in fact formally equivalent, as is now recognized.

stream lines *relative to the cylinder* are, as is well known, steady and beautifully simple; but the actual path of the particles of this fluid as they are put into motion by the neighbourhood of the cylinder and finally left at rest in a displaced position after the cylinder has passed on, is a very different affair, as may be seen from the traces given in Maxwell's paper.

"As your proof-sheets and queries have forced me back on this subject, I hope to arrange for obtaining some early investigation of the corresponding magnetic problem, viz. the displacement and strain in a rotational æther due to an electron rushing through it, on the usual kind of working hypothesis that the electron is representable as if it were a small charged body."

P.S. "On second thoughts the total displacement of position of an element of æther at P due to an electron e passing near it at distance h with uniform velocity v , is readily worked out. For it



takes place around the axis of motion of the electron; and, if velocity of æther = $k \times$ magnetic force, we have

$$\begin{aligned} \text{arc of displacement} &= k \int \frac{eu \sin \theta}{r^2} dt, \text{ where } udt = dx \\ &= k \int \frac{e}{h} \sin \theta \cdot d\theta \\ &= \frac{2ke}{h}, \text{ independent of } u; \end{aligned}$$

and the angle turned through in traversing this arc = $\frac{2ke}{h^2}$.

"This rotational *total* displacemement of the element of æther, arising from a motion *differentially* irrotational, is greatest when h is smallest. When h is but little greater than the effective radius a of the electron, the analysis is not adequate, for the electron cannot then be treated as a point.

"It will, however, still give the order of magnitude of the rotation; so putting $e=10^{-21}$, $h=a=10^{-14}$ cm. it gives $2 \cdot 10^7 k$ radians*. If this is to be small, k must be at least of the order 10^{-9} ; so that unit c.g.s. magnetic field would be represented by an æther-velocity of not more than 10^{-9} cm. per sec. This would require the density of æther to exceed say 10^{16} . But observe that the total arcual displacement, being proportional to h^{-1} , can mend itself (as is of course

* The numerical index of each of these estimates would by me be reduced by unity; but it does not matter for the main lines of the argument.—O. L.

known) by an irrotational motion of the æther having velocity-potential of form $A \log h$; thus in its aggregate amount it involves no elastic reaction, and disappears spontaneously by irrotational flow when the electron has passed on. Thus, even with much lower effective inertia of the æther, the passage of the electron would not involve any permanent elastic dislocation in its surroundings, though very near it there might be increased elastic reaction equivalent to increase of its effective size: it being understood that so long as we do not know how the *core* of the electron is keyed together, the nature of its own penetration through æther—whether it be analogous to the motion of a vortex ring or otherwise—is beyond our cognisance, and must remain a matter of immediate hypothesis.”

From this note a deduction at once follows, thus:—

The angular displacement at any distance h from path of electron, after it has passed, is $\phi = 2ke/h^2$; so the linear velocity of æther circulation at that place, while the electron is passing, is

$$w = kH = \frac{keu}{h^2} = \frac{1}{2}\phi u.$$

Everywhere, except close to an electron, ϕ is extremely small; but I see no reason for attributing to ϕ any but a finite value (*i. e.* something comparable to a radian) quite close to an electron. If that is permissible, the above little calculation is confirmatory of the hypothesis that w_0 , the rotation speed round the equator, is of the same order as u , the translation speed of an electron. But Larmor supposes that the angular shift must be small everywhere. In that case w_0 would be less than u , and the estimate of ætherial density, together with the values of the other extraordinarily large quantities in this paper, would have to be correspondingly *increased*.

APPENDIX C.

The following brief note from Professor J. J. Thomson is of special interest:—

“Cavendish Laboratory, 13 March, 1907.

“Many thanks for letting me see the proof of your paper. I have held for some time the view that the mass of bodies is the mass of æther they carry along with them, and that this mass can be carried along by the lines of electric force as well as the bodies these lines connect. The amount carried by the small negative end is small compared with that carried by the lines of electric force, while the amount of the æther carried by the large positive end exceeds that carried by the lines; hence the mass of the corpuscle seems wholly electrical, while that of the atom does not. I think, however, both masses are identical in origin, being masses of portions of æther. I went into this view at some length in my ‘Electricity and Matter,’ and have been confirmed in it by later consideration. Of course it requires a very dense æther.”

It is satisfactory thus to find a fair consensus of leading opinion in favour of some such proposition as I have endeavoured to argue out, and make more precise, in the above paper.

XL. *The Influence of the Velocity of the α particle upon the Stopping Power of the Substance through which it passes.*
By W. H. BRAGG, M.A., Elder Professor of Mathematics and Physics in the University of Adelaide.*

A PAPER "On the α particles of Radium and their loss of range in passing through various atoms and molecules" was published by Mr. Kleeman and myself in the *Philosophical Magazine* for September 1905. On p. 337 of this paper it was pointed out that the ionization curve obtained when a metal sheet was placed over the radium did not show exactly the same drop from the normal throughout its entire length: the slower α particles were a little less affected than the swifter. That is to say, the α particle ejected by RaC lost rather more of its range in going through a plate placed over the radium than the particle from Ra itself; the former particle being the swifter of the two. We remarked that this point deserved further enquiry.

The question has been further discussed in a paper by Kucera and Masek, "Ueber die Strahlung des Radiotellurs II.," pp. 630-640, of No. 19 of the *Physikalischen Zeitschrift* for 1906. In § 7 of their paper they state that the stopping-power of Al is lowered by 10 per cent., and that of Pt by 12.5 per cent. if the rays, instead of entering the metal directly after leaving the radiotellurium, first pass through 1.9 cm. of air. Since the α particle would lose about 11 per cent. of its velocity in crossing such a layer, the authors suggest that probably the stopping-power is proportional to the velocity of the α particle (see also paragraph 7 of their summing up).

This hypothesis will of course explain the effects observed by Kleeman and myself. But in the form in which it is stated by Kucera and Masek it seems to me to be inconsistent with certain other experiments made by Rutherford, McClung, and Levin, and indeed by the authors themselves.

I think this is clearly seen if the various results are set out as follows:—

1. Rutherford has shown (*Phil. Mag.* Aug. 1906) that the α particle loses energy at a uniform rate on its way through aluminium. Rutherford measured the velocity of the particle directly; so that his experiments may be said to be absolute as compared with those made by Kleeman and myself. Our measurements were relative, since we determined the energy

* Communicated by the Author. Read before the Australasian Association for the Advancement of Science, Jan. 9, 1907.

indirectly, by finding the thickness of the air stratum which the particle could still penetrate.

2. It has been shown by McClung, by Levin, and by Kucera and Masek (§ 1 of the paper quoted), that each layer of Al foil which is laid upon a radioactive surface diminishes the range of the α particle by an equal amount. That is to say, if one layer cuts off $\cdot 5$ cm., two cut off 1 cm., three $1\cdot 5$ cm., and so on. Taken in conjunction with Rutherford's experiment this implies that in air, as in aluminium, the loss of energy in crossing a given layer is independent of the speed of the particle, for if the loss of range in air is proportional to the thickness of the foil, then the relation between speed and stopping-power must be the same in metal as in air.

3. Kucera and Masek describe experiments from which it would appear that the range lost by an α particle in going through two sheets of dissimilar metals such as Al and Au or Al and Pt (§ 6 of the paper quoted), is independent of the order in which it passes through them. Now if it crosses the Al first, its velocity whilst passing through this metal is greater than when passing through the gold, and *vice versa*. Since speed is a matter of indifference when it crosses the Al, it must be so also when crossing the Au or Pt. Therefore for these metals also the stopping-power is independent of the speed of the particle.

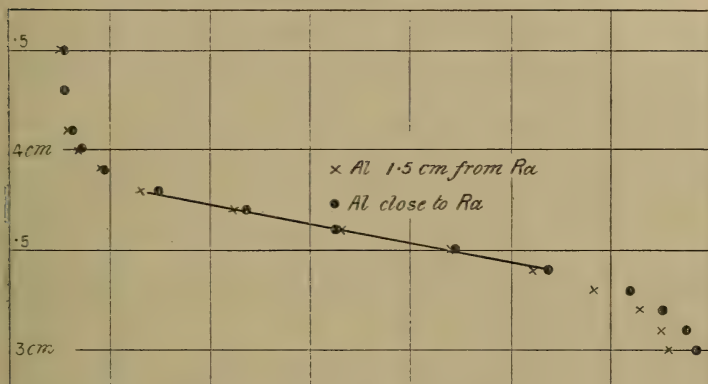
We have thus reached a conclusion which is directly opposed to that of Kucera and Masek; and it is clear that there must be an error somewhere.

Some experiments which I have made recently seem to suggest a probable explanation of the inconsistencies. In agreement with the work of Kucera and Masek, and with the original experiments of Kleeman and myself, they show that the stopping-power of a metal sheet is not independent of the speed. But they show also that this effect is not equally great in different metals; it is small in Al, but larger in Au and Pt. It follows that the reversal of two layers of different metals must alter the stopping-power of the combination; and this I find to be well shown under suitable circumstances.

Let us first consider some experiments with aluminium. The behaviour of this metal is of special interest since it was used by Rutherford in the research already mentioned (Phil. Mag. Aug. 1906). My experiments show, on the whole, that the velocity of the α particle has only a small influence on the loss of range in going through a given sheet. It is fairly clear that there is some influence; but the experimental errors are often large enough to hide it.

In fig. 1 are two ionization curves, due to the α particles of RaC. One of these shows the readings taken when a certain Al sheet, equivalent to about 3.2 cm. of air, was placed close to the radium; the other, the readings when the

Fig. 1.—Showing that the stopping-power of a certain Al sheet is not altered by varying its distance from the radium.



sheet was 1.5 cm. from the radium. The two curves almost lie on top of one another; yet in the first case the α particles entered the plate at a higher speed than in the second, when they had to cross 1.5 cm. of air before reaching the plate.

In a second experiment the drop of the curve due to placing eight layers of Al foil over the radium was first measured. The average range was cut down from 6.82 cm. to 6.04 cm., or .78 cm. When a sheet of tin-foil was placed over the radium so as to reduce the speed of the α particle, the same Al foil cut down the range from 5.22 to 4.425, or .795 cm. The speed of the α particle was again reduced by the addition of a second sheet of tin-foil. The Al then cut down the range from 3.63 cm. to 2.86 cm., or .77 cm. Thus no obvious variation occurred in the effect of the Al foils, though the speed of the α particle differed considerably in the three cases. This agrees with the results of McClung and Levin, and with the experiment described in the first paragraph of the paper of Kucera and Masek already quoted.

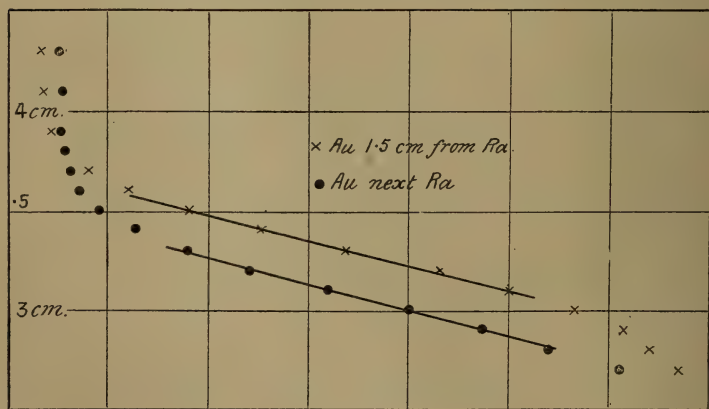
On the other hand, I found that when the Al foil was placed over the radium and the whole ionization curve was obtained, it did not show the same drop in all four sections. The drop for RaC was found to be .79 cm. (agreeing with the results already given), whereas that for Ra was only .725 (3.185 to 2.460). There is here a difference of over

half a millimetre, which is well outside the limits of experimental error, provided there is no mistake in the cause to which it is ascribed. At this point I ought to say that the method, in which the whole ionization curve of radium is used, generally shows more of the effect under discussion than the methods referred to earlier in this paper. When the loss of range of the α particles of RaC is compared with the loss of range of those of Ra, more difference between the two losses is generally found than if the former loss is compared with that of α particle of RaC whose velocity has been cut down so as to equal the velocity of the particles of Ra itself. I have not yet been able to find a reason for this: the work is of course on the extreme boundary of what can be done under present experimental conditions. There is certainly no justification at present for supposing that there is here any proof that the α particles of RaC and of Ra are dissimilar in nature or condition; but when greater accuracy is attainable it will be proper to make a careful examination of the question.

This point being reserved, it will be clear that on the whole the stopping-power of aluminium relative to air is at least nearly independent of the velocity of the α particle.

The effects differ considerably in the case of gold. In fig. 2 are shown two curves representing the results of an

Fig. 2.—Showing that the stopping-power of a certain gold sheet is altered by varying its distance from the radium.



experiment which differed from that recorded in fig. 1 in one particular only, viz. that Au was used in place of Al.

But in fig. 1 the two curves are almost identical; in fig. 2 they are $\cdot 23$ cm. apart.

In another experiment 28 thicknesses of gold-leaf were placed over the radium, and the whole ionization curve was determined. It was found that the drop for RaC was 1.13 cm. for RaA 1.01 cm., for the emanation $\cdot 98$ cm., and for Ra $\cdot 93$ cm.

In yet another experiment the velocity of the α particles of RaC was cut down by putting an Al sheet over the radium, so that the average range became 3.63; the same gold-foils then cut it down to 2.70, the drop being therefore $\cdot 93$ cm. In this case, therefore, the two methods agreed, and there was not, as usual, a greater effect obtained when using the particles of Ra than when using those of RaC whose velocity had been reduced to that of the particles from Ra.

Thus it is clear that if the effect under discussion is small in the case of Al, it is larger in the case of Au. The stopping-power of Au increases with the speed of the α particle.

It is now of some interest to consider certain results obtained by Kleeman and myself two years ago; they relate to the drop in various parts of the full radium ionization curve, and are set out in the following table:—

TABLE.

Showing drop in various parts of the ionization curve when different metal sheets were placed over the radium.

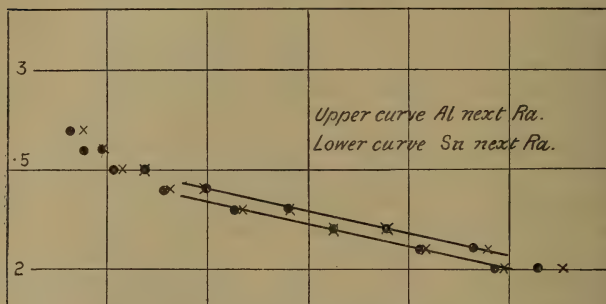
	Ra C.	Ra A.	Emanation.	Ra.
1. Al	2.31	2.25		
2. Sn	1.92	1.77	1.67	
3. Sn	1.91	1.81		
4. Ag	$\cdot 86$	$\cdot 81$	$\cdot 80$	$\cdot 73$
5. Au	1.63	1.46	1.36	

From these results it appears to be not improbable that the effect under consideration increases with atomic weight.

If the stopping-powers of metals are affected in different degrees by variations in the speed of the particle, then the stopping-power of a pair of metal sheets, say Al and Au, must depend on the order in which the α particle passes through them. This deduction furnishes a good test of the accuracy of the preceding experiments and arguments. I have therefore tried to examine the matter carefully; the more so, as Kucera and Masek did not find the effect.

In one experiment I used two thicknesses of tin-foil equivalent to about 3 cm. of air, and 16 thicknesses of Al foil equivalent to about 1.5 cm. of air. When the Al foil was next the Ra the reduction of range was less by .06 cm. than when the Sn foil was next the Ra. The readings are given in the following table and plotted in fig. 3. The foils were

Fig. 3.—Showing that the stopping-power of a pair of metal sheets Al and Sn is altered by inverting the sheets.



reversed three times, so that the curve was found four times, as recorded in columns I., II., III. and IV.

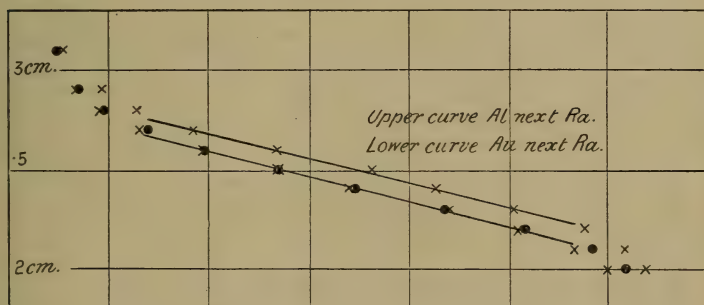
Distance from radium to ionization chamber.	I. Sn next radium.	II. Al next radium.	III. Sn next radium.	IV. Al next radium.
2.0	495	554	487	529
2.1	416	479	411	464
2.2	327	379	324	378
2.3	234	283	227	279
2.4	163	196	155	197
2.5	115	137	104	137
2.6	89	94	75	92
2.7	76	69	63	67
3.0	63	53	45	47
3.3	56	45	45	43

The figures in the first and third columns agree closely; so also do those in the second and fourth. Thus the errors of experiment are much less than the effect observed; this is also obvious from the figure.

Al and Au show the effect better than Al and Sn. The results given in the following table were obtained with an Au sheet equivalent to about 3.4 cm. of air and the same eight thicknesses of Al foil. They are plotted in fig. 4. The two curves are a whole millimetre apart.

Distance from radium to ionization chamber.	I. Au next radium.	II. Al next radium.	III. Au next radium.
2.0	598	637	616
2.1	564	614	584
2.2	509	576	514
2.3	440	505	435
2.4	340	426	347
2.5	271	364	268
2.6	194	267	196
2.7	132	185	140
2.8	91	128	96
2.9	68	94	69
3.1	48	57	50
3.3	44	47	38
3.5	38	43	37

Fig. 4.—Showing that the stopping-power of a pair of metal sheets Au and Al is altered by inverting the sheets.



In another experiment, in which the Al was equivalent to about 3.2 cm. and the gold consisted of the 28 foils already referred to, and equivalent to about 1 cm. of air, the average range of the α particle of RaC after passing through Au first and Al afterwards was 2.575, whereas when it passed through the metals in the inverse order it was 2.655. In this case there was therefore a drop of .08 cm.

It is of course a differential effect that is observed in these cases. When the Au is placed above the Al it is equivalent to taking it nearly 3 cm. further away from the radium, making, that is to say, the same difference between the velocities of the α particles used in the two cases as exists between the velocities of the particles of Ra and RaC. Now it has already been stated that this set of gold-foils cut down

the range of the RaC particles by 1.13 and of the Ra particles by .93, the difference being .2 cm. This means that when the effects of gold are compared with those of air, we obtain a larger difference than when they are compared with those of aluminium; from which we should expect that the stopping-power of the latter metal varies to some extent with the speed of the α particle.

To sum up, it is clear that the stopping-power of a metal increases with the speed of the α particle, and the more so the greater the atomic weight of the metal. The effects are small, and not easy to measure accurately; so that, although there is not a complete agreement between the results obtained by different methods, yet it is possible that the inconsistencies may disappear with more perfect measurements.

It was observed by Madame Curie that when a pair of metal sheets was used as one of the walls of the ionization chamber, the ionization current was varied by the inversion of the sheets. Rutherford ('Radioactivity,' p. 188) ascribed this to differences in the secondary radiation from the various metals. Kucera and Masek have rejected Rutherford's explanation, and supposed the effect to be due to a differential scattering of the α rays (*Phys. Zeit.* vii. 19. pp. 650-654). But it is clear that an effect of this kind ought to occur if the range of the α particle after passing through the sheets is altered by the inversion; and before ascribing it to secondary radiation or scattering of the α rays it must be made clear that it cannot be completely explained in this direct way.

It is not easy to calculate the change that should be expected in Madame Curie's form of the experiment. For when a radioactive plate is placed at a little distance from an aperture of about the same size as the plate, geometrical difficulties are introduced into the calculation, since it is necessary to find the solid angles subtended by the aperture at various points of the plate. The difficulties largely disappear if the metal sheets are laid directly on the radioactive surface, and the ionization chamber is large enough in every way to allow the α particles to complete their paths therein.

I have carried out an experiment under these conditions, using a deposit of RaC on a small piece of foil, a sheet of Al equivalent to 3.35 cm. of air, and the 28 gold-foils already referred to. The metals were inverted at regular intervals of four minutes, and the successive readings were:—

	Al next RaC.	Au next RaC.
1	339	
2		309
3	305	
4		253
5	246	
6		214

$$\text{Now } \sqrt{(339 \times 305)}/309 = 1.04; 3.05/\sqrt{(309 \times 253)} = 1.09$$

$$\sqrt{(305 \times 246)}/253 = 1.08; 2.46/\sqrt{(253 \times 214)} = 1.06$$

The mean of these results is 1.07. Thus the ionization when the Au was uppermost is 7 per cent greater than when it was next the RaC.

Now, as already stated, when these two particular metal sheets were inverted, there was a difference in the range of the α particle after passing through them equal to .08 cm. In this case the metals were at a distance of 1.5 cm. from the radium. In order to calculate from this result a value of the difference in the ionizations, for comparison with the experimental value of 7 per cent., it is necessary to allow for the fact that when the latter value was obtained the metals were in contact with the radioactive surface, and, moreover, many of the rays traversed the plates at a considerable inclination to the normal. To do this it is necessary to make some assumption as to the law connecting stopping-power with speed. For this reason, and because the calculation is rather lengthy, and the data not so reliable as might be wished, I will only state here that I found the calculated difference in the ionizations to be about 11 per cent. Therefore in this particular experiment the difference in range when the plates were inverted was quite enough to account for the observed difference in the ionizations.

This may also be seen on general considerations. The α particle that passed normally through the plates completed afterwards a range of about 2.5 cm. in air. The difference of range on inversion was at least .08 cm. Thus the difference in the ionization caused by the normal rays was at least .08/2.5 or 3 per cent. But many of the α particles crossed the plates obliquely; indeed, the relative importance of the oblique to the normal radiation is a striking feature of the α rays. In the case of this important part of the radiation the paths through the metals were longer, and therefore the differences of range on inversion must have been greater. On the other hand, the subsequent paths in the air were less; and therefore the differences of range were of still greater

importance, relatively. It is clear, therefore, that the observed difference of 7 per cent. in the ionizations can easily be accounted for.

This is of course but one instance. Yet so far as it goes it leads to the conclusion that the effects observed by Madame Curie, and by Kucera and Masek, can be explained by ascribing them to the slight alteration in the stopping-power of two metals when inverted. We should expect, on this hypothesis, the alteration to be relatively greater the thicker the plates, the greater the distance from the radioactive material, and probably also the greater the contrast between the atomic weights of the metals. This agrees very well with actual observation.

Kucera and Masek ascribe the effects to a difference in the amount of the scattering of the rays by the two metals. I do not quite follow their argument as illustrated by the diagram in § 3 of their third paper (*Phys. Zeit.* vii. 19. pp. 650-654). They point out that if the scattering power of two metals is unequal, then there must be a difference in the paths of the outside rays of the pencil due to inversion of the plates. But the scattering is small, and this differential effect can only be of the second order of small quantities. Also the outside rays are not fairly representative of all the rays. And again, they compare the passage of a bundle of α rays through air or metal to the passage of a ray of light through a somewhat turbid medium. But the effect of two such media upon a ray of light which passes through them in succession is independent of the order in which it does so.

If the stopping-power of a substance depends upon the speed of the α particle, it is clear that the proportionality between the stopping-power and the square root of the atomic weight cannot be exact, except possibly at some particular speed. But the stopping-power does not vary much with the speed; and the general truth of the law is not affected by any of the results here described.

It may be added that when the stopping-power of some metals were measured by Kleeman and myself, we placed the sheets about 1.5 cm. from the radium. They were usually equivalent to about 3 cm. of air, so that they were fairly in the centre of the range of the particle of RaC, and the average velocity of the particle when crossing them was nearly equal to the average velocity of the particle throughout its whole course.

XLI. *On the Saturation Constants according to van der Waals's Equation.* By JOHN P. DALTON, M.A., *Carnegie Research Scholar* *.

A. *Deduction of Constants.*

FOR the deduction of the saturation constants θ , π , ω_1 and ω_2 , the following different methods are available and have been used :—

1. M. Planck's method for the equation of Clausius may be adapted to that of van der Waals. For this purpose an independent variable ϕ is chosen, such that

$$3\omega_1 - 1 = 3r \sin^2 \frac{1}{2}\phi, \quad \text{and} \quad 3\omega_2 - 1 = 3r \cos^2 \frac{1}{2}\phi.$$

Since

$$\frac{8\theta}{3\omega_1 - 1} - \frac{3}{\omega_1^2} = \pi = \frac{8\theta}{3\omega_2 - 1} - \frac{3}{\omega_2^2},$$

hence

$$r = -\frac{2}{3} \frac{1}{\sin^2 \phi} \left[\frac{\cos \phi + \sin^2 \phi \cdot \log \tan \frac{1}{2}\phi}{\log \tan \frac{1}{2}\phi + \cos \phi} \right].$$

Also

$$\pi = \frac{3\omega_2\omega_1 - \omega_2 - \omega_1}{\omega_1^2\omega_2^2}.$$

and

$$\theta = \frac{(\omega_1 + \omega_2)(3\omega_2 - 1)(3\omega_1 - 1)}{8\omega_1^2\omega_2^2}.$$

When $\phi = 90^\circ$, $\omega_1 = 1 = \omega_2$, hence values of ϕ must so be chosen that $90^\circ > \phi > 0$.

This method was used by A. Stoletow ‡ who calculated pressures only, and gave the following results :—

ϕ .	θ .	π .
80°	0.9960	0.984
70	0.9861	0.945
65	0.9780	0.914
60	0.9677	0.876
55	0.9549	0.829
50	0.9396	0.776

* Communicated by Prof. J. P. Kuenen, Ph.D.

† M. Planck, *Wied. Ann.* Bd. xiii. pp. 535-543 (1881).

‡ A. Stoletow, *J. de russ. phys. Ges.* xiv. p. 167 (1882).

518 Mr. J. P. Dalton on the Saturation Constants
and also by A. Batschinski *, who calculated volumes only:—

ϕ .	θ .	ω_1 .	ω_2 .
90°	1	1	1
40°	0.89894	0.6021	2.3625
25°	0.80247	0.5190	4.1103
15°	0.69682	0.4659	7.9828
8°	0.58213	0.4275	19.585
4°	0.48222	0.4028	57.30
2°	0.40712	0.3877	178.8
0° 15'	0.30723	0.3709	1974

2. The constants may be obtained from the *pv*-isothermals using Maxwell's criterion. This method was employed by H. Hilton †, who gives the following table:—

θ .	π .	ω_1 .	ω_2 .
0.937	0.773	0.660	1.84
0.875	0.587	0.573	2.53
0.844	0.486	0.545	3.31
0.750	0.281	0.495	4.83
0.625	0.090	0.441	—
0.500	0.048	0.418	—

3. They may also be obtained graphically by means of the thermodynamic potential. E. Riecke ‡ using this method obtained the following:—

θ .	π .	ω_1 .	ω_2 .
0.60	0.109	0.432	16.60
0.65	0.137	0.448	10.90
0.70	0.199	0.466	7.75
0.75	0.288	0.489	5.53
0.80	0.384	0.517	4.14
0.85	0.507	0.553	3.05
0.90	0.650	0.602	2.32
0.95	0.807	0.693	1.78

4. A second graphical method is given by the use of the free energy. This method was adopted by Kamerlingh

* A. Batschinski, *Zeits. f. phys. Chemie*, xli. p. 743 (1902).

† H. Hilton, *Phil. Mag.* [6] i. p. 579, and ii. p. 803 (1901).

‡ E. Riecke, *Wied. Ann.* liii. p. 379 (1894).

Onnes *, but he does not communicate any results. It was used independently by the author. Isothermal ψ , v -curves were constructed by means of the equation

$$-\frac{1}{p_K v_K} \psi = \frac{8\theta}{3} \log_e(3\omega - 1) + \frac{3}{\omega},$$

and a double-tangent drawn to each curve. The slope of these tangents gave the values of π . As it was difficult to read volumes exactly from the curve, values of π were substituted in the "reduced" equation of state, and the volumes determined by calculation. The tabulated results were:—

θ .	π .	ω_1 .	ω_2 .
0.50	0.0280	0.4067	45.62
0.60	0.0869	0.4326	16.72
0.65	0.1364	0.4485	11.16
0.70	0.2007	0.4671	7.799
0.75	0.2825	0.4896	5.642
0.80	0.3840	0.5174	4.162
0.85	0.5058	0.5532	3.113
0.90	0.6494	0.6029	2.337
0.95	0.8095	0.6854	1.744
1.00	1.0000	1.0000	1.000

5. Clausius's method † applied by him to the equation

$$\frac{p}{RT} = \frac{1}{v - \alpha} - \frac{27(\alpha + \beta)}{8\theta(v + \beta)^2},$$

may be easily adapted to the present purpose and the table given by him ‡ (which appears to be very accurate) may be used. He calculates at intervals of $\theta = 0.01$ the corresponding values of $\frac{\Pi}{\Pi_K}$, $\frac{w}{w_K}$, and $\frac{W}{W_K}$, where $\Pi = p/RT$, $w = v_1 - \alpha$, and $W = v_2 - \alpha$.

By comparing his equation with that of van der Waals expressed in the form

$$\frac{p}{RT} = \frac{1}{v - b} - \frac{a}{RTv^2},$$

it will be seen that the saturation constants π , ω_1 , and ω_2

* Kamerlingh Onnes, *Comm. Leid.* no. 66, p. 10 (1900).

† R. Clausius, *Wied. Ann.* Bd. xiv. pp. 279-290 and 692-704 (1881).

‡ *Ibid.* pp. 694-5.

according to van der Waals's equation may be derived from Clausius's table by means of the relations

$$\pi = \theta \frac{\Pi}{\Pi},$$

$$\omega_1 = \frac{2 \frac{v}{w_K} + 1}{3},$$

and

$$\omega_2 = \frac{2 \frac{W}{W_K} + 1}{3}.$$

As these constants are frequently used, the full table is here given:—

θ .	π .	ω_1 .	ω_2 .
0.20	0.000001180	0.35584	448520
1	2646	0.35714	211560
2	5478	0.35846	107120
3	0.000010626	0.35980	57670
4	19536	0.36116	32760
5	34175	0.36254	19509
6	57226	0.36395	12111
7	92232	0.36537	7801.7
8	0.00014367	0.36682	5193.5
9	21701	0.36830	3560.3
0.30	31881	0.36980	2505.9
1	45688	0.37133	1806.1
2	64006	0.37288	1330.1
3	87849	0.37446	998.60
4	0.0011835	0.37608	733.13
5	15673	0.37772	592.61
6	20435	0.37939	466.98
7	26265	0.38109	372.93
8	33318	0.38283	301.49
9	41753	0.38460	246.49
0.40	51744	0.38641	203.63
1	63464	0.38825	169.83
2	77095	0.39013	142.89
3	92833	0.39206	121.20
4	0.011085	0.39402	103.58
5	13134	0.39602	89.147
6	15451	0.39807	77.227
7	18054	0.40017	67.300
8	20966	0.40231	58.969
9	24205	0.40451	51.947
0.50	27789	0.40675	45.984
1	31738	0.40905	40.891
2	36073	0.41141	36.517
3	40812	0.41383	32.741
4	45975	0.41630	29.465
5	51581	0.41884	26.609
6	57646	0.42145	24.109
7	64188	0.42413	21.913
8	71230	0.42687	19.973
9	78783	0.42971	18.256

θ .	π .	ω_1 .	ω_2 .
0.60	0.086868	0.43262	16.730
1	95508	0.43560	15.366
2	0.10470	0.43869	14.147
3	11448	0.44186	13.052
4	12485	0.44513	12.067
5	13584	0.44851	11.177
6	14746	0.45201	10.371
7	15971	0.45561	9.6387
8	17262	0.45934	8.9733
9	18620	0.46319	8.3660
0.70	20046	0.46719	7.8113
1	21541	0.47135	7.3027
2	23108	0.47565	6.8355
3	24747	0.48013	6.4057
4	26459	0.48478	6.0093
5	28246	0.48963	5.6431
6	30109	0.49469	5.3041
7	32048	0.49998	4.9896
8	34064	0.50551	4.6978
9	36160	0.51131	4.4259
0.80	38335	0.51741	4.1725
1	40593	0.52382	3.9356
2	42933	0.53058	3.7141
3	45353	0.53773	3.5064
4	47859	0.54531	3.3113
5	50449	0.55336	3.1276
6	53126	0.56195	2.9545
7	55886	0.57117	2.7910
8	58736	0.58107	2.6361
9	61674	0.59176	2.4889
0.90	64700	0.60339	2.3487
1	67816	0.61615	2.2151
2	71024	0.63019	2.0867
3	74319	0.64592	1.9633
4	77708	0.66368	1.8438
5	81188	0.68412	1.7271
6	84762	0.70817	1.6117
7	88429	0.73746	1.4957
8	92192	0.77535	1.3757
9	96048	0.83028	1.2418
1.00	1.00000	1.00000	1.0000

B. Comparison with Empirical Formulæ.

(1) VAPOUR-PRESSURE LAW.

The values of π given above were used to test the well-known empirical vapour-pressure law of van der Waals, viz.,

$$\log \pi = f \left(1 - \frac{1}{\theta} \right),$$

where f is a constant. Onnes* found that the pressures he obtained by the "free energy" method obeyed the above law

* *Loc. cit.*

with "fair approximation," and gave f (at low temperatures) a value = 3.4, when $\log \pi$ is expressed in natural logarithms.

In the following, the constant has been calculated for both ordinary and natural logarithms :—

θ	$-\log_{10} \pi$	$-\left(1 - \frac{1}{\theta}\right)$	f'	f
0.20	5.9281	4.0000	1.4820	3.4124
.25	4.4663	3.0000	1.4888	3.4280
.30	3.4965	2.3333	1.4985	3.4504
.35	2.8049	1.8571	1.5103	3.4776
.40	2.2861	1.5000	1.5241	3.5094
.45	1.8816	1.2222	1.5395	3.5449
.50	1.5561	1.0000	1.5561	3.5830
.55	1.2875	.81818	1.5736	3.6234
.60	1.0611	.66667	1.5917	3.6650
.65	.86697	.53846	1.6101	3.7073
.70	.69797	.42857	1.6286	3.7500
.75	.54904	.33333	1.6471	3.7925
.80	.41640	.25000	1.6656	3.8352
.85	.29715	.17647	1.6839	3.8773
.90	.18910	.11111	1.7019	3.9188
.95	.09051	.05263	1.7197	3.9597
1.00	.00000	.00000	—	—

Fig. 1 shows graphically the deviation of f from a constant.

(2) THE LAW OF CAILLETET AND MATHIAS.

Onnes * found that for the saturation volumes determined by him the law of the diameter held "very well up to quite low temperatures," and A. Batschinski † obtained by plotting mean density against temperature a line that did not "differ much from a straight line."

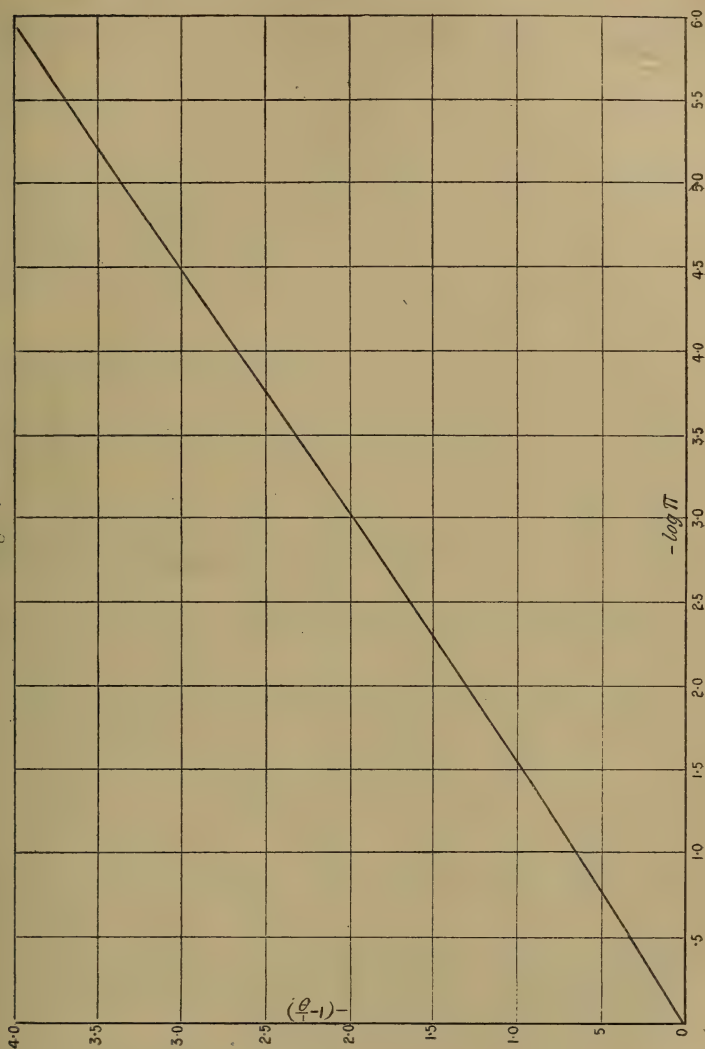
From the table of saturation constants given above, the following figures were derived :—

θ	$\frac{1}{\omega_1} + \frac{1}{\omega_2}$	θ	$\frac{1}{\omega_1} + \frac{1}{\omega_2}$
[0.00	3.0000]	.60	2.3713
0.20	2.8103	.65	2.3191
.25	2.7584	.70	2.2685
.30	2.7046	.75	2.2196
.35	2.6492	.80	2.1724
.40	2.5928	.85	2.1268
.45	2.5363	.90	2.0831
.50	2.4802	.95	2.0407
.55	2.4251	1.00	2.0000

* *Loc. cit.*

† *Loc. cit.*

Fig. 1.



It is known * that the equation

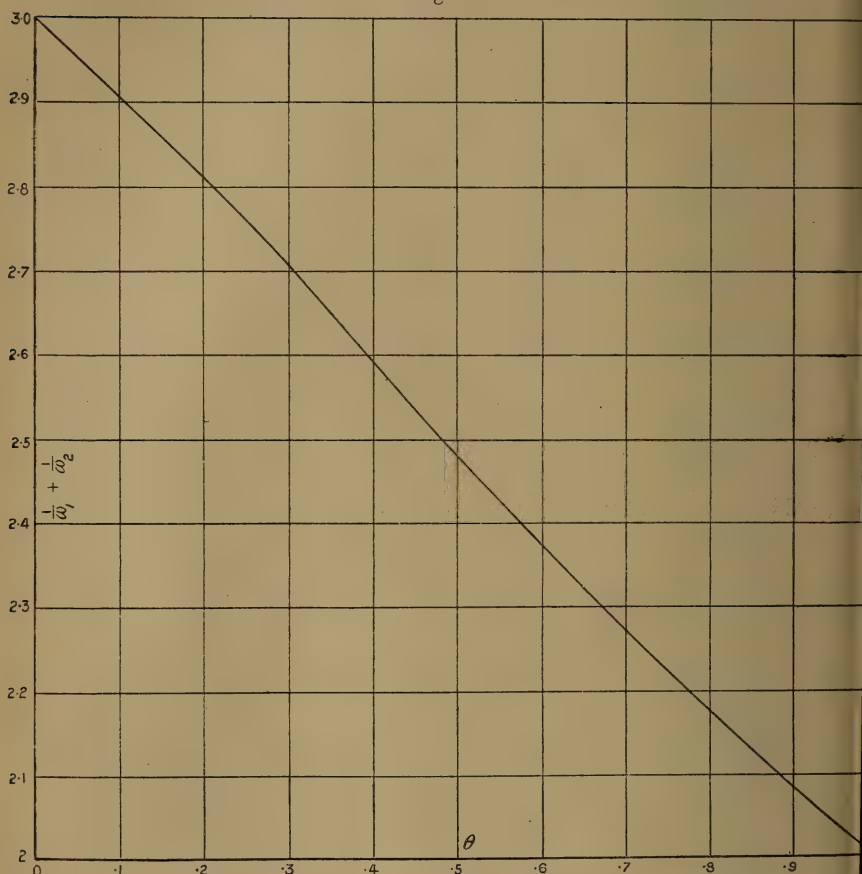
$$\frac{1}{\omega_1} + \frac{1}{\omega_2} = a + b\theta$$

is not an accurate expression of the results of observation, but the curvature shown by plotting the foregoing numbers

* S. Young, Phil. Mag. [5] 1. pp. 291-305 (1900).

(fig. 2) is more marked than that given by experimental results.

Fig. 2.



S. Young (*loc. cit.*) finds as a more correct formula

$$D_t = D_0 + \alpha t + \beta t^2.$$

For all substances α is negative; β is zero only when the ratio of the actual to the theoretical density at the critical point has the normal value 3.77. When this ratio < 3.77 , β is positive and the curve is convex to the temperature-axis. This is quite in accordance with fig. 2, where the ratio has the value 2.67.

University College, Dundee,
February, 1907.

XLII. *On the Variation of Specific Heats with Temperature and Density.* By JOHN P. DALTON, M.A., *Carnegie Research Scholar* *.

A. *According to van der Waals's Equation.*

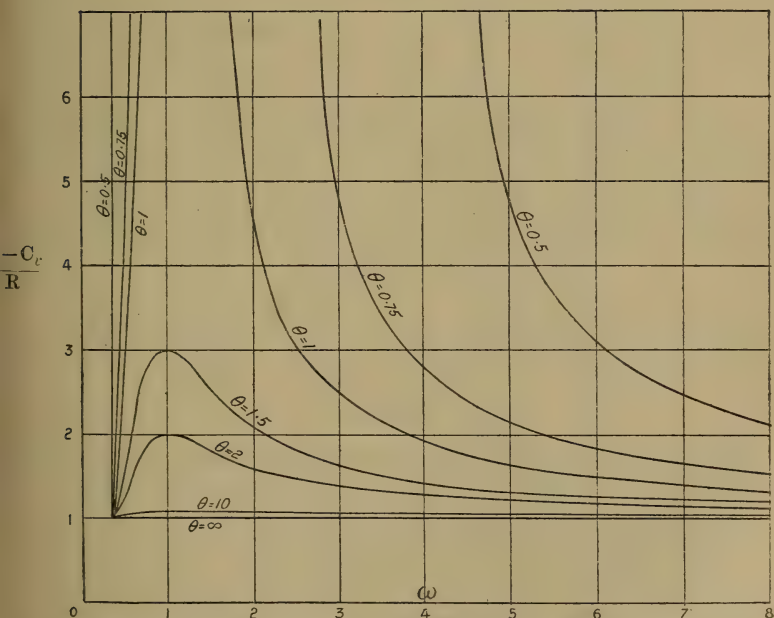
IN order to determine the effect on the specific heats of deviations from the gas-laws, van der Waals's equation, shall be used. This gives the relation †

$$\frac{C_p - C_v}{R} = \frac{RT}{RT - \frac{2a}{v^3}(v-b)^2},$$

or, using reduced magnitudes,

$$\frac{C_p - C_v}{R} = \frac{4\theta\omega^3}{4\theta\omega^3 - (3\omega - 1)^2}.$$

Fig. 1.



From this equation the accompanying set of isothermal $\left(\frac{C_p - C_v}{R}, \omega\right)$ -curves were derived (fig. 1). (The portions of the curves corresponding to the unstable parts of the

* Communicated by Prof. J. P. Kuenen, Ph.D.

† Van der Waals, *Contin.* i. p. 131 (1899).

pv-isothermals between the points of maximum and minimum pressure have been omitted as of no practical importance.)

Since C_v does not vary with the volume ($\frac{\partial C_v}{\partial v} = T \frac{\partial^2 p}{\partial T^2} = 0$), the same curves must also determine the variation of C_p and of C_p/C_v with the volume. From the diagram it appears that C_p is independent of the volume at large volumes and at high temperatures; as the temperature falls, C_p increases; at temperatures above the critical it reaches a maximum at a volume equal to the critical volume; and at temperatures below the critical it becomes infinite at two volumes, one of which is greater, and the other less than the critical volume. These points where $C_p = \infty$ correspond to the points of maximum and minimum pressures in the *pv*-isothermals, *i. e.* where $\left(\frac{\partial p}{\partial v}\right)_T = 0^*$. They may be determined from the relation†

$$(3\omega - 1)^2 = 4\omega^3\theta.$$

For the isothermals shown in the figure $C_p = \infty$ when

$\theta = 0.5$	$\omega_1 = 0.5$	$\omega_2 = 3.732$
$\theta = 0.75$	$\omega_1 = 0.605$	$\omega_2 = 2.137$
$\theta = 1.0$	$\omega_1 = 1.0$	$\omega_2 = 1.0$

van der Waals's equation is notoriously incorrect, but the general features of fig. 1 may still be the same as for real substances. The variations of C_v are known to be small and to take place only between narrow limits; that quantity, therefore, cannot disturb the general features of the diagram.

B. Experimental Evidence.

Owing to the scantiness of data available only a partial and incomplete comparison of the foregoing results with direct experimental results is possible.

(1) ETHER.

The author has calculated the variations of $C_p - C_v$ and C_p/C_v in the case of ether, and, as will be seen, the results afford striking confirmation of the above deductions.

Data were obtained from papers by Ramsay and Young, "On the Thermal Properties of Ethyl Oxide"‡, and "On the Continuous Transition from the Liquid to the Gaseous State of Matter at all Temperatures"§, and by Perman and Ramsay, "Ein Versuch, die adiabatischen Verhältnisse des Äthyloxyds zu bestimmen"||.

* Cf. *Cont.* i. p. 133. † Cf. H. Hilton, *Phil. Mag.* [6] i. p. 584 (1901).

‡ Ramsay & Young, *Phil. Trans.* clxxviii. A, pp. 57-93 (1887).

§ Ramsay & Young, *Phil. Mag.* [5] xxiii. pp. 435-458 (1887).

|| Perman, Ramsay, & Rose-Innes, *Zeits. f. phys. Chemie*, xxiii. pp. 385 *et seq.*

$$C_p - C_v.$$

For the calculation of $C_p - C_v$ the equation

$$C_p - C_v = -T \frac{\left(\frac{\partial p}{\partial T}\right)_v^2}{\left(\frac{\partial p}{\partial v}\right)_T}$$

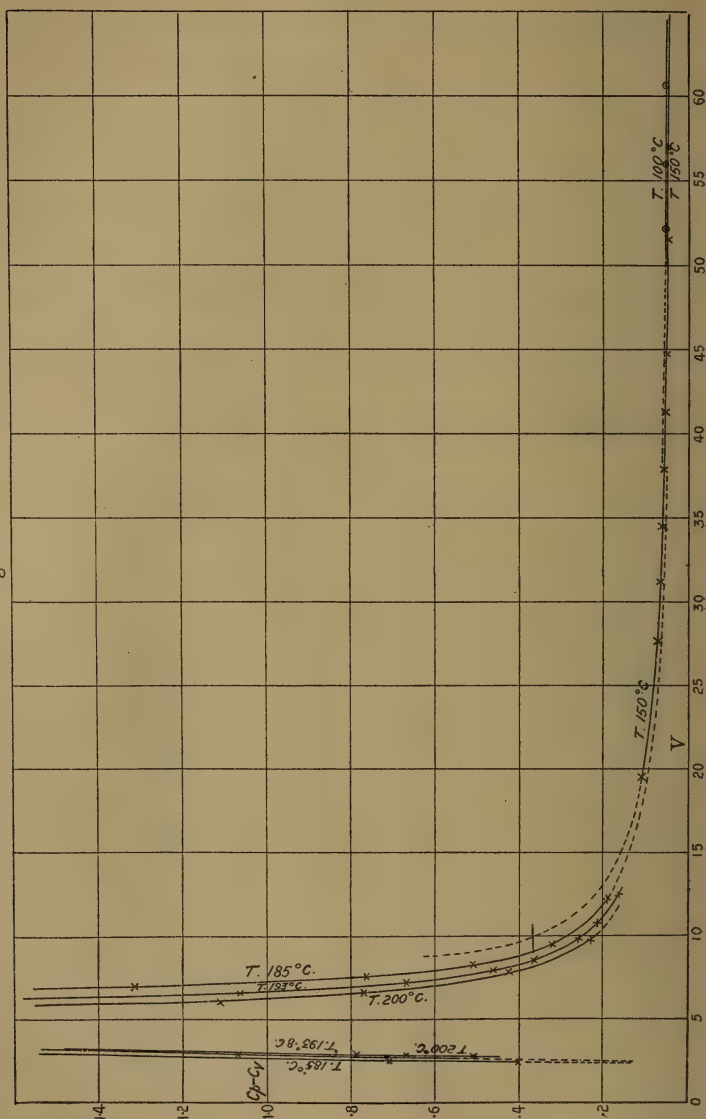
was used. Values of $\left(\frac{\partial p}{\partial T}\right)_v$ were obtained by interpolation from the tables given by Ramsay and Young (*loc. cit.*). For the determination of $\left(\frac{\partial p}{\partial v}\right)_T$, isothermals were drawn; the tangents at the required points gave the corresponding values of $\left(\frac{\partial p}{\partial v}\right)_T$.

The following results were thus obtained:—

T. 100° C.		T. 193°·8 C.	
V.	$C_p - C_v$.	V.	$C_p - C_v$.
61·67	·0523	12·42	·1640
55·97	·0528	10·75	·2157
52·20	·0536	9·76	·2600
50·35	·0515	8·45	·3684
		7·81	·4895
		7·16	·6787
		6·52	1·0747
		5·88	2·2573
		4·92	9·5409
		4·60	21·225
		3·798	21·673
		3·508	16·706
		3·217	8·9250
		2·929	3·8364
		2·856	2·6041
		2·785	1·7425
		2·713	1·0933
		2·641	·8008
		2·584	·7080
T. 150° C.		T. 200° C.	
V.	$C_p - C_v$.	V.	$C_p - C_v$.
65·31	·0401	9·76	·2332
56·81	·0460	8·45	·3356
51·56	·0484	7·81	·4279
44·67	·0543	6·52	·7711
41·26	·0553	5·88	1·1079
37·84	·0602	5·24	1·7769
34·46	·0630	2·75	·7861
31·07	·0672	2·7	·7214
27·68	·0740	2·6	·6138
24·29	·0876	2·5	·5267
19·47	·1145		
T. 185° C.			
V.	$C_p - C_v$.		
12·06	·1897		
10·44	·2632		
9·48	·3205		
8·21	·5090		
7·58	·7625		
6·96	1·3072		
6·33	2·1100		
2·424	·7048		
2·376	·5045		
2·330	·4466		
2·262	·4025		

V is the volume of 1 gram in cubic centimetres, and $C_p - C_v$ is expressed in calories.

Fig. 2.



When plotted with $C_p - C_v$ as ordinates and V as abscissæ the foregoing results give the set of curves shown in fig. 2; and as might naturally be expected, the diagram agrees in

general outline with that deduced from van der Waals's equation.

$$\frac{C_p}{C_v} \text{ for Ether.}$$

By a well-known thermodynamical relation

$$\gamma = \frac{C_p}{C_v} = \frac{E_\eta}{E_T},$$

where E_η and E_T are the adiabatic and isothermal elasticities respectively ; and

$$E_\eta = - \frac{S^2}{V},$$

where S is the velocity of sound in the vapour, and V is its specific volume ; then

$$\gamma = \frac{S^2}{V^2 \left(\frac{\partial p}{\partial V} \right)_T}.$$

This equation was used for the calculation of γ , values of S^2 being taken from the data given by Ramsay and Perman (*loc. cit.*).

The following table (p. 530) gives the results as obtained.

Fig. 3 (p. 531) represents these results graphically, and, with the exception of the left branch of the curve for $T=200^\circ\text{C}$., they are fairly regular.

It is evident that as the temperature increases while the volume is kept constant, the value of γ falls ; as the density increases at constant temperature, γ also increases at first slowly and then more rapidly ; at volumes smaller than the critical volume γ decreases rapidly with increase of density.

The general resemblance between the $C_p - C_v$ and γ diagrams is marked, and must be due to the comparative smallness of the variation of C_v .

T. 100° C.		T. 193°·8 C.	
V.	γ .	V.	γ .
71·22	1·271	12·42	1·504
61·67	1·217	10·75	1·660
55·97	1·276	9·76	1·789
52·20	1·276	8·45	1·991
50·35	1·274	7·81	2·321
1·638	2·447	7·16	2·822
1·610	1·424	6·52	3·812
T. 150° C.		5·88	7·098
65·61	1·124	4·92	21·696
56·81	1·182	4·60	48·863
51·76	1·274	3·798	44·47
44·67	1·329	3·508	34·13
41·26	1·335	3·217	17·23
37·84	1·306	2·929	7·064
37·07	1·480	2·856	4·829
34·46	1·278	2·785	4·288
32·10	1·307	2·713	2·709
31·07	1·306	2·641	2·262
29·13	1·266	T. 200° C.	
27·68	1·289	10·06	1·574
25·24	1·382	8·45	1·803
24·29	1·449	7·81	2·032
23·31	1·432	7·39	2·295
21·38	1·547	6·73	2·916
19·47	1·569	6·06	3·486
T. 185° C.		5·88	3·559
12·06	1·703	5·24	4·909
10·44	1·868	2·80	3·265(?)
9·48	1·922	2·63	2·948(?)
8·21	2·460	2·52	2·885(?)
7·58	3·106		
6·96	4·606		
2·424	3·136		
2·376	2·351		
2·330	2·320		

Variation of C_v with Volume.

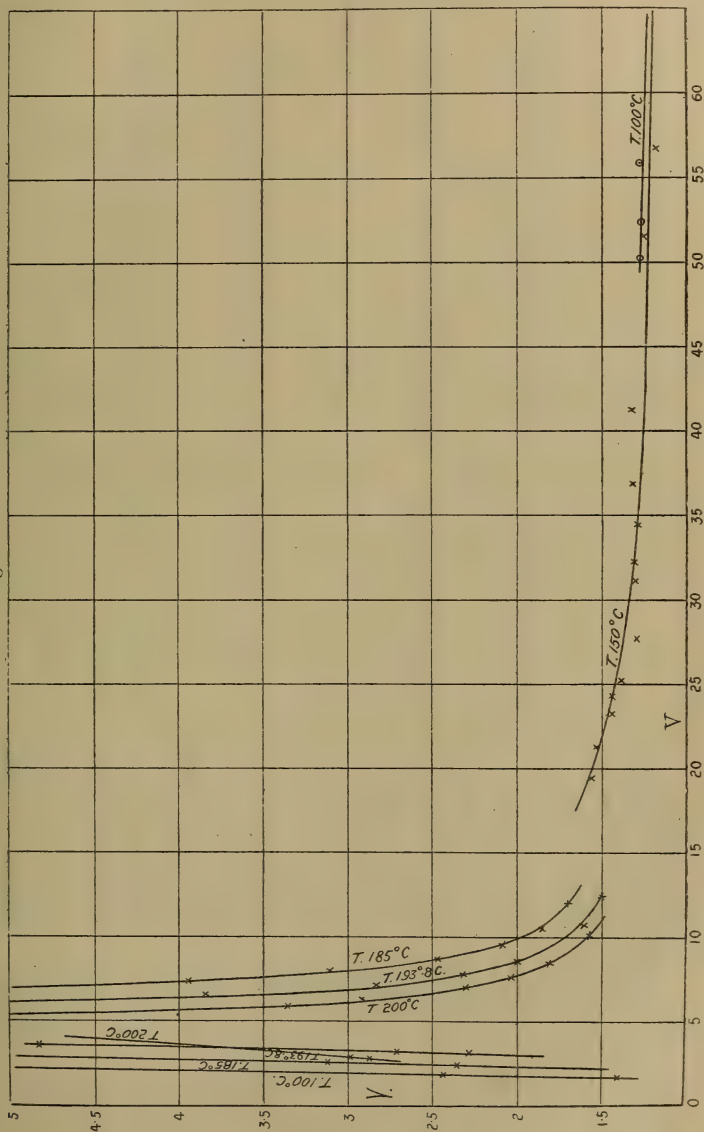
C_v was calculated by means of the equation

$$C_v = \frac{T \left(\frac{\partial p}{\partial T} \right)_v^2}{\left(\frac{\partial p}{\partial v} \right)_T - \left(\frac{\partial p}{\partial v} \right)_\eta};$$

but the resulting curves, though fairly well marked in general

outline, were somewhat irregular. "Smoothed" values of $C_p - C_v$ and γ were, therefore, obtained graphically, and from

Fig. 3.



them C_v was obtained. The following table with fig. 4 (p. 533) give the results :—

T. 100° C.		T. 193°·8 C.	
V.	C_v .	V.	C_v .
71·22	·1713	12·42	·3254
61·67	·2217	10·75	·3320
55·97	·1938	9·76	·3470
52·2	·1915	8·45	·3684
50·35	·1910	7·81	·3703
		7·16	·3725
		6·52	·3822
		5·88	·4080
		4·92	·4610
		4·60	·4434
		3·80	·4836
		3·51	·5042
		3·22	·5499
		2·93	·6328
		2·86	·6477
		2·79	·6001
		2·71	·4967
		2·64	·4576
		2·58	·4702
T. 150° C.		T. 200° C.	
V.	C_v .	V.	C_v .
65·61	·2005	9·76	·3989
56·81	·2044	8·45	·4179
51·56	·2104	7·81	·4148
44·67	·2217	6·52	·4439
41·26	·2212	5·88	·4326
37·84	·2234	5·24	·4546
34·46	·2136	2·80	·3500(?)
31·07	·2063	2·63	·3189(?)
27·68	·1974	2·52	·2856(?)
24·29	·1991		
19·47	·2000		
T. 185° C.			
V.	C_v .		
12·06	·2920		
10·44	·3032		
9·48	·3476		
8·21	·3486		
6·96	·3625		
6·33	·3857		
2·42	·2500		
2·38	·3827		
2·33	·3381		
2·26	·3042		

If the gas laws were obeyed $C_p - C_v$ for ether would be 0·0268 cal. E. Wiedemann* gives for ether $\gamma_\infty = 1·078$, hence $C_{v=\infty}$ should be 0·3436 cal. This seems to be incompatible with the above results; there is probably some constant error in the experimental data used for calculating γ which, however, does not influence their relative value.

For ether the following deductions may be made:—

(1) In the liquid state C_v varies with the volume more than with the temperature, of which it is practically independent.

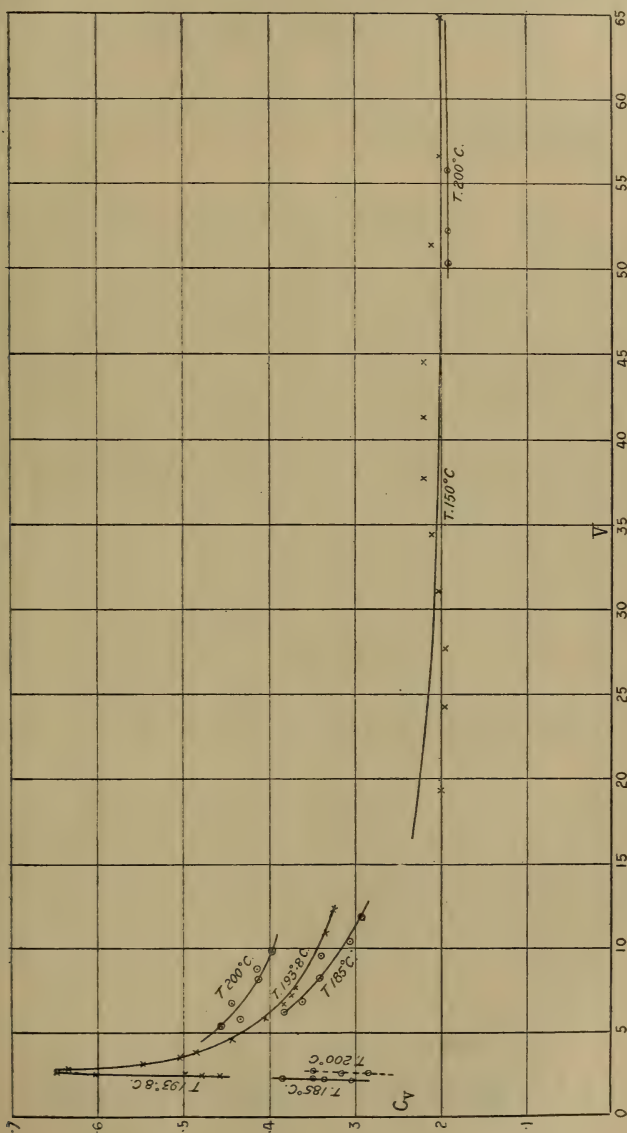
(2) In the gaseous state and at large volumes C_v is independent of the volume, and becomes a function of the temperature only. This conclusion is supported by Young's

* E. Wiedemann, Wied. Ann. Bd. ii. pp. 195–217 (1877).

researches * on Isopentane. He finds that at large volumes

$\left(\frac{\partial p}{\partial T}\right)_v$ is constant; hence $\frac{\partial C_v}{\partial v} = T \frac{\partial^2 p}{\partial T^2} = 0$.

Fig. 4.



* S. Young, Proc. Phys. Soc. xiii. pp. 602-657 (1894).

(3) With increasing density C_v at first increases; in the neighbourhood of the critical volume it attains a maximum; and as the density is still further increased it diminishes. This is in accordance with the figures given for isopentane by C. Dieterici*; but at low temperatures the variation given by him for C_v appears to be much too rapid. It is also in accordance with the calculation of C_v for isopentane by M. Reinganum†, who finds for C_v a maximum between $v=2.4$ c.c. and $v=10$ c.c.

(2) AIR.

In a paper dealing with "the thermodynamic properties of air," A. W. Witkowski‡ has deduced the variation of γ with the temperature along lines of equal volume, and has given the general features of the (γ, θ) diagram; but with the exception of the results obtained at large volumes and at high temperatures, *i.e.* in the state of a perfect gas, his conclusions are at variance with those derived from a study of van der Waals's equation and of the behaviour of ether. According to Witkowski, the ratio C_p/C_v first rises with increase of temperature; at $T=-120^\circ\text{C}$. ($\theta=1.15$) it attains a maximum which increases with the pressure; at higher temperatures it falls again and gradually becomes constant. Since at $\theta=1.15$ there is a maximum value of γ for every volume, $\frac{\partial\gamma}{\partial\theta}$ must vanish at that temperature for all volumes.

Hence, in the isothermal (γ, v) -diagram, the isothermal $\theta=1.15$ must be crossed by another at all volumes. Of such crossings neither fig. 1 nor fig. 3 gives any indication.

Witkowski§ also gives the (C_p, p) - and (C_v, p) - diagrams for air. At small pressures the former agrees with the results obtained for ether, but when p no longer $\propto \frac{1}{v}$ direct comparison is not possible. C_v is given as a linear function of the pressure. This is certainly not the case for ether; at large volumes C_v is independent of the pressure, but at smaller volumes the curves are concave to the pressure axis.

E. H. Amagat|| calculates γ for air at $T=50^\circ\text{C}$. by two methods:—

- (1) from C_p as determined by Lussana, and
- (2) from C_v as determined by Joly.

* C. Dieterici, *Ann. d. Physik*, xii. pp. 154–185 (1903).

† M. Reinganum, *Ann. d. Physik*, xviii. p. 1017 (1905).

‡ Witkowski, *Phil. Mag.* [5] xlii. pp. 1–37 (1896).

§ *Loc. cit.*

|| E. H. Amagat, *C. R.* cxxii. pp. 66–70 (1896).

The two sets of results are unfortunately inconsistent, and no conclusion can be drawn from them.

S. Lussana* finds that for air " C_p with the pressure increases; it reaches a maximum, and then decreases." This, as far it goes, confirms the results obtained for ether.

(3) CARBON DIOXIDE.

E. H. Amagat† obtaining C_v from Joly's empirical formulæ, calculated γ for CO_2 by means of the equation

$$\gamma = \frac{C_v + T \left(\frac{\partial p}{\partial T} \right)_v \left(\frac{\partial v}{\partial T} \right)_p}{C_v}.$$

The conclusions at which he arrives confirm the results shown in figs. 1 and 3. He says that for CO_2 :

" γ doit aussi passer par un maximum peu éloigné des limites du Tableau.... A une température plus rapprochée de la température critique ces variations seraient encore plus prononcées.... Au contraire, à des températures de plus en plus élevées, ces variations s'affaiblissent graduellement et le maximum par lequel passent les valeurs de γ finirait par disparaître."

S. Lussana‡ gives the following results for CO_2 :—

- (1) With increasing pressure C_p increases with the pressure less rapidly at low temperatures than at high; and
- (2) At atmospheric pressure C_p increases with increasing temperature; but, with increasing pressure, it is very soon greater for lower temperatures than for higher;

and in a further communication § he finds agreement between his results and those deduced by Amagat.

With the exception of the irregular behaviour of C_p at atmospheric pressure observed by Lussana, his results afford further confirmation of the results deduced from the equation of state.

University College, Dundee,
February 1907.

* S. Lussana, *Beibl.* Bd. xxiii. p. 245 (1899).

† E. H. Amagat, *C. R.* cxxi. pp. 863-6 (1895).

‡ S. Lussana, *Beibl.* Bd. xx. pp. 522-3 (1896).

§ S. Lussana, *Beibl.* Bd. xxii. pp. 548-9 (1898).

XLIII. *On the Specific Heat of Saturated Vapours.*
By JOHN P. DALTON, M.A., Carnegie Research Scholar*.

A. According to van der Waals's Equation.

A WELL-KNOWN thermodynamic relation is expressed by the equation

$$h_s = C_v + T \left(\frac{\partial p}{\partial T} \right)_v \frac{dv}{dT},$$

where h_s is the specific heat of the saturated vapour, and $\frac{dv}{dT}$ is calculated along the saturation line.

Hence

$$h_s = C_v + \frac{RT}{v-b} \frac{dv}{dT},$$

or, in "reduced" magnitudes,

$$h_s = C_v + \frac{3R\theta}{3\omega-1} \frac{d\omega}{d\theta}.$$

At "inversion" points, *i. e.* at points where the specific heat of the saturated vapour changes its sign, $h_s=0$. The curve of inversion points is, therefore, determined by the equation

$$-\frac{C_v}{R} = \frac{3}{3\omega-1} \theta \frac{d\omega}{d\theta}.$$

Now, the law of corresponding states does not involve the correspondence of thermal quantities. But, considering that C_v does not vary with the volume, and probably comparatively little with the temperature, C_v may be assumed constant and equal to $C_{v=\infty}$. At infinite volume the gas laws are obeyed, hence

$$R = C_{v=\infty}(\gamma_\infty - 1),$$

where $\gamma_\infty = C_p/C_v$ at large volume.

The equation to the curve of inversion points may now be written in terms of γ ,

$$\gamma_\infty = 1 - \frac{3\omega-1}{3} \frac{1}{\theta} \frac{d\theta}{d\omega}.$$

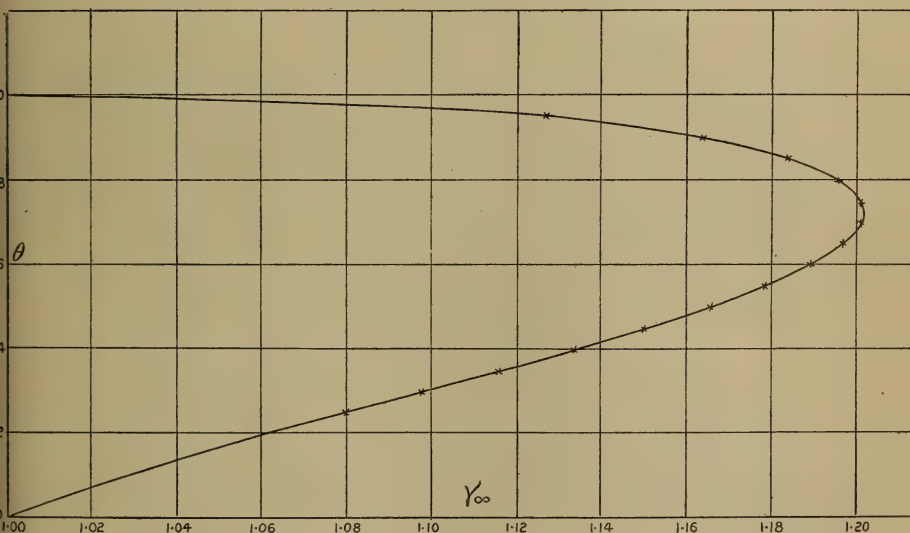
Using the saturation constants given in a previous paper, $\frac{d\theta}{d\omega}$ was calculated by means of Stirling's interpolation formula, and the right member of the above equation evaluated.

* Communicated by Prof. J. P. Kuenen, Ph.D.

θ .	γ_{∞} .	θ .	γ_{∞} .
0.25	1.0804	0.65	1.1972
.30	1.0983	.70	1.2013
.35	1.1164	.75	1.2011
.40	1.1341	.80	1.1955
.45	1.1509	.85	1.1838
.50	1.1660	.90	1.1637
.55	1.1790	.95	1.1267
.60	1.1896	1.00	1.0000

and on plotting these values the curve shown in fig. 1 was obtained.

Fig. 1.



From this it appears that the specific heat of saturated vapours whose $\gamma_{\infty} > 1.202$ must be of invariable sign, viz. negative. No "inversion" can occur in monatomic vapours such as Hg, A, He whose $\gamma_{\infty} = 1.67$, or in such diatomic gases as H₂, O₂, N₂ whose $\gamma_{\infty} = 1.41$, and the specific heat of their saturated vapours must always be negative. For vapours such as ether, chloroform, benzene, &c. whose $\gamma_{\infty} < 1.202$ there must be *two* inversion temperatures which become further apart as γ_{∞} gets smaller.

From fig. 1 the following table of inversion points was derived:—

γ_{∞} .	Inversion Temperatures.	
	θ_1 .	θ_2 .
1.00	0.000	1.00
1.02	0.07	0.995
1.04	0.14	0.99
1.06	0.195	0.985
1.08	0.25	0.98
1.10	0.305	0.970
1.12	0.36	0.96
1.14	0.42	0.935
1.16	0.48	0.905
1.18	0.555	0.86
1.20	0.68	0.765
1.202	0.725	0.725

Besides the fact that there must be two inversion points or none at all (except in the limiting case when the two coincide), it is also worth noting that the region to the right of the curve is the region of negative specific heats, to the left, *i. e.* enclosed by the curve, of positive; while on the curve itself the specific heat is zero. Hence

- (1) Saturated vapours of substances whose $\gamma_{\infty} > 1.202$ have always a negative specific heat ;
- (2) Saturated vapours of substances whose $\gamma_{\infty} < 1.202$ have a specific heat that, as the temperature rises, is first negative, then positive, and finally negative again ;
- (3) In the limiting case when $\gamma_{\infty} = 1.202$ the two inversion points coincide; the specific heat is then always negative, except at one point when it becomes zero.

Kamerlingh Onnes*, in a paper dealing with the reduced ψ -surface, makes use of the equation (which is not quite exact)

$$\frac{\theta}{\omega} \frac{d\omega}{d\theta} = - \frac{1}{\gamma - 1},$$

for the determination of inversion temperatures according to van der Waals's equation, and finds that according to it the limit of γ_{∞} is 1.195 if inversion is to be possible. He does not seem to note, however, that every saturated vapour that changes its sign once, must do so a second time, and his division of substances into classes does not appear to be entirely in accordance with the above results.

* Kamerlingh Onnes, *Comm. Leid.* No. 66 (1900).

B. *Experimental Evidence.*

As in the case of ordinary specific heats one might expect the foregoing results to hold qualitatively for real substances; but, although observations of inversion points are neither sufficiently numerous nor accurate to afford a satisfactory test, an attempt was made to obtain a curve more closely adapted to experimental results.

For the saturation constants the known properties of a normal substance may be used instead of the equation of state. Isopentane was taken as a typically normal substance. Its properties have been fully investigated by S. Young*, whose data have here been used.

The equation to the curve of inversion points is

$$-C_v = T \left(\frac{\partial p}{\partial T} \right)_v \frac{dv}{dT}.$$

The assumption was made that for all values of θ , $\frac{C_v}{C_{v=\infty}}$ along the saturation line is the same function of θ , so that

$$C_v = C_{v=\infty} \phi(\theta),$$

$$\text{or} \quad C_v = \frac{R}{\gamma_\infty - 1} \phi(\theta).$$

The above equation may therefore be rewritten in terms of γ ,

$$\gamma_\infty = 1 - \frac{R}{T \left(\frac{\partial p}{\partial T} \right)_v \frac{dv}{dT}} \phi(\theta).$$

C. Dieterici†, as a result of calorimetric experiment, has determined C_v for isopentane along the saturation line, and has also calculated $T \left(\frac{\partial p}{\partial T} \right)_v \frac{dv}{dT}$; in the following, the values given by him have been used. $\phi(\theta)$ may therefore be evaluated for various values of θ , if $C_{v=\infty}$ is known. For this purpose, $C_p - C_v$ was calculated from Young's data‡ at a temperature of 30° C., and as near the saturation volume as possible. This gave a value $C_p - C_v = 0.028$. C_v at that temperature is given by Dieterici = 0.361; hence, γ at that point = 1.08. This was assumed to be equal to γ_∞ . From the gas laws it follows that $(C_p - C_v)_\infty = 0.0276$ for isopentane; and since $\gamma_\infty = 1.08$, $C_{v=\infty}$ must be 0.345. By using the

* S. Young, Proc. Phys. Soc. xiii. pp. 602-657 (1894).

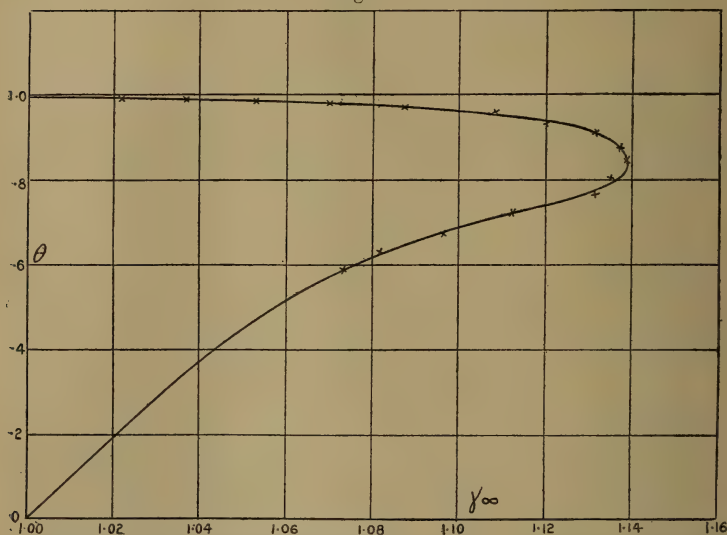
† C. Dieterici, *Ann. d. Physik*, xii. pp. 602-657 (1903).

‡ *Loc. cit.*

above equation, and determining $\phi(\theta)$ by this method the following results (plotted in fig. 2) were obtained:—

θ .	γ_{∞} .	θ .	γ_{∞} .
0.5924	1.0735	0.9180	1.1310
0.6359	1.0816	0.9397	1.1201
0.6793	1.0966	0.9614	1.1089
0.7227	1.1121	0.9722	1.0879
0.7661	1.1314	0.9831	1.0708
0.8094	1.1350	0.9896	1.0533
0.8529	1.1387	0.9961	1.0363
0.8746	1.1377	0.9983	1.0219
0.8963	1.1368		

Fig. 2.



The general features of the diagram show fair agreement with that deduced from the equation of state. In this case, however, a still lower value of γ_{∞} is necessary to render possible an inversion point. From the diagram the following "Table of Inversion Temperatures for Normal Substances" has been derived:—

γ_{∞} .	θ_1 .	θ_2 .
1.02	0.19	0.995
1.04	0.375	0.99
1.06	0.515	0.985
1.08	0.62	0.98
1.10	0.69	0.96
1.12	0.74	0.94
1.139	0.835	0.835

For normal saturated vapours, therefore, we might expect

- (1) a negative specific heat for $\gamma_{\infty} > 1.139$;
- (2) a specific heat first negative, then positive, and finally negative for $\gamma_{\infty} < 1.139$; and
- (3) a negative specific heat vanishing at $\theta = 0.835$ for $\gamma_{\infty} = 1.139$.

J. D. van der Waals* divides vapours into two classes: (1) when $\gamma < 1.07$ the sign of the specific heat is positive, and (2) when $\gamma > 1.08$ it is negative. When γ does not differ much from 1.08, nothing definite can be asserted about the sign of the specific heat.

Observations of actual inversion points are hardly sufficiently numerous to enable a direct comparison to be made between the foregoing results and experimental data. The few, however, that are known are here given.

	γ_{∞} .	θ observed.	θ_1 or θ_2 from Table.
Acetone.....	1.132 (Wiedemann)	0.92 (Cazin).	0.91
Benzene.....	1.129 (— do. —)	{ 0.67 (do.). 0.70 (Dupré). }	0.77
Chloroform ...	1.139 (do.)		
	1.110 (Müller)	0.74 (do. do.).	0.825
			0.72

The degree of correspondence is sufficient to make one regret that further comparison is at present impossible; it justifies, however, a rejection of L. Natanson's† "first approximation" that inversion occurs in various vapours at corresponding temperatures, and of his subsequent deduction of an inversion point for hydrogen.

Abnormal Substances.

An attempt was made to extend the foregoing method to abnormal substances, of which methyl alcohol was taken as an example. Data for the calculation of $T \left(\frac{\partial p}{\partial T} \right)_v \frac{dv}{dT}$ were obtained from the researches of Ramsay and Young‡. In this case, however, no data are available for the determination of $\phi(\theta)$. C_v was therefore considered constant, and equal to $C_{v=\infty}$. The resulting $(\gamma_{\infty}, \theta)$ curve was similar to that shown in fig. 2, but the maximum value of γ_{∞} on the curve was much lower than that for normal substances. It would seem,

* van der Waals, *Beibl.* Bd. ii. p. 331 (1878).

† L. Natanson, *Zeits. f. phys. Chem.* xvii. p. 276 (1895).

‡ Ramsay and Young, *Phil. Trans.* clxxviii. A. pp. 57-93 (1887) and *Phil. Mag.* [5] xxiv. pp. 196-212 (1887).

then, that abnormality diminishes the possibility of the existence of an inversion point by lowering the value of γ_∞ above which inversion is impossible. How far this conclusion is affected by the assumption $\phi(\theta) = \text{constant} = 1$ it is at present impossible to say; and in the elucidation of this point there is still plenty of scope for further research.

University College, Dundee,
February 1907.

XLIV. On the Relation of the Field at the Point of an Electrified Needle to the Mechanical Force resulting. By F. B. YOUNG, B.A.*

THE present research was carried out in the Physical Laboratory of University College, Bristol, under the direction of Professor Chattock, in order to obtain a correction factor for certain values found by him for the field which was just sufficient to produce discharge at the extremity of a needle †. These values were obtained by measuring the pull of an electrified plate on an earth-connected needle opposite and normal to the plate and protruding through an earth-connected surface. The needle being cylindrical and ground to a hemispherical point this pull (P) is a function of the field (f) at the centre of the point.

Fig. 1 is a section of the point, R being its radius, S the electrical density at the centre of the point, and σS the density at any other part of its surface. The pull on the surface of the annulus shown, resolved parallel to the axis of the point, is

$$4\pi^2(\sigma S)^2 R^2 \sin \theta \cos \theta d\theta;$$

whence

$$\begin{aligned} P &= 2\pi^2 R^2 S^2 \int_0^{\frac{\pi}{2}} \sigma^2 \sin 2\theta \cdot d\theta \\ &= \frac{R^2 f^2}{8} \int_0^{\frac{\pi}{2}} \sigma^2 \sin 2\theta d\theta; \\ f &= a \frac{\sqrt{8P}}{R}, \quad \dots \dots \dots (1) \end{aligned}$$

where

$$a = \frac{1}{\sqrt{\int_0^{\frac{\pi}{2}} \sigma^2 \sin 2\theta d\theta}}.$$

* Communicated by Prof. A. P. Chattock.

† Phil. Mag. Sept. 1891.

If it be assumed that the density on the hemisphere is sensibly uniform, $a=1$. The values of f were calculated on this assumption in the paper referred to, and it was the object of the present investigation to find how far that assumption held good.

Fig. 1.

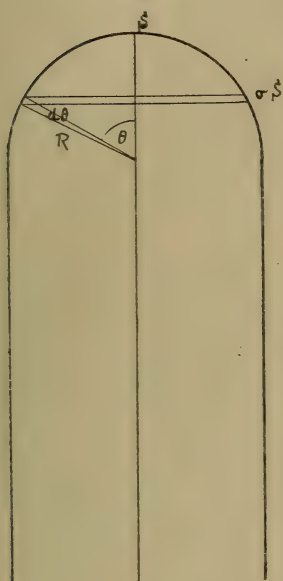
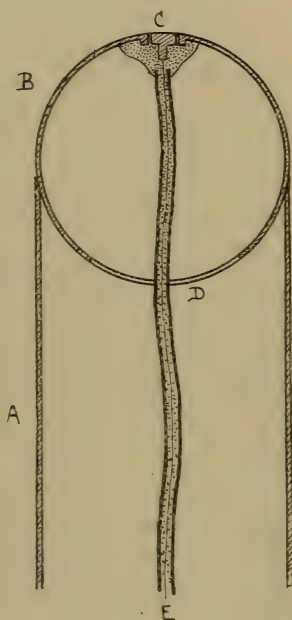


Fig. 2.



The investigation involved the experimental determination of the distribution of the charge round the hemisphere. For this purpose magnified models of the needle and plate were made; the sewing-needles employed in the above-mentioned research were represented by two models about two metres in length and with diameters of about 13 cms. and 8 cms. respectively, which were fixed upright on the laboratory floor, whilst the plate employed was a screen about 3 metres square suspended horizontally above.

Of the two model needles employed the smaller was more perfectly made and gave more consistent results than the other. A section of it is shown in fig. 2. A is a metal tube the end of which was turned to fit accurately a brass sphere B of the same diameter, so that only a half of the sphere projected. In the sphere a hole 8 mm. in diameter was drilled to receive a plug C 7.5 mm. in diameter; the plug was

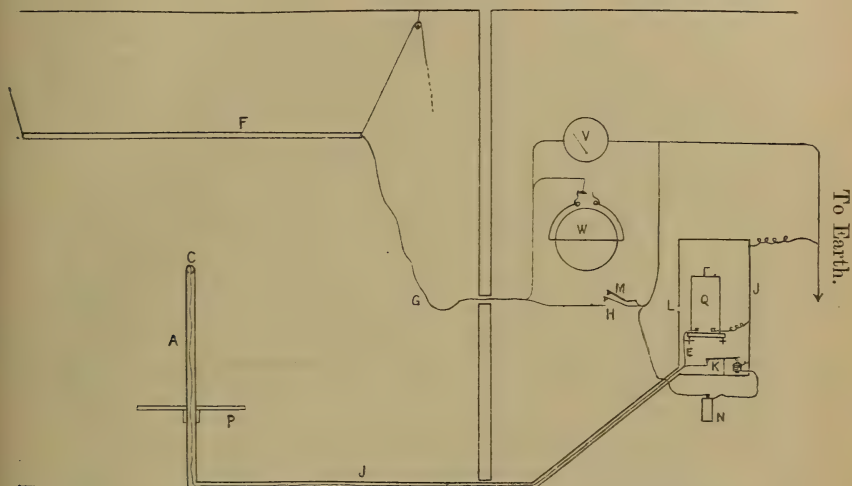
supported and insulated with paraffin-wax. Directly opposite C was a slit D, about 4 cms. in length. To the plug, which served as a proof-plane, was soldered a fine copper wire E which passed through D to an electrometer. The wire was insulated with rubber tubing, which was surrounded by an earth-connected sheath of tinfoil; it was held in position by means of a plastic mixture of paraffin-wax and Prout's glue with which the rubber tubing was filled. This arrangement served to make the electrostatic capacity of the wire constant and independent of its position relative to the cylinder A. On the great circle passing through C and D was scratched a scale of centimetres and with the aid of a cardboard gauge which marked the position of the extremity of the needle's axis the proof-plane could be moved, by rotating the sphere in its socket, to a known distance from its central position.

The mode of employment of this arrangement may be briefly stated as follows:—a charge was induced on the needle-point by charging the screen to a certain constant potential and the portion of the charge residing on the proof-plane was measured by means of a Dolezalek electrometer. The process was repeated for various positions of the proof-plane, and in this way the values of σ were obtained for various values of θ . The advantages of the form of apparatus employed were two-fold: the surface of the sphere was unbroken except for the narrow ring of insulating material, whilst, since the proof-plane was a fixture, any irregularity of density caused by this discontinuity or by any imperfection in the setting of the proof-plane entered equally into each reading.

The general arrangement of the apparatus is shown in fig. 3. The screen F consisted of a wooden frame covered with brown paper, which in its turn was covered with silvered paper in order to form a conducting surface. The whole was suspended near the ceiling by means of pulleys and silk cords. A wire G from the screen passed through the wall into an adjoining room, where it was connected to a Wimshurst machine W through a key which made contact with either prime conductor at will. The difference of potential between the screen and the earth was measured by means of a Kelvin Electrostatic Voltmeter V. The screen could be immediately brought to earth potential by means of the discharging-key H. A large Leyden jar, not shown in the figure, was attached to the wire G in order to increase the capacity of the screen and so to render less rapid the fall of potential due to leakage. The wire E from the proof-plane of the needle A was carried

on paraffin blocks along the floor into the adjoining room to one terminal of the electrometer Q, the other terminal of which was connected to earth. The whole of the wire and the electrometer were enclosed in a metallic earth-connected

Fig. 3.



case J of which the needle A itself formed a part, so that the only part of the system which was exposed to external electrical influences was the surface of the proof-plane itself. This casing was found very necessary. The deflexions of the electrometer were read through a window L in the case protected with wire-netting. The proof-plane together with the attached electrometer quadrants were connected to earth through the key K, which was normally closed but which could be temporarily opened by means of an electromagnet controlled from the key M placed upon the discharging-key H. One terminal of the magnet was connected to earth through the case J, whilst the other was connected through a dry-cell N to M; M when closed made contact with earth, thus completing the circuit and opening K. The permanent connexion of the electromagnetic circuit to earth was found important.

The method of taking the readings was as follows :—The screen F was charged positively by means of the Wimshurst rather above the required potential (considerably over 6000 volts) and was then allowed to fall to that potential by leakage accelerated, if necessary, by touching the wire G with a piece of wood. A charge was induced upon the end

of the needle A in which the proof-plane C shared as though an integral part of the needle, being in metallic connexion with it through the key K. The screen having reached the required potential, the keys M and H were closed in rapid succession. When the screen was discharged the induced charge on the needle was released; the greater part of that charge returned to earth, but the portion residing on the proof-plane, finding its way of escape cut off by the opening of K, distributed itself over the proof-plane, wire and quadrants, and produced a throw of the electrometer-needle. A similar throw was produced in the opposite direction by charging the screen negatively instead of positively, and the average of these and two similar throws was taken. The sphere was next rotated to bring the proof-plane successively to various angular distances from its central position, and at each position the above operations were repeated. In this way was obtained a series of mean deflexions which, provided the capacity of the wire E and the potential of the electrometer-needle remained constant, were proportional to the density of the charge upon the proof-plane in its successive positions.

To correct for any possible want of symmetry in the needle a second series of observations was made, in which the sphere was rotated in such a direction as to lower the proof-plane on the opposite side, whilst to guard against a possible distortion of the field by surrounding objects a third and fourth series were made for which the proof-plane revolved in a different vertical plane. As a further precaution the sign of the charge on the electrometer-needle was reversed for two of the

TABLE I.

θ .	σ .				
	A.	B.	C.	D.	Mean.
0	—	—	—	—	1.0
.262	.995	.993	.995	.988	.993
.524	.967	.971	.978	.968	.973
.785	.937	.933	.933	.934	.935
1.047	.880	.885	.883	.884	.883
1.309	.804	.807	.805	.818	.809

four series. The complete set of observations is given in Table I.; the consistency of the series seems to show that there

were no serious defects in the arrangements made. The mean values of these series in terms of the deflexion for the central position as unity gave the values of σ for various values of θ , from which was plotted a curve of relative density (curve A in fig. 4).

Fig. 4.



The constancy of the potential of the electrometer-needle was at first tested by a frequent application of a Clark cell which was connected to the quadrants through a switch actuated from the outside of the case C by means of a silk thread. This method, however, involved a disturbance and a possible change of capacity of the proof-plane connexions; and it was found preferable, if somewhat less speedy, to take readings for the central position of the proof-plane alternately with those for other positions, and to compare each of the latter with the mean of the preceding and succeeding readings. If any considerable drift occurred in the needle's potential the readings were discarded. The throw of the needle was taken in preference to steady deflexions partly in order to obtain larger readings and partly to lessen the lapse of time between discharge and reading, and consequently the possibility of errors due to slight leakage &c.

Tests were applied for the purpose of ascertaining whether the capacity of the connecting wire was affected by the displacement of the sphere. With a thin plate of paraffin-wax and tinfoil a condenser was made whose capacity was approximately equal to that of the proof-plane system, so that, when added to that system, it reduced the deflexions to about a half of their original magnitude. The values of σ were then found as before for two or three values of θ . If the capacity of the proof-plane system is constant the values so obtained should agree with those previously found; if the values differ slightly $\sigma = \sigma_1 (1 + 2\sigma_2 - \sigma_1)$ approximately, where σ_1 and σ_2 are the apparent values of σ without and with the condenser respectively. It was found that the results agreed well within the limits of experimental error except for the position of the proof-plane most remote from the centre, where a change of capacity of nearly 2 per cent. seemed to be indicated. In this position the wire and its covering were in a somewhat strained position, so that a slight change of capacity was possible. The corrected value for the value of θ in question does not, however, fit so well into the density curve as the uncorrected value, and the difference may well be accounted for by the diminution of the deflexions which rendered the values of σ_2 liable to errors twice as great as those of σ_1 . The test, indeed, cannot be regarded as giving more than a general indication that no progressive rise or fall of capacity occurs as θ is increased; the smoothness of the curve and the consistency of the four series may be taken as evidence that no irregular fluctuations occurred which were not of negligible magnitude. The question of constancy of capacity was important as it probably constituted the weakest link in the chain of experimental proof.

In order to integrate $\int_0^{\frac{\pi}{2}} \sigma^2 \sin 2\theta \cdot d\theta$ a second curve was developed from curve A by plotting the value of $\sigma^2 \sin 2\theta$ with θ . The area of this curve, which gave the required integral, was found to be 0.85. The ordinates of curve A' are proportional to the contributions of the annular elements of the hemispherical surface to the pull P for the corresponding values of θ , so that the curve serves to show the relative importance of these contributions. The part of the surface which contributes most to the pull is that for which θ approximates to $\frac{\pi}{4}$. Hence the assumption of uniform density will give an approximate result only in so far as the

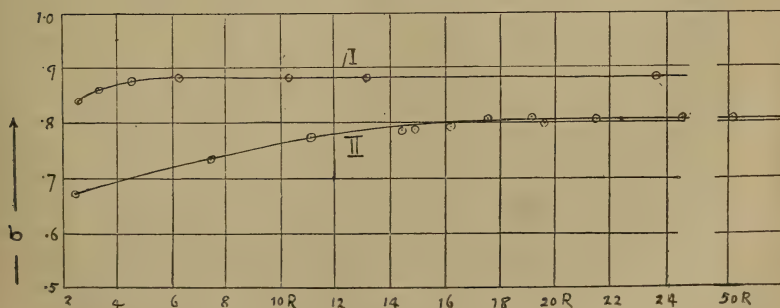
density does not vary between $\theta=0$ and $\theta=\frac{\pi}{4}$; whilst curve A shows a change of density within these limits which is by no means negligible. The value of a in equation (1) is thus $\frac{1}{\sqrt{.85}}=1.085$, and the values of f obtained in the previous research must be multiplied by that number.

The curve A in fig. 6 is the density curve obtained with the larger sphere; the density appears to fall off a little more rapidly, and the correction factor derived from this curve is about 2 per cent. higher than that obtained from the small sphere. The discrepancy is no doubt due to the imperfectly spherical shape of this sphere. The value of a may therefore be taken as very close to 1.09.

It was desired in the above determinations to have the needle sufficiently long and at a sufficient distance from the screen to leave the distribution of the charge practically independent of any conditions other than the shape of the needle-point. To ensure that the screen was at a sufficient distance, observations were taken whilst the screen was at a distance of 50 cm., and again when the distance was increased to about 90 cm.; the two sets of observations were found to agree within the limits of experimental error. Professor Chattock, however, in view of possible future work, wished to obtain the minimum values of the length of needle and distance of screen for which the distribution was still dependent only on the shape of the needle.

The screen was therefore lowered a little at a time, and for each position of the screen the relative density for a certain value of θ ($\theta=1.047$) was ascertained. Though the absolute

Fig. 5.



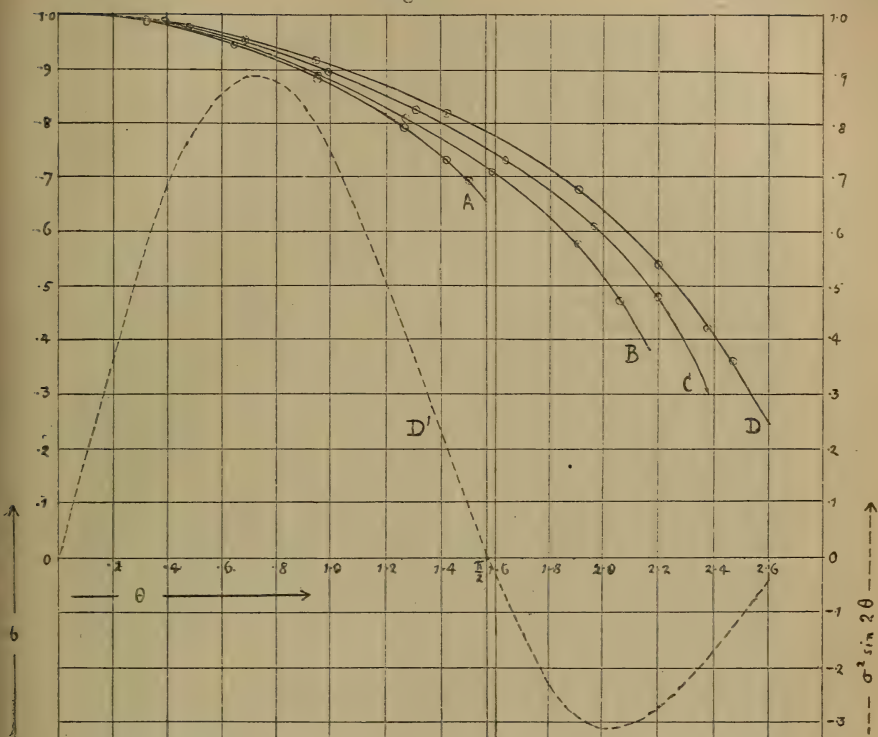
density naturally rose as the distance decreased, the relative density σ remained constant to a certain limit beyond which

it fell with increasing rapidity. The value of σ is plotted (curve I. in fig. 5) with the distance of the screen expressed as a multiple of R , the radius of the hemisphere, and the curve shows that for distances of about $5.5 R$ and upwards the distribution is undisturbed. For the purpose of testing the influence of the length of the needle upon the density curve, a small circular screen was made by stretching paper covered with metal foil over a hoop about a metre in diameter. This was mounted on a wooden clip by means of which it could be clamped upon the needle concentrically with it and in a horizontal plane (P in fig. 3). A wire connected the screen to the earthed casing so that when the screen was raised the needle was virtually made shorter. The screen A was maintained at a distance of about $16 R$, whilst the screen P was moved to various heights on the needle. The value of σ for a constant value of θ ($\theta = 1.309$) was found for each position of the screen, and curve II. in fig. 5 was plotted, the abscissæ representing the length of the needle which protruded through P in terms of R . The curve shows that the distribution of the charge is practically independent of the length of the needle if that length exceeds $18 R$. (The length of the needle used in determining the correction factor a was over $50 R$.)

If the end of a platinum wire is fused in the oxy-coal gas flame it will, under the influence of surface tension, assume a very truly spherical surface. Unfortunately, however, the end is rarely hemispherical since the molten platinum usually forms a blob at the end of the wire. The manufacture of such points is much simpler than the grinding of steel ones, and it was therefore desirable to determine to what extent the value of a was affected by the blob. Three stalks of various diameters were made; and the large sphere was mounted upon these in succession to represent a blob.

The curves A , B , C , and D in fig. 6 are the density curves obtained for stalks whose radii are respectively R , $.82 R$, $.68 R$, and $.51 R$. The tension curve D' is the one derived from D , and shows how the charge beneath the hemisphere diminishes the pull of the screen upon the needle. The density of the curve falls off less rapidly as the stalk becomes narrower, and there is consequently an increase in the pull on the upper portion of the sphere which partially but not wholly counteracts the back-pull upon the under surface. In calculating the value of a from the derived curves the negative area was of course subtracted from the positive area.

Fig. 6.



Collected Results.

In Table II. are given the values of a obtained for various values of R'/R , where R' is the radius of the stalk and R that of the sphere.

TABLE II.

R'/R .	a .
1.00	1.09
.82	1.14
.68	1.16
.51	1.22

XLV. *Osmotic Pressure from the Standpoint of the Kinetic Theory.* By T. MARTIN LOWRY, D.Sc.*

VAN 'T HOFF'S GAS-THEORY OF OSMOTIC PRESSURE.

IN any discussion of the origin and mechanism of osmotic pressure it is necessary to take into account, as a factor of fundamental importance, the fact—now fully established by experiment†—that the osmotic pressure of a dilute solution can be calculated with a very close approach to accuracy by means of the formula $PV = RT$, which expresses the relationship between the pressure and volume of a gas. Not only is the osmotic pressure proportional to the concentration of the solution and to its absolute temperature, but the constant R has the same value in the two cases, so that the osmotic pressure is actually identical in magnitude with the pressure exerted by a gas of equal molecular concentration. This result could be predicted on general grounds in the case of gaseous osmotic pressures (Arrhenius, *Zeit. phys. Chem.* 1889, iii. p. 119; Ramsay, *Phil. Mag.* 1894, xxxviii. p. 206); but the conditions prevailing in a liquid are so far different that the gas-analogy, though suggestive, would be utterly insufficient to justify the application of the gas-laws to osmotic pressure unless these could be verified experimentally or established on an independent theoretical basis.

When, however, the discovery was made by van 't Hoff (*Zeit. phys. Chem.* 1887, i. pp. 481–508) that osmotic pressure in liquids is actually identical in magnitude with the pressure of a gas exposed to similar conditions of temperature and pressure, it was generally assumed that the two phenomena must be essentially similar in character, and that the osmotic pressure produced by dissolving sugar in water was due, as in gases, to the bombardment of semi-permeable membrane by the molecules of the solute (van 't Hoff, *loc. cit.* p. 482; B. A. Report, 1890, p. 336; *Zeit. phys. Chem.* 1890, v. pp. 174–176). Van 't Hoff's theory had the merit of giving a simple quantitative explanation of osmotic pressure, and was not unreasonable when applied to the osmotic pressure of gases dissolved in liquid (as well as in gaseous) solvents. The conception was, however, not easy to apply to the more ordinary cases of osmotic pressure, such as those afforded by aqueous sugar solutions. In most cases the solute molecules have a very

* Communicated by the Faraday Society (Abstract of a contribution to a general discussion on "Osmotic Pressure" held on Jan. 29, 1907).

† Pfeffer, *Osmotische Untersuchungen*, Leipzig, 1877; Morse & Fraser, *Amer. Chem. Journ.* 1905, xxxiv. p. 1 (1906); compare also the experiments of Griffiths on the freezing-points of dilute aqueous solutions.

slight mobility, and in the absence of the solvent are unable to produce any hydrostatic pressure whatever upon the walls of the containing vessel. It is therefore scarcely reasonable to attribute the whole action of the solution to the relatively inert solute, whilst neglecting the very active part played by the solvent (Fitzgerald, B. A. Report, 1890, p. 327).

THE MECHANISM OF OSMOTIC PRESSURE

There can at the present time be no doubt that osmotic pressure depends essentially on the phenomena of selective solubility. The palladium membrane used by Ramsay to develop a gaseous osmotic pressure acts in virtue of its ability to absorb or dissolve hydrogen, but not nitrogen; the amount of hydrogen absorbed depends on the pressure, and equilibrium is attained when the partial pressure of the hydrogen inside the vessel is equal to its total pressure outside. A water-membrane has been used by Nernst to develop an osmotic pressure between ether and an ethereal solution of benzene, the former being soluble and the latter insoluble in water. Copper ferrocyanide, which absorbs or dissolves water and certain salts but not sugars, forms an efficient semi-permeable membrane for aqueous sugar solutions, but not for salt solutions. The presence of the sugar diminishes the solubility of the water in the membrane, and a flow of liquid is set up because the membrane when saturated with regard to the water on one side is supersaturated with regard to the solution on the other side. The amount of water taken up by the membrane can, however, be increased by compressing the liquid, and it is thus possible to counterbalance the decrease of solubility due to the sugar, and by equalizing the solubility on the two sides of the membrane to stop the flow of solvent into the solution. The pressure required to equalize the solubilities and stop the flow is the so-called "osmotic pressure," and it may again be urged that there is no *a priori* reason for supposing that this would be the same as that produced by a gas of equal molecular concentration.

Pickering's theory (*Ber.* 1891, xxiv. p. 3639) that the action of the semi-permeable membrane depends on the relative size of the molecules, has not been confirmed. The similar mechanical theory of Sutherland (*Phil. Mag.* 1897 [5] xliv. pp. 493-498), that the membrane consists of "meshes" through which water but not sugar can pass appears to be equally untenable and need not be discussed here. The theories of Traube and of Armstrong are referred to later.

POYNTING'S THEORY.

Much interest was aroused by the publication in 1896 of a paper by Professor Poynting (*Phil. Mag.* [5] xlii. pp. 289-301) in which an explanation was given of the osmotic pressure of liquids which afforded an alternative to van't Hoff's gas-theory, and had the advantage that the active part in osmosis was assigned to the solvent rather than to the solute. In many respects the view he advocated is very similar to that described below, but the two theories differ in one essential point, and consequently lead to widely different explanations of the origin of osmotic pressure. In particular, Poynting was led to assume that osmotic pressure was due to the formation of labile hydrates; and it became necessary, as Whetham pointed out (*Nature*, Oct. 15, 1896), to assume that all substances which gave a normal osmotic pressure were monohydrated, whilst the double osmotic pressure of binary salts might be ascribed to the formation of a dihydrate or of two monohydrated ions*. The invariable formation of loose monohydrates was so improbable, and is so far in contradiction to recent work on the hydrates present in solution, that Poynting's theory has failed to secure general acceptance as an explanation of the osmotic phenomena.

OSMOTIC PRESSURE AS A KINETIC PHENOMENON.

The theory of osmotic pressure now described formed the subject of a paper read before the Chemical Society of the Central Technical College as long ago as May 1896; but it was only recently that it was recognized as being sufficiently novel to warrant further publicity.

The starting-point of the theory is a consideration of the conditions prevailing at the surface of separation of the solution and the semi-permeable membrane, which may be either a layer of copper ferrocyanide or merely the boundary between liquid and vapour or liquid and ice. The simplest of these cases is undoubtedly that which involves the equilibrium between liquid and vapour. In this case the kinetic theory postulates a continual process of evaporation, whereby rapidly moving particles are constantly escaping from the surface of the liquid into the vapour space. This is balanced by the condensation of practically all the molecules of the vapour that impinge on the liquid surface. When the vapour reaches a certain concentration the rate of condensation becomes equal to the rate of evaporation, and a condition of equilibrium is attained, not because evaporation has

* Compare I. Traube, *Ann. Phys. Chem.* 1897, ii. 62. pp. 490-506.

ceased, but because it is neutralized by an equal and opposite process of condensation. In the case of a non-volatile liquid the mobility of the relatively heavy molecules or molecular complexes is so small that very few are able to escape, and the maximum vapour-pressure is inappreciable.

If now a solution be prepared by mixing a volatile and a non-volatile liquid, or by dissolving a non-volatile solid in a volatile liquid, the surface will contain both kinds of molecules. If one of the solvent molecules be struck by a rapidly moving molecule from the interior of the liquid, it will be projected into the vapour space. If, however, one of the non-volatile molecules be struck, it will be unable to escape, and the solvent particle will rebound in much the same way as if it had struck the wall of the containing vessel. The rate of evaporation is therefore reduced by the addition to the solvent of a non-volatile solute. On the other hand, *it is probable that the presence of the non-volatile molecules would not interfere with the rate of condensation of the vapour.* This point is of fundamental importance, as the opposite view was advocated by Poynting, who supposed that condensation and evaporation would be checked to an equal extent, just as if the surface had been covered by a plate of perforated zinc. It must be remembered, however, that a considerable upward velocity is required before a molecule can escape from the surface of a liquid, and that a molecule descending with even the smallest downward velocity would have little chance of escaping when once it came within the range of attraction of the liquid. Even if, on reaching the liquid surface, a vapour molecule should strike against a non-volatile molecule of the solute, the attraction of the neighbouring molecules of the solvent would be sufficient to hold it, and thus ensure its condensation.

It need scarcely be pointed out that similar conditions would prevail at the surface separating liquid and ice, or at the surface of one of the more conventional semi-permeable membranes. In the former case the presence of a non-isomorphous solute might prevent the adhesion to the ice of a molecule of solvent moving towards it but separated from it by a molecule of solute. On the other hand, it would not prevent the melting off or dissolution of an ice molecule if the average kinetic energy (*i. e.* temperature) of the ice were raised by the latent heat of crystallization of other molecules passing from the liquid to the solid state. In the case of a membrane such as copper ferrocyanide the solute would check the escape of solvent molecules from the solution into the membrane, but would not oppose the return of wanderers

migrating from the membrane back into the solution. This would lead to a disturbance of the equilibrium between the solvent and the liquids on either side, and would be sufficient to produce an osmotic flow and a consequent osmotic pressure.

DEDUCTION OF THE GAS-FORMULA FROM THE KINETIC THEORY OF OSMOTIC PRESSURE.

Of the theories of osmotic pressure that have hitherto been put forward, that of van 't Hoff is the one that leads most directly to a simple quantitative explanation of the phenomena. Poynting, in developing his theory quantitatively, considered that solute molecules and stable compounds of solvent and solute would merely alter the effective surface of the liquid without changing the relative rates of evaporation and condensation; only in the case of labile compounds was it considered possible that the rate of evaporation might be checked without altering the rate of condensation; on this basis a quantitative explanation was only possible on the impracticable assumption that all ordinary solutes are loosely monohydrated in aqueous solutions. The more recent theory that osmotic pressure depends on a disturbance in the equilibrium between simple and polymerized solvent molecules, *e. g.*, $n\text{H}_2\text{O} \rightleftharpoons \text{H}_{2n}\text{O}_n$ (Armstrong, *Proc. Roy. Soc.* 1906, A. lxxviii. pp. 264–271), has also, as yet, failed to yield a quantitative interpretation of the pressures produced.

The quantitative interpretation of the kinetic theory of osmotic pressure follows at once from Nernst's proof of the relationship between osmotic pressure and vapour pressure ('Theoretical Chemistry,' pp. 124–129). Thus if N be the number of gram-molecules of solvent, and n the number of gram-molecules of solute in unit volume of the solution, the validity of the gas-formula for osmotic pressure is readily deducible from the equation

$$\frac{N}{N+n} = \frac{p'}{p},$$

where p is the vapour-pressure of the solvent, and p' that of the solution*. If the simple assumption is made that the molecules of solvent are uniformly distributed in the surface layer, and that the spacing or packing is the same as in the pure solvent, it follows at once that the rates of evaporation from unit surface of solvent and solution will be in the ratio

* The value of N is determined by the molecular weight of the solvent as it exists in the vapour, and not by its molecular weight in the liquid state.

$N + n$ to N , and that this will also be the ratio of the vapour-pressures, as required by the above formula. The argument is, of course, identical with that used by Poynting to show that each molecule of solute must destroy the mobility of a molecule of solvent; but, whereas he was led to assume the regular formation of labile monohydrates, the theory given above merely postulates that the mobility of a solvent molecule is destroyed when its place in the surface of the liquid is occupied by a molecule of the solute.

SURFACE STRUCTURE OF LIQUIDS.

It will at once be noticed that in its simplest form the kinetic theory of osmotic pressure would indicate that the pressures calculated from the gas-formula might be subject to a small correction for the volume changes accompanying dissolution. Whether such corrections are necessary can only be determined by experiment, but the evidence now available points to a very close agreement between the values observed and calculated for dilute solutions. If this identity should be confirmed, it will be possible to deduce from observations of osmotic pressure some information in reference to the surface structure of liquids, since if the agreement is exact there must be an equally exact replacement of solvent by solute in the surface of the liquid. Thus in view of the different masses of the molecules that may be dissolved in the same solute and yield identical osmotic pressures, there must be a considerable spacing between the actual molecules in the surface. Again, it may be noted that this exact replacement does not take place in the interior of the liquid where the molecular volumes of different solutes differ widely, and when calculated in the conventional way may even have a negative value. It is, however, by no means improbable that the marshalling of the molecules on the frontiers of the liquid may be governed by a stricter discipline than that which prevails in the interior, and that the surface molecules may even be forced to conform to the exact regulations which govern the replacement of molecules in solid solutions or isomorphous mixtures.

FORMATION OF COMPLEXES.

The formation in the solution of loose complexes of solvent with solute or of solvent molecules with one another has not yet been referred to, but presents no difficulty in the development of the theory. Such complexes are usually formed without any large change of volume, and under the stricter

conditions prevailing at the surface no alteration in the area of surface occupied need result from the linking up of the "residual affinities" of the molecules. Neither need it be supposed that the rate of evaporation would be affected otherwise than by a general reduction of mobility* due to the chemical attraction of the molecules and producing equal effects in solvent and in dilute solution. Thus, as there is no change of energy involved in the replacement of one solvent molecule in a complex by another from outside, a rapidly moving particle impinging on a complex might drive a combined solvent molecule into the vapour space and itself occupy the vacant position in the complex, the effects produced being much the same as if no complex existed. A similar statement would apply to molecules of solvent attached to the solute in the form of labile hydrates or compounds; a free solvent molecule impinging on a combined molecule in the surface of the liquid might drive it out and take its place; but if it should impinge on the nuclear solute molecule it would be repulsed and driven back into the interior, just as it would be by an uncombined molecule of solute.

CONCENTRATED SOLUTIONS.

No attempt has been made in the above to account quantitatively for the osmotic pressure of concentrated solutions. Even in the case of gases, the equation $PV = RT$ only applies strictly to a material gas within narrow limits of pressure; but Morse and Fraser's experiments indicate a much wider applicability when the formula is applied to osmotic pressures, provided only that for concentrated solutions V is interpreted as the volume of solvent used to dissolve the solute and not the total volume of the solution. It need scarcely be pointed out that this modification is very similar in type to the co-volume correction in van der Waals's equation.

OSMOTIC PRESSURE AND SURFACE TENSION.

In view of the close relationship that has been indicated between osmotic pressure and surface structure it would not be surprising that a relationship should exist between surface tension and osmotic pressure. Such a relationship has been postulated theoretically by Traube, who supposes that osmotic pressure depends on a tendency to equalize the surface tension of the two liquids, and has been confirmed experimentally by

* Compare the reduction of mobility caused by liquid cohesion in a gas and represented by the quantity $\frac{a}{v^2}$ in van der Waals's equation.

Batelli and Stephanini (*Atti R. Accad. Lincei*, 1905, (v.) 14. ii. pp. 3-14), who find that solutions of equal surface tension have equal osmotic pressures, even in cases in which the solutions are not nominally equimolecular.

In conclusion it may be pointed out that whilst the views advocated above were comparatively novel ten years ago, the idea that osmotic pressure depends on the activity of the solvent rather than on that of the solute has now become widely accepted, and has been advocated not only by Poynting, but also by Armstrong (*Enc. Brit.* xxvi. p. 739), Beilby (B. A. Report, South Africa, 1905, p. 301), and others. At the present time, therefore, the only points for which any degree of novelty can be claimed are in reference to the mechanism by which the activity of the solvent at the surface of the liquid is reduced by the "blocking action" of the solute operating in one direction only, and to the possibility of deducing from the osmotic phenomena information as to the surface structure of liquids.

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XLVI. *Notices respecting New Books.*

Les Ultramicroscopes et les Objets Ultramicroscopiques. Par A. COTTON et H. MOUTON. Paris: Masson & Cie. 1906. Pp. 232.

THE important part played by the microscope in almost every department of science, pure and applied, readily accounts for the large amount of attention which this instrument has received and for the rapid improvements in its construction—improvements which have rendered it one of the most perfect instruments at our disposal. The limits beyond which a microscope of the ordinary type ceases to be serviceable—limits imposed on it by diffraction phenomena—having been reached, investigators began to turn their attention to means whereby objects too tiny to be visible under ordinary conditions could be so rendered, by the use of special devices, although such devices do not enable us to study the exact form or minute structure of the object viewed.

An account of the most recent researches in this direction will be found in Messrs. Cotton and Mouton's excellent little treatise, excellent alike from the point of view of the selection and arrangement of subject-matter, and that of simplicity of treatment. The subject is introduced by a very clear sketch of the physical theory of the microscope and the limits of visibility. The use of radiations of short wave-length is then dealt with, and in the next chapter is given an account of the conditions under which ultramicroscopic objects may be rendered visible, and a description of the special instruments devised by Siedentopf and Zsigmondy, and

by the authors themselves. The remainder of the work deals with ultra-microscopic objects. A chapter is devoted to the study of the minute particles embedded in transparent solids—such as particles of gold in ruby-glass—and those forming a thin deposit on their surfaces—such as metallic deposits on glass. Particles contained in liquids are next dealt with. Three chapters are devoted to the study of colloidal solutions. It is shown how the properties of such solutions may be accounted for by supposing them to possess a granular structure. The electrical behaviour of such solutions is studied in detail, and an account is given of the indirect methods which enable us to form some sort of notion regarding the form and properties of colloidal granules. The concluding chapter deals with the ultra-microscopic particles of interest to biologists.

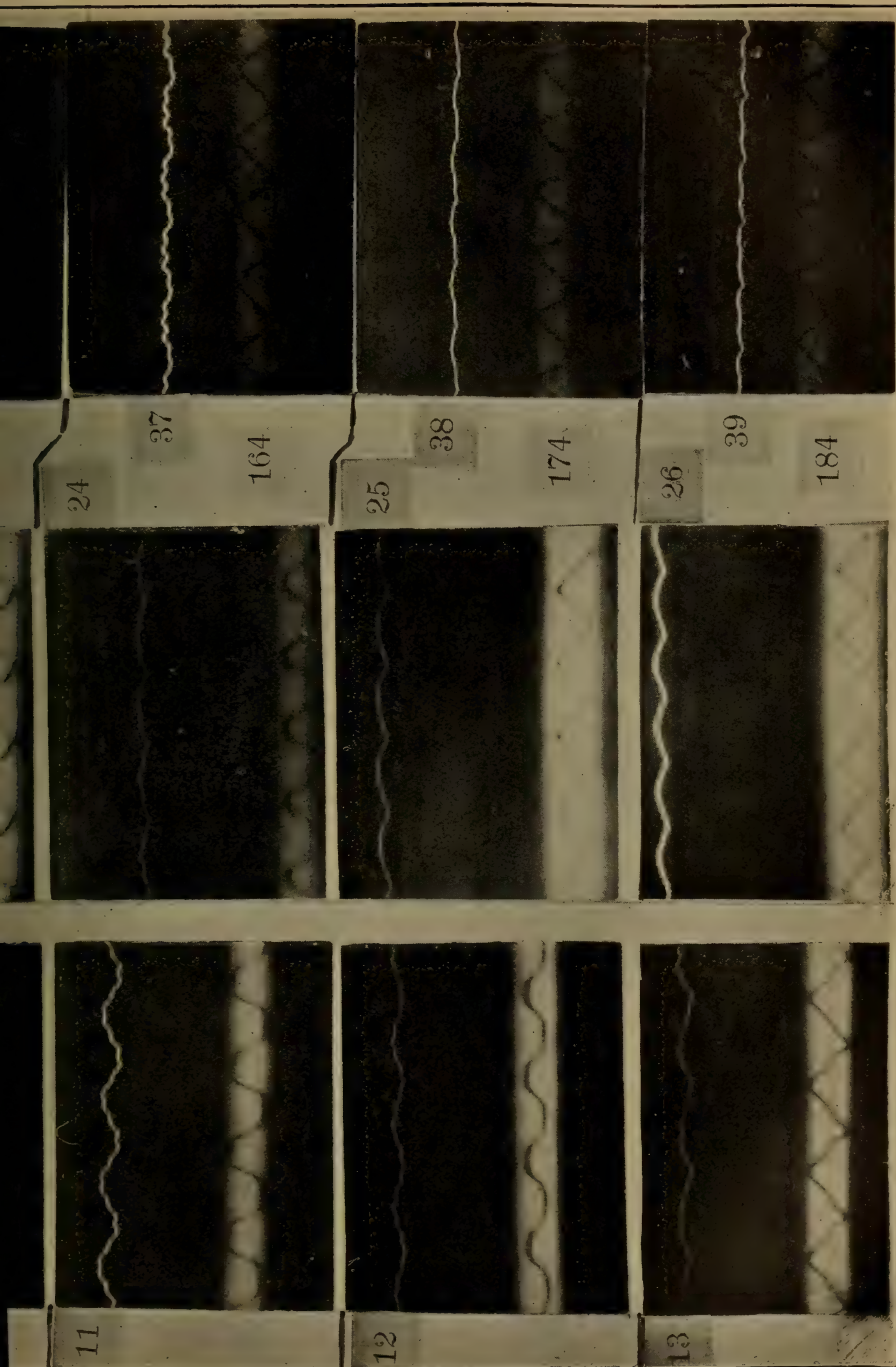
The authors are to be congratulated on having produced so readable and simply written a book on a very difficult subject, and no person interested in microscopic work can afford to be without it.

Lectures on the Theory of Functions of Real Variables. Vol. I.
By JAMES PIERPONT. Ginn & Co.: Boston, New York, &c. 1905.

THE distinguished Professor of Mathematics in Yale University has in these published lectures given an altogether unique book to the English-speaking student of pure mathematics. The idea at the basis of the treatment of what the ordinarily trained student will soon recognize as his familiar friends in the infinitesimal calculus is rigorous demonstration. More particularly stated, the problem is to examine the condition under which the recognized theorems and processes can be applied correctly. The foundations of the whole doctrine of mathematical continuity must be laid firm and sure; and to effect this the author is compelled to begin with the modern theory of rational and irrational numbers. Following mainly along the lines initiated by Cantor, he devotes several chapters to the theory of the elementary functions; and it is not till Chapter VIII. that the subject of differentiation of functions of one variable is taken up. On the basis of the doctrine of limits already established, the differential coefficients of the fundamental formulæ of differentiation are deduced with rigour, an important educative feature being the indication in certain cases of the incorrectness of the more usual methods of establishing these formulæ. From Chapter XII. to the last chapter (XVI.) integration is discussed with the same uniform method—clear, rigorous, and brief. At times, indeed, the discussion seems almost too brief, partaking more of the character of lecture notes than of a systematic treatise. This, however, has its advantages in making the reader complete the demonstration in his own way. There can be but one opinion as to the value of the work; and the publication of the second volume will be awaited with interest and expectation.



The photograph shows a pleochroic halo in a rock-section; magnified 300 diameters; light polarized. The halo-sphere extends across flakes of biotite mutually inclined, as is shown by the unequal absorption of the rays.



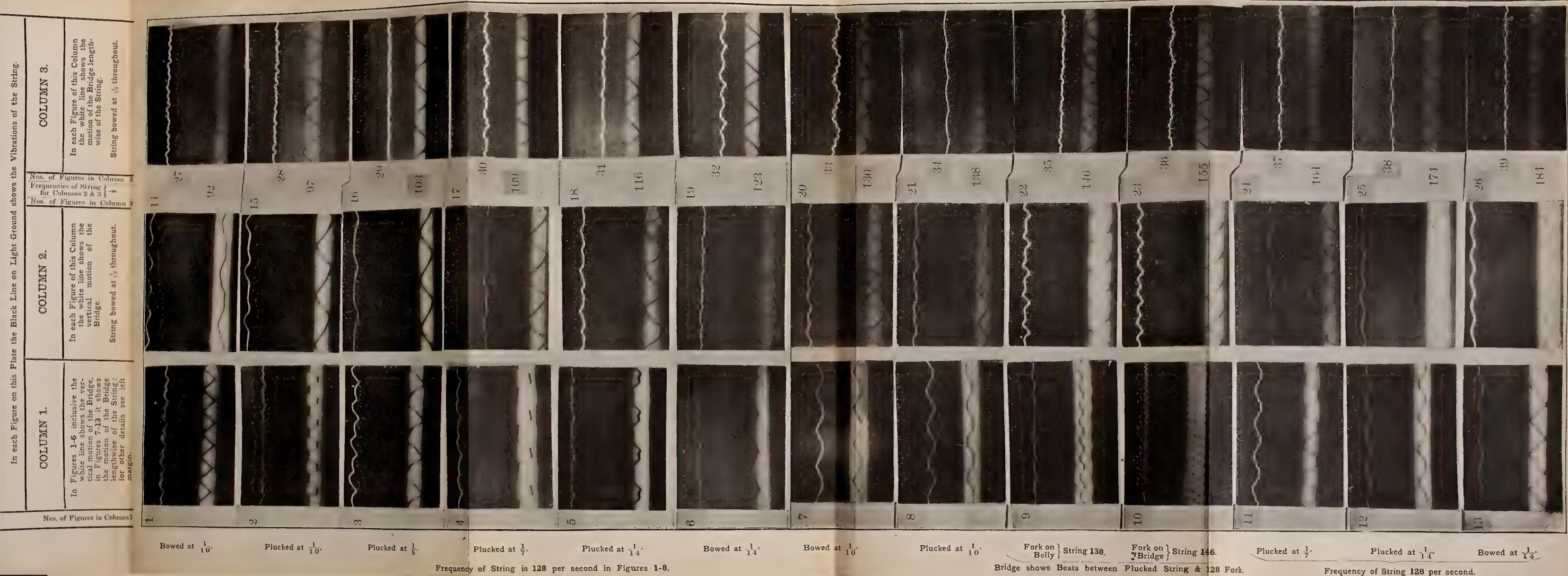
Plucked at $\frac{1}{7}$.

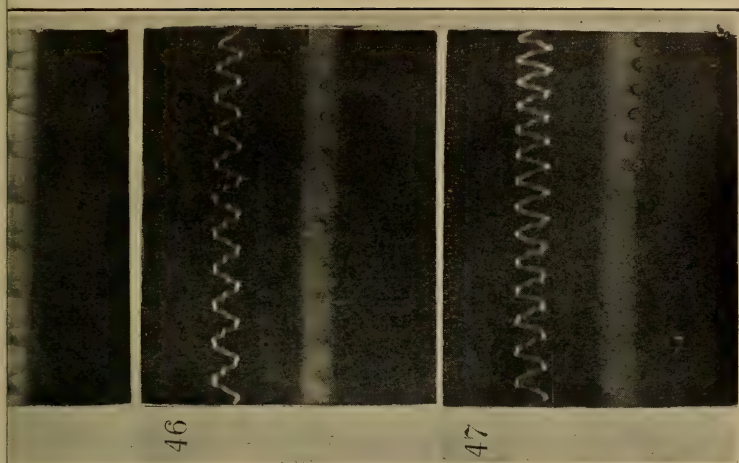
Plucked at $\frac{1}{14}$.

Bowed at $\frac{1}{14}$.

Fork.

Frequency of String 128 per second.



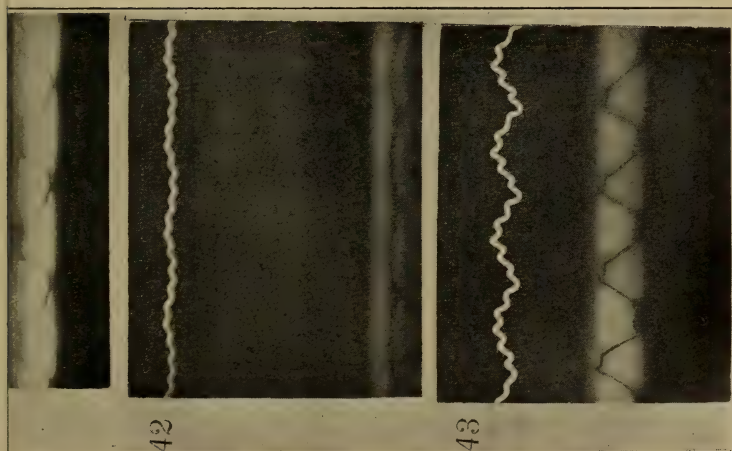


$\frac{1}{7}$.

Plucked at $\frac{1}{10}$.

Plucked at $\frac{1}{7}$.

Load '008 of String.



$\frac{1}{0}$.

Bowed at $\frac{1}{10}$.

Plucked at $\frac{1}{10}$.

Load '004 of String.

String loaded at Centre and of Frequency 128 throughout; the white line shows the motion of Bridge lengthwise of String.

COLUMN 1.

Plates shot fast.

Nos. of
Figures in Col. 1.

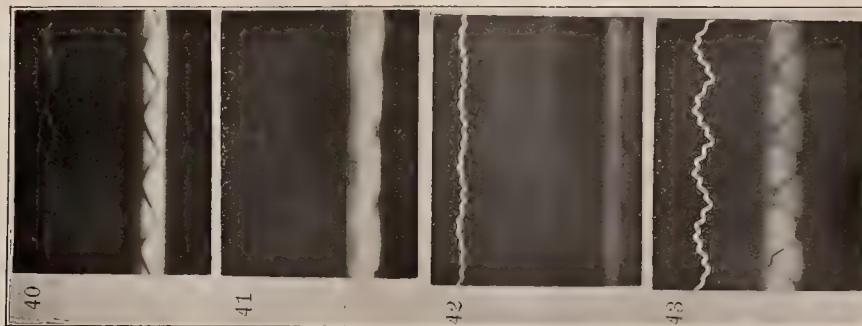
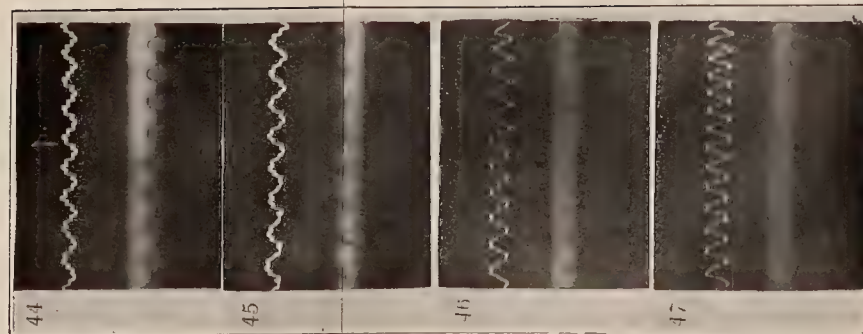
Loads &c. for Col. 1.

COLUMN 2.

Plates shot slow.

Nos. of
Figures in Col. 2.

Loads &c. for Col. 2.



Period = 0.1835 sec.

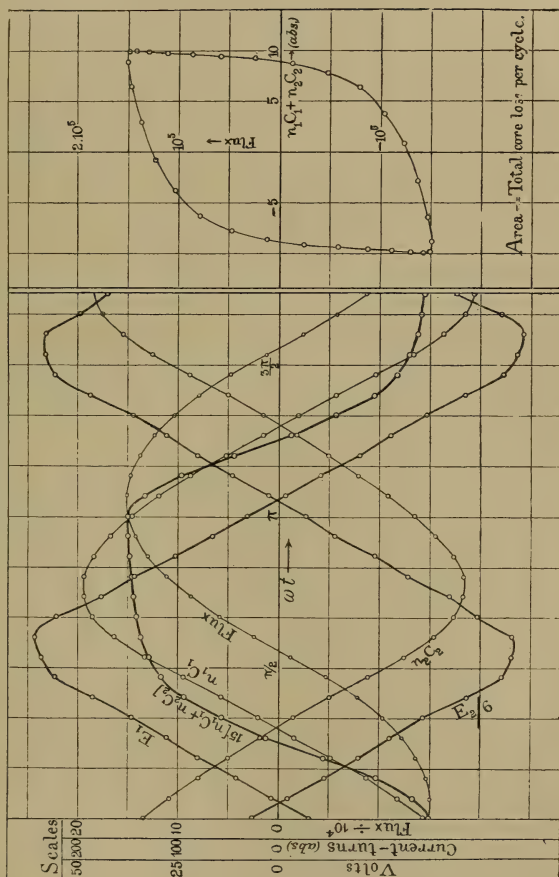


FIG. 2.—TRANSFORMER DIAGRAM. No load. Period = 0.02015 sec.

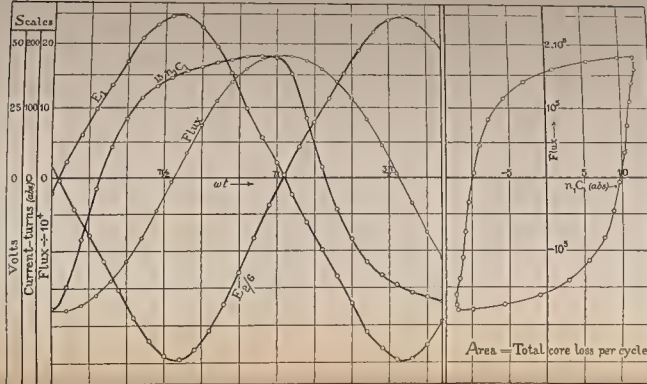


FIG. 3.—TRANSFORMER DIAGRAM. Non-inductive load. Period = 0.02053 sec.

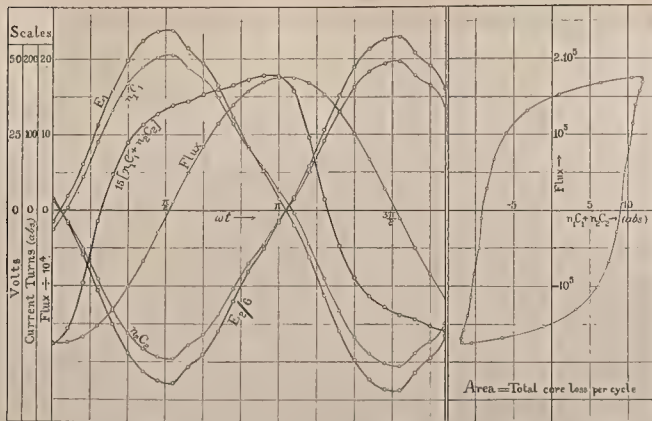
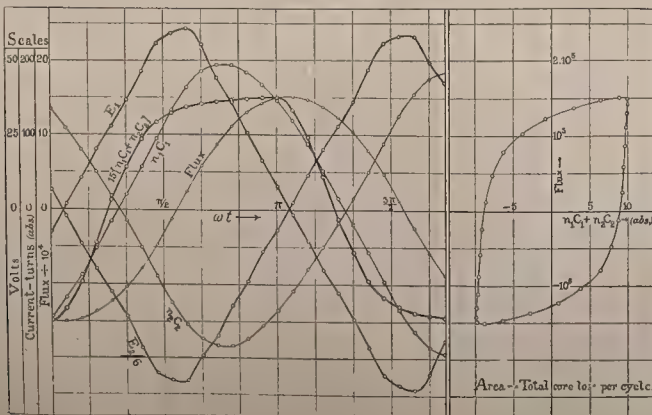


FIG. 4.—TRANSFORMER DIAGRAM. Inductive load; power factor = .73. Period = 0.01835 sec.





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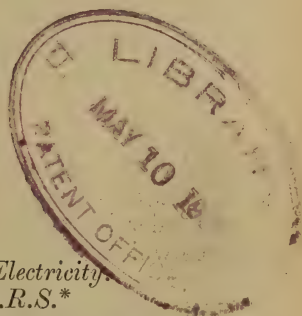
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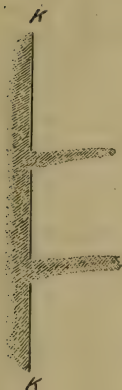


XLVII. *On Rays of Positive Electricity.*

By J. J. THOMSON, M.A., F.R.S.*

IN 1886 Goldstein discovered that when the cathode in a discharge-tube is perforated, rays pass through the openings and produce luminosity in the gas behind the cathode; the colour of the light depends on the gas with which the tube is filled and coincides with the colour of the velvety glow which occurs immediately in front of the cathode. The appearance of these rays is indicated in fig. 1, the anode being to the left of the cathode KK. Since the rays appeared through narrow channels in the cathode, Goldstein called them "Kanalstrahlen": now that we know more about their nature, "positive rays" would, I think, be a more appropriate name. Goldstein showed that a magnetic force which would deflect cathode rays to a very considerable extent was quite without effect on the "Kanalstrahlen." By using intense magnetic fields, W. Wien showed that these rays could be deflected, and that the deflexion was in the

Fig. 1.



opposite direction to that of the cathode rays, indicating that these rays carry a positive charge of electricity. This was confirmed by measuring the electrical charge received by a vessel into which the rays passed through a small hole, and also by observing the direction in which

* Communicated by the Author.

they are deflected by an electric force. By measuring the deflexions under magnetic and electric forces, Wien found by the usual methods the value of e/m and the velocity of the rays. He found for the maximum value of e/m the value of 10^4 , which is the same as that for an atom of hydrogen in the electrolysis of solutions. A valuable summary of the properties of these rays is contained in a paper by Ewers*.

As these rays seem the most promising subjects for investigating the nature of positive electricity, I have made a series of determinations of the values of e/m for positive rays under different conditions. The results of these I will now proceed to describe.

Apparatus.

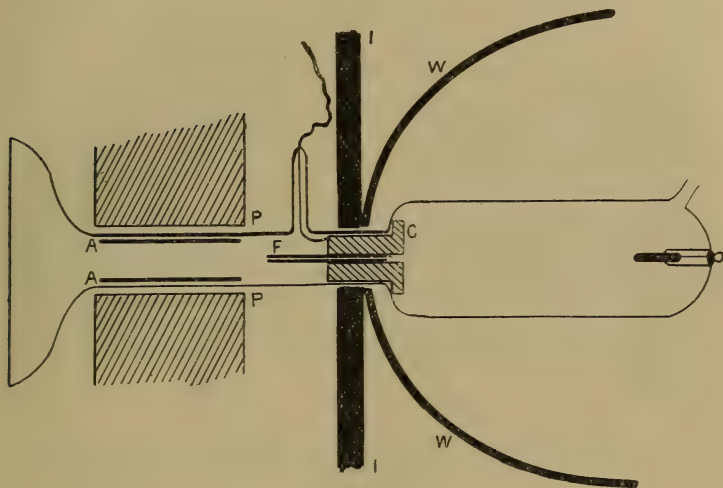
Screen used to detect the rays.—The rays were detected and their position determined by the phosphorescence they produced on a screen at the end of the discharge-tube. A considerable number of substances were examined to find the one which would fluoresce most brightly under the action of the rays. As the result of these trials, Willemite was selected. This was ground to a very fine powder and dusted uniformly over a flat plate of glass. Considerable trouble was found in obtaining a suitable substance to make the powder adhere to the glass. All gums &c. when bombarded by the rays are liable to give off gas; this renders them useless for work in vacuum-tubes. The method finally adopted was to smear a thin layer of "water-glass" (sodium-silicate) over the glass plate, and then dust the powdered Willemite over this layer and allow the water-glass to dry slowly before fastening the plate to the end of the tube.

The form of tube adopted is shown in fig. 2. A hole is bored through the cathode, and this hole leads to a very fine tube F. The bore of this tube is made as fine as possible so as to get a small well-defined fluorescent patch on the screen. These tubes were either carefully made glass tubes, or else the hollow thin needles used for hypodermic injections, which I find answer excellently for this purpose. After getting through the needle, the positive rays on their way down the tube pass between two parallel aluminium plates A, A. These plates are vertical, so that when they are maintained at different potentials the rays are subject to a horizontal electric force, which produces a horizontal deflexion of the patch of light on the screen. The part of the tube containing the parallel aluminium plates is narrowed as much as possible, and passes between the poles P, P of a powerful electromagnet of the Du Bois type. The poles of this magnet

* *Jahrbuch der Radioaktivität*, iii. p. 291 (1906).

are as close together as the glass tube will permit, and are arranged so that the lines of magnetic force are horizontal and at right angles to the path of the rays. The magnetic force produces a vertical deflexion of the patch of phosphorescence on the screen. To bend the positive rays it is necessary to use strong magnetic fields, and if any of the lines of force were to stray into the discharge-tube in front

Fig. 2.



of the cathode, they would distort the discharge in that part of the tube. This distortion might affect the position of the phosphorescent patch on the screen, so that unless we shield the discharge-tube we cannot be sure that the displacement of the phosphorescence is entirely due to the electric and magnetic fields acting on the positive rays after they have emerged from behind the cathode.

To screen off the magnetic field, the tube was placed in a soft iron vessel *W* with a hole knocked in the bottom, through which the part of the tube behind the cathode was pushed. Behind the vessel a thick plate of soft iron with a hole bored through it was placed, and behind this again as many thin plates of soft iron, such as are used for transformers, as there was room for were packed. When this was done it was found that the magnet produced no perceptible effect on the discharge in front of the cathode.

The object of the experiments was to determine the value of e/m by observing the deflexion produced by magnetic and electric fields. When the rays were undeflected they produced a bright spot on the screen ; when the rays passed

through electric and magnetic fields the spot was not simply deflected to another place, but was drawn out into bands or patches, sometimes covering a considerable area. To determine the velocity of the rays and the value of e/m , it was necessary to have a record of the shape of these patches. This might have been done by substituting a photographic plate for the Willemite screen. This, however, was not the method adopted, as, in addition to other inconveniences, it involves opening the tube and repumping for each observation, a procedure which would have involved a great expenditure of time. The method actually adopted was as follows:—The tube was placed in a dark room from which all light was carefully excluded, the tube itself being painted over so that no light escaped from it. Under these circumstances the phosphorescence on the screen appeared bright and its boundaries well defined. The observer traced in Indian ink on the outside of the thin flat screen the outline of the phosphorescence. When this had been satisfactorily accomplished the discharge was stopped, the light admitted into the room, and the pattern on the screen transferred to tracing-paper; the deviations were then measured on these tracings.

Calculation of the Magnetic and Electric Deviation of the Rays.

If we assume the electric field to be uniform between the plates and zero outside them, then we can easily show that x , the horizontal deflexion of a ray whose charge is e , mass m , and velocity v , is given by the equation

$$x = \frac{1}{2} X \frac{e}{mv^2} l(l + 2d),$$

where X is the force between the plates, l the length of path of the rays between the plates, and d the distance of the screen from the nearer end of the parallel plates.

To find the deflexion due to the magnetic field, we have, if ρ is the radius of curvature of the path at a point where the magnetic force is H ,

$$\frac{mv^2}{\rho} = H e v,$$

or

$$\frac{1}{\rho} = \frac{e}{mv} H.$$

If y is the vertical displacement of the particle, we have

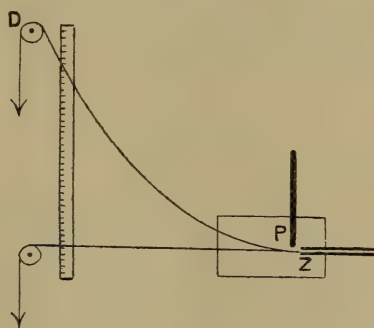
$$\frac{1}{\rho} = \frac{d^2y}{dz^2} \text{ approximately,}$$

where z is measured along the path of the ray. Hence

$$\frac{d^2y}{dz^2} = \frac{e}{mv} H; \\ y = \frac{e}{mv} \left[\int_0^{l+d} \int_0^z H dz \right]. \quad . \quad . \quad . \quad (1)$$

In these strong fields there are considerable variations of H along the path, so that to calculate the integrals we should have to map out the value of H along the path of the ray. This would be a very laborious process, and it was rendered unnecessary by the following simple method, which, while not involving anything like the labour of the direct method, gives much more accurate results. The method is shown in fig. 3.

Fig. 3.



The part of the tube through which the rays pass was cut off, and a metal rod placed so that its tip Z coincided with the aperture of the narrow tube through which the positive rays had emerged. A very fine wire soldered to the end of this tube passed over a light pulley, and carried a weight at the free end. The pulley was supported by a screw by means of which it could be raised or lowered; a known current passed through the wire, entering it at Z and leaving it through the pulley. The pulley was first placed so that the path of the stretched wire when undeflected by a magnetic field coincided with the path of the undeflected rays. A vertical scale whose edge was at the same distance from the

opening through which the rays emerge as the screen on which the phosphorescence had been observed, was placed just behind the wire, and was read by a reading microscope with a micrometer eyepiece. When the magnetic field was put on, the wire was deflected; and if T is the tension of the wire, ρ the radius of curvature into which it is bent, i the current through the wire,

$$\frac{T}{\rho} = Hi;$$

or, if y_1 is the vertical displacement of the wire,

$$\frac{d^2 y_1}{dz^2} = \frac{i}{T} \cdot H.$$

Now if $\frac{dy_1}{dz} = 0$ when $z = 0$ we have, if y_1 is the displacement of the wire at the scale,

$$y_1 = \frac{i}{T} \int_0^l \cdot \int_0^z H dz. \quad . \quad . \quad . \quad . \quad (2)$$

Hence, comparing (1) and (2) we have

$$\frac{y}{y_1} = \frac{\frac{e}{mv}}{\frac{i}{T}}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

a relation from which the magnetic force is eliminated. To ensure that the tangent to the wire is horizontal when $z = 0$, the following method is used. P is a chisel-edge carried by a screw and placed about 1 mm. in front of the fixed end of the wire; this is adjusted so that when the magnetic field is not on the wire just touches the edge: this can be ascertained by making the contact with the wire complete an electric circuit in which a bell is placed. When the magnetic field is put on the wire is pulled off from the edge, and the tangent at $z = 0$ is no longer horizontal; it can, however, be brought horizontal by raising or lowering the pulley D until the wire is again in contact with P , which can be ascertained again by the ringing of the bell. Then y_1 is the vertical distance between the point where the wire now crosses the edge of the scale and the point where it crossed it before the magnetic field was put on. Since y , y_1 , i , and T can easily be measured, equation (3) gives us the value of e/mv , while the deflexion under the electric force gives the value of e/mv^2 .

If y is the vertical displacement of the patch of phosphorescent light on the screen produced by the magnetic field, x the horizontal displacement due to the electrostatic field, we see that

$$y = \frac{y_1}{(i/T)} \frac{e}{mv} = B \frac{e}{mv},$$

$$x = A \frac{e}{mv^2},$$

where A and B are constants depending on the position of the screen and the magnitudes of the electric and magnetic forces. These quantities can be calculated by means of the equations just given.

Since

$$\frac{y}{x} = \frac{B}{A} v,$$

$$\frac{y^2}{x} = \frac{B^2}{A} \frac{e}{m}.$$

We see that if the pencil is made up of rays having a constant velocity but having all values of e/m up to a maximum value, the spot of light will be spread out by the magnetic and electric fields into a straight line extending a finite distance from the origin. While if it is made up of two sets of

Fig. 4.



Fig. 5.



rays, one having the velocity v_1 the other the velocity v_2 , the spot will be drawn out into two straight lines as in fig. 4.

If e/m is constant and the velocities have all values up to a maximum, the spot of light will be spread out into a portion of a parabola, as indicated in fig. 5.

We shall later on give examples of each of these cases.

The discharge was produced by means of a large induction-coil, giving a spark of about 50 cm. in air, with a vibrating make and break apparatus. Many tubes were used in the course of the investigation, the dimensions of these varied slightly. The distance of the screen from the hole from which the rays emerged was about 9 cm., the length of the parallel plates about 3 cm., and the distance between them .3 cm.

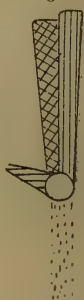
Properties of the Positive Rays when the Pressure is not exceedingly low.

The appearance of the phosphorescent patch after deflexion in the electric and magnetic fields depends greatly upon the pressure of the gas. I will begin by considering the case when the pressure is comparatively high, say of the order of $1/50$ of a millimetre. At these pressures, though the walls of the tube in front of the cathode were covered with bright phosphorescence and the dark space extended right up to the walls of the tube and was several centimetres thick, traces of the positive column could be detected in the neighbourhood of the anode. I will first take the case where the tube was filled with air. Special precautions were taken to free the air from hydrogen; it was carefully dried, and a subsidiary discharge-tube having a cathode made of the liquid alloy of sodium and potassium was fused on to the main tube. When the discharge passes from such a cathode it absorbs hydrogen. The discharge was sent through this tube at the lowest pressure at which enough light was produced in the gas to give a visible spectrum, until the hydrogen lines disappeared and the only lines visible were those of nitrogen and mercury vapour. This pressure was a little higher than that used for the investigation of the positive rays, but a pump or two was sufficient to bring the pressure down to this value. The appearance of the phosphorescence on the screen when the rays were deflected by magnetic and electric forces separately and conjointly is shown in fig. 6.

The deflexion under magnetic force alone is indicated by vertical shading, under electric force alone by horizontal shading, and under the two combined by cross shading.

The spot of phosphorescence is drawn out into a band on either side of its original position. The upper portion, which is very much the brighter, is deflected in the direction which indicates that the phosphorescence is produced by rays having a positive charge; the lower portion (indicated by dots in the figure), which though faint is quite perceptible on the Willemite screen, is deflected as if the rays carried a negative charge. The length of the lower portion is somewhat shorter than that of the upper one, but is quite comparable with it. The intensity of the luminosity in the upper portion is at these pressures quite continuous; no abrupt variations such as would show themselves as bright patches could be detected, although, as will be seen later on, these make their appearance at lower pressures. Considering for the present the upper portion, the straightness of the edges shows that the velocity of the rays is approxi-

Fig. 6.



mately constant, while the values of e/m range from zero at the undeflected portion to the value approximately equal to 10^4 at the top of the deflected band. This value of e/m is equal to that for a charged hydrogen atom, and moreover there was no specially great luminosity in the positions corresponding to $e/m=10^4/14$ and $10^4/16$, the values for rays carried by nitrogen or oxygen atoms, though these places were carefully scrutinised. As hydrogen when present as an impurity in the tube has a tendency to accumulate near the cathode, the following experiment was tried to see whether the Kanalstrahlen were produced from traces of hydrogen in the tube. The discharge was sent through the tube in the opposite direction, *i. e.*, so that the perforated electrode was the anode, the electric and magnetic fields being kept on. When the discharge passed in this way there was of course no luminosity on the screen; on reversing the coil again so that the perforated electrode was the cathode, the luminosity flashed out instantly, presenting exactly the same appearance as it had done when the tube had been running for some time with the perforated electrode as cathode.

The fact that a spot of light produced by the undeflected positive rays is under the action of electric and magnetic forces drawn out into a continuous band was observed by W. Wien, who was the first to measure the deflexion of the positive rays under electric and magnetic forces. The values of e/m obtained from the deflexions of various parts of this band range continuously from zero, the value corresponding to the undeflected portion, to 10^4 , the value corresponding to those most deflected. Wien explained this by the hypothesis that the charged particles which make up the positive rays act as nuclei round which molecules of the gas through which the rays pass condense, so that very complex systems made up of a very large number of molecules get mixed up with the particles forming the positive rays, and that it is these heavy and cumbrous systems which give rise to that part of the luminosity which is only slightly deflected. I think that the constancy of the velocity of the rays, indicated by the straight edges of the deflected band, is a strong argument against this explanation, and that the existence of the negative rays is conclusive against it. These negatively electrified rays, which form the faintly luminous portion of the phosphorescence indicated in fig. 6, are not cathode rays. The magnitude of their deflexion shows that the ratio of e/m for these rays, instead of being as great as 1.7×10^7 , the value for cathode rays, is less than 10^4 . The particles forming these rays are thus comparable in size with those which form the positive rays. The existence of these negatively electrified rays suggests

at once an explanation, which I think is the true one, of the continuous band into which the spot of phosphorescence is drawn out by the electric and magnetic fields. The values of e/m which are determined by this method are really the mean values of e/m , while the particle is in the electric and magnetic fields. If the particles are for a part of their course through these fields without charge, they will not during this part of their course be deflected, and in consequence the deflexions observed on the screen, and consequently the values of e/m , will be smaller than if the particle had retained its charge during the whole of its career. Thus, suppose that some of the particles constituting the positive rays, after starting with a positive charge, get this charge neutralized by attracting to them a negatively electrified corpuscle: the mass of the corpuscle is so small in comparison with that of the particle constituting the positive ray, that the addition of the particle will not appreciably diminish the velocity of the positive particle. Some of these neutralized particles may get positively ionized again by collision, while others may get a negative charge by the adhesion to them of another corpuscle, and this process might be repeated during the course of the particle. Thus there would be among the rays some which were for part of their course unelectrified, at other parts positively electrified, and at other parts negatively electrified. Thus the mean value of e/m might have all values ranging from α , its initial value, to $-\alpha'$, where α' might be only a little less than α . This is just what we observe, and when we remember that the gas through which the rays are passing is ionized, and contains a large number of corpuscles, it is, I think, what we should expect.

At very low pressures, when there are very few ions in the gas, this continuous band stretching from the origin is replaced by discontinuous patches.

Positive Rays in Hydrogen.

In hydrogen, when the pressure is not too low, the brightness of the phosphorescent patch is greater than in air at the same pressure; the shape of the deflected phosphorescence is markedly different from that in air. In air, the deflected phosphorescence is usually a straight band, whereas in hydrogen the boundary of the most deflected side is distinctly curved and is concave to the undeflected position. The appearance of the deflected phosphorescence is indicated in fig. 7.

The result indicated in fig. 8, which was also obtained with hydrogen, shows that we have here a mixture of two bands, as indicated in fig. 4, the two bands being produced by carriers having different maximum values of e/m . The greatest value

of e/m obtained with hydrogen was the same as in air, 1.2×10^4 , the velocity was 1.8×10^8 cm./sec. The presence

Fig. 7.



Fig. 8.



of the second band indicates that mixed with these we have another set of carriers, for which the maximum value e/m is half that in the other band, *i. e.* 5×10^3 . The curvature of the boundary generally observed is due to the admixture of these two rays.

Positive Rays in Helium.

In helium the phosphorescence is bright and the deflected patch has in general the curved outline observed in hydrogen. I was fortunate enough, however, to find a stage in which

Fig. 9.



the deflected patch was split up into two distinct bands, as shown in fig. 9. The maximum value of e/m in the band *a* was 1.2×10^4 , the same as in air and hydrogen, and the velocity was 1.8×10^8 ; while the maximum value of e/m in band *b* was almost exactly one quarter of that in *a* (*i. e.* 2.9×10^3). As the atomic weight of helium is four times that

of hydrogen, this result indicates that the carriers which produce the band *b* are atoms of helium. This result is interesting because it is the only case (apart from hydrogen) in which I have found values of e/m corresponding to the atomic weight of the gas; and even in the case of helium, when the pressure in the discharge-tube is very low and the electric field very intense, the characteristic rays with $e/m = 2.9 \times 10^3$ sometimes disappear and, as in all the gases I have tried, we get two sets of rays, for one set of which $e/m = 10^4$ and for the other 5×10^3 .

Although the helium had been carefully purified from hydrogen, the band *a* (for which $e/m = 10^4$) was generally the brighter of the two. The case of helium is an interesting one; for the class of positive rays, known as the α rays, which are given off by radioactive substances, would *a priori* seem to consist most probably of helium, since helium is one of the products of disintegration of these substances. The value of e/m for these substances is 5×10^3 , where we have seen that in helium it is possible to obtain rays for which $e/m = 2.9 \times 10^3$. It is true that, at very low pressures and with strong electric

fields, we get rays for which $e/m = 5 \times 10^3$; but this is not a peculiarity of helium: all the gases which I have tried show exactly the same effect.

Argon.

When the discharge passed through argon the effects observed were very similar to those occurring in air. The sides were perhaps a little more curved, and there was a tendency for bright spots to develop. The measurements of the electric and magnetic deflexion of these spots gave $e/m = 10^4$, the value obtained for other cases. There was no appreciable increase of luminosity in the positions corresponding to $e/m = 10^4/40$, as there would have been if an appreciable number of the carriers had been argon atoms.

Positive Rays in Gases at very low pressures.

As the pressure of the gas in the discharge-tube is gradually reduced, the appearance of the deflected phosphorescence changes: instead of forming a continuous band, the phosphorescence breaks up into two isolated patches; that part of the phosphorescence in which the deflexion was very small disappears, as also does the phosphorescence produced by the negatively electrified portion of the rays.

In the earlier experiments considerable difficulty was experienced in working at these very low pressures; for when the pressure was reduced sufficiently to get the effects just described, the discharge passed through the tube with such difficulty, that in a very few seconds after this stage was reached sparks passed from the inside to the outside of the tube, perforating the glass and destroying the vacuum. In spite of all precautions, such as earthing the cathode and all conductors in its neighbourhood, perforation took place too quickly to permit measurements of the deflexion of the phosphorescence.

This difficulty was overcome by taking advantage of the fact that, when the cathode is made of a very electropositive metal, the discharge passes with much greater ease than when the cathode is made of aluminium or platinum. The electropositive metals used for the cathode were (1) the liquid alloy of sodium and potassium which was smeared over the cathode, and (2) calcium, a thin plate of which was affixed to the front of the cathode. With these cathodes the pressure in the tube could be reduced to very low values without making the discharge so difficult as to lead to perforation of the tube by sparking, and accurate measurements of the position of the patches of phosphorescence could be obtained at leisure.

The results obtained at these low pressures are very interesting. Whatever kind of gas may be used to fill the

tube, or whatever the nature of the electrode, the deflected phosphorescence splits up into two patches. For one of these patches the maximum value of e/m is about 10^4 , the value for the hydrogen atom ; while the value for the other patch is



about 5×10^3 , the value for α particles or the hydrogen molecule. Examples of the appearance of this phosphorescence are given in figs. 10, 11, 12 ; in fig. 12 the magnetic force was reversed.

The differences in the appearance are due to differences in the pressure rather than to differences in the gas ; for at slightly higher pressures than that corresponding to fig. 12, the appearance shown in figs. 10 and 11 can be obtained in air. In all these cases the more deflected patch corresponds to a value of about 10^4 for e/m , while e/m for the less deflected patch is about 5×10^3 .

It will be noticed that in fig. 11 there is no trace in the helium tube of rays for which $e/m = 2.5 \times 10^3$, which were found in helium tubes at higher pressures ; at intermediate pressures there are *three* distinct patches in helium, for the first of which $e/m = 10^4$, for the second $e/m = 5 \times 10^3$, and for the third $e/m = 2.5 \times 10^3$ approximately. Helium is a case where there are characteristic rays—*i. e.*, rays for which $e/m = 10^4/M$, where M is the atomic weight of the gas, when the discharge potential is comparatively small, and not when, as at very low pressures, the discharge potential is very large. I think it very probable that if we could produce the positive rays with much smaller potential differences than those used in these experiments, we might get the characteristic rays for other gases. I am at present investigating with this object the positive rays produced when the perforated cathode is, as in Wehnelt's method, coated with lime, when a potential difference of 100 volts or less is able to produce positive rays. The interest of the experiments at very low pressures lies in the fact that in this case the rays are the same whatever gas may be used to fill the tube ; the characteristic rays of the gas disappear, and we get the same kind of carriers for all substances.

I would especially call attention to the simplicity of the effects produced at these low pressures : only two patches of

phosphorescence are visible. This is, I think, an important matter in connexion with the interpretation of these results ; for at these low pressures we have to deal not only with the gas with which the tube was originally filled, but also with the gas which is given off by the electrodes and the walls of the tube during the discharge ; and it might be urged that at these low pressures the tube contained nothing but hydrogen given out by the electrodes. I do not think this explanation is feasible, for the following reasons :—

(1) The gas developed during the discharge is not wholly hydrogen : if the discharge is kept passing long enough to develop so much gas that the discharge through the gas is sufficiently luminous to be observed by a spectroscope, the spectrum always showed, in addition to the hydrogen lines, the nitrogen bands ; indeed, the latter were generally the most conspicuous part of the spectrum. If the phosphorescent screen on which the positive rays impinge is observed during the time this gas is being given off, the changes which take place in the appearance of the screen are as follows :—If, to begin with, the pressure is so low that the phosphorescent patches are reduced to two bright spots, then, as the pressure begins to go up owing to the evolution of the gas, the deflexion of the spots increases. This is owing to the reduction in the velocity of the rays consequent upon the reduction of the potential difference between the terminals of the tube, as at this stage an increase in the pressure facilitates the passage of the discharge. In addition to the increase in the displacement, there is an increase in the area of the spots giving a greater range of values of e/m ; this is owing to the increase in the number of collisions made by the particles in the rays on their way to the screen. As more and more gas is evolved, the patches get larger and finally overlap ; the existence of the second patch being indicated by a diminution in the brightness of the phosphorescence at places outside its boundary. As the pressure increases the luminosity gets more and more continuous, and we finally get to the continuous band as shown in fig. 6. At this stage it is probable that there may be enough luminosity to give a spectrum showing the nitrogen lines, indicating that a considerable part of the gas in the tube is air. It is especially to be noted that during this process, when gas was coming into the tube, there has been no development of patches in the phosphorescence indicating the presence of new rays ; on the contrary, one type of carrier—that corresponding to $e/m = 5 \times 10^3$ —has disappeared. The presence of the nitrogen bands in the spectrum shows that nitrogen is carrying part of the discharge, and yet there are no rays characteristic of

nitrogen to be observed on the screen ; a proof, it seems to me, that different gases may be made by strong electric fields to give off the same kind of carriers of positive electricity.

Another result which shows that the positive rays are the same even although the gases are different is the following. The tube was pumped until the pressure was much too low for the discharge to pass, then small quantities of the following gases were put into the tube : air, carbonic oxide, hydrogen, helium, neon (for which I am indebted to the kindness of Sir James Dewar); the quantity admitted was adjusted so that it was sufficient to cause the discharge to pass and yet did not raise the pressure beyond the point where the phosphorescence is discontinuous. In every case there were patches corresponding to $e/m = 10^4$, $e/m = 5 \times 10^3$, and except with helium these were the only patches ; in helium, in addition to the two already mentioned, there was a third patch for which $e/m = 2.5 \times 10^3$.

I also tried another method of ensuring that at these low pressures there were other gases besides hydrogen in the tube. I filled the tube with helium, and after exhausting to a fairly low pressure by means of the mercury pump, I performed the last stages of the exhaustion by means of charcoal cooled with liquid air. This charcoal absorbs very little helium in comparison with other gases ; so that it is certain that there was helium in the tube. The appearance of the phosphorescent screen of tubes exhausted in this way did not differ from those exhausted solely by the pump.

The most obvious explanation of these effects seems to me to be that under very intense electric fields different substances give out particles charged with positive electricity, and that these particles are independent of the nature of the gas from which they originate. These particles are, as far as we know at present, of two kinds : for one kind e/m has the value of 10^4 , that of an atom of hydrogen ; for the other kind e/m has half this value, *i. e.* it has the same value as for the α particles from radioactive substances.

This agreement in the maximum value of e/m at different pressures is a proof that this is a true maximum, and that there are not other more deflected rays not strong enough to produce visible phosphorescence ; for if this were the case—*i. e.*, if the value of e/m for a particle that had never lost its charge temporarily by collision were greater than 10^4 —we should expect to get larger values for e/m at low pressures than at high.

I have much pleasure in thanking my assistant Mr. E. Everett for the assistance he has given me in these experiments.

XLVIII. Note on a Spectrum of the Röntgen Rays from a Focus Tube, and the relatively Selective Absorption of Röntgen Rays in certain Metals. By JOHN MEAD ADAMS.*

[Plate XIV.]

IN the course of a research† upon the transmission of Röntgen rays through metallic sheets, it became necessary to test by direct experiment Röntgen's theory that an ordinary beam of Röntgen rays is heterogeneous and that substances show selective absorption toward the different kinds of rays; and furthermore, to ascertain whether the selective absorption, if it exists, follows the same law for all substances, in other words whether the absorption of different substances is relatively selective.

To obtain a direct answer to these questions a spectrum of Röntgen rays was sought by the following method:—A Röntgen-ray tube (see fig. 1, Pl. XIV.) was prepared, like an ordinary focus-tube in all essential respects except for the target. The target consisted of a strip of platinum 6·3 cms. long by 1·3 cms. wide, bent into a circular arc of 5 cms. radius and placed in the tube in the position indicated by T in the figure. A thick lead screen was set up in front of the tube (that is, facing the concave side of the target) in a plane parallel to the axis of the tube, and about 18 cms. distant from it. At a point opposite to the target this screen was pierced by a small hole about 0·15 cm. in diameter, with bevelled edge. A photographic plate or a fluorescent screen could be placed parallel to the lead screen and about as far in front of it as the axis of the tube was behind it. When the tube was in operation under these circumstances, an observer at the fluorescent screen perceived a bright spot upon it—the image of the spot on the target where the cathode discharge from the electrode C focussed, formed according to the principle of the pin-hole camera. A magnetic field was then applied to the tube in the neighbourhood of the electrode C, in direction perpendicular to the plane of the paper, and of such magnitude that the cathode discharge was spread into a spectrum‡ along the concave surface of the target. The bright spot upon the fluorescent screen was now drawn out into a band, and it was to be expected that the Röntgen rays

* Communicated by Professor J. Trowbridge.

† Adams, *Proc. Amer. Acad.* xlii. p. 671 (1907).

‡ Birkeland, *Comptes Rendus*, cxxiii. p. 492 (1896).

at different points along this band would show different properties, since they were produced by cathode particles which differed from one another, presumably in velocity.

That a more or less complete separation of the Röntgen rays into a spectrum was actually effected was made plain by dividing the band into halves lengthwise, and photographing it while one-half was covered by a sheet of copper 0.0044 cm. thick, the other half being uncovered. Fig. 2 shows a photograph thus obtained. The width of the band was limited by brass strips with bevelled edges about 0.3 cm. apart. The Röntgen rays produced by the least deviated cathode particles are at A, those produced by the most deviated cathode particles at B. The difference in absorbing power of the copper at the two ends of the band is plain. The variations in intensity along the uncovered half of the spectrum give a rough idea of the distribution of the different kinds of rays.

The existence of relatively selective absorption in the case of one pair of metals, aluminium and silver, was shown by photographing one half of the spectrum through a sheet of one metal, the other half through a sheet of the other. The aluminium was 0.16 cm. thick, the silver 0.002 cm. Fig. 3 shows a set of these photographs. In that figure, CD is a comparison spectrum, photographed without the interposition of any metal. In the photograph EF, the silver was on the left and the aluminium on the right. The thicknesses of the two sheets were so chosen that the rays at E, corresponding to the least deviated cathode particles, were equally absorbed in them. Under these circumstances, the rays at F were transmitted by the silver in much greater quantity than by the aluminium. To show that this effect was not to be explained by a difference in thickness of either sheet at different points along the spectrum, the sheets were turned in their own plane through two right angles, and the photograph GH was then made, showing the effect unchanged. Secondary radiation from the sheets was not present in appreciable quantity at the photographic plate; for if it had been, it would have resulted in a blurring of the boundaries of the shadows on the plate, the distance between the latter and the sheets being about 2 cms.

Another pair of metals, aluminium and tin, has been found to show relatively selective absorption to some extent, but the effect has not yet been obtained sufficiently well marked for reproduction.

The conclusions to be drawn from these experiments are the following :—

- (1) The beam of Röntgen rays from a focus-tube which yields a magnetic cathode spectrum is heterogeneous.
- (2) A metallic sheet shows selective absorption of the different rays.
- (3) This selective absorption does not follow the same law in all metals: in certain pairs of metals the absorption is relatively selective.

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XLIX. *The Rate of Recovery of Residual Charge in Electric Condensers.* By FRED. T. TROUTON, F.R.S., and SIDNEY RUSS, B.Sc.*

[Plate XV.]

THE analogy between the phenomenon of residual charge in Leyden jars and the recovery from elastic overstrain in solids has often been pointed out. The theoretical investigations of Boltzmann and the experiments of Kohlrausch and Hopkinson have thrown considerable light on the subject.

In view, however, of recent work done by Mr. A. O. Rankine†, and subsequently confirmed by Mr. Phillips‡, on the recovery of solids from overstrain by a method in which the strain was kept constant while the stress disappears, it seemed desirable to experimentally examine the relations governing the rate at which residual charge makes its appearance in a discharged condenser under conditions analogous to those of their experiments.

The explanation of the phenomenon of residual charge has been referred§ to a heterogeneity in the structure of the dielectric due to its consisting of parts having diverse conducting and dielectric properties, all however subject to simple linear laws. This hypothesis leads to an exponential expression for the rate of recovery of the residual charge. This

* Communicated by the Physical Society: read March 8, 1907.

† Phil. Mag. [6] vol. viii. 1904, p. 538.

‡ Phil. Mag. [6] vol. ix. 1905, p. 513.

§ Maxwell, vol. i. p. 414.

explanation is on somewhat the same general lines as and leads to analogous results to that which has been advanced in the case of recovery from elastic overstrain, and which is based on the assumption of the material consisting in part of viscous and in part of purely elastic constituents. This theory of elastic overstrain leads to an exponential expression for the recovery from the strain in terms of the time. Now no such expression fits experimental evidence, but on the other hand the recovery can be well represented by the logarithm of a quantity proportional to the time of recovery. Thus in Rankine's experiments where a constant strain is maintained, the stress while decreasing is given by

$$S = S_0(1 - K \log (pt + 1)),$$

where S represents the stress at any moment required to preserve a constant strain in the stretched substance.

Our work was undertaken in the first place to examine whether the exponential law of the recovery of the Residual Charge was justified by experimental data, and failing this, to attempt to find a law of the above type agreeing with observation.

The examination of the rate of recovery was carried out in two ways.

Where the Residual Charges were great enough the current as it came out of the recuperating dielectric was simply passed through a sensitive galvanometer and the rate of recovery thus noted.

This method was found quite feasible in the case of a large condenser, the dielectric of which was celluloid, but was unsuited for mica condensers of the ordinary standard type.

For such, a method was devised in which the plates of the condenser were allowed to charge up to a certain difference of potential (due to the Residual Charge coming out of the dielectric) which was measured by an electrometer, and kept at that difference of potential by inserting a variable resistance in parallel with the condenser.

The difficulty of getting a resistance as high as that required, and one which at the same time could be easily adjusted, was surmounted by adopting ionized air as the material of which the resistance was made. The cross section of this resistance was arranged so as to be easily and quickly varied as alteration in the resistance was required.

*Rate of Recovery, or Current at Constant Difference
of Potential.*

In these experiments a constant difference of potential was maintained between the two coatings of the condenser, and the rate at which the Residual Charge leaked out under these conditions was found.

It is clear that if a constant deflexion on the scale of the electrometer can be maintained while the Residual Charge is appearing, then a constant difference of potential exists between the coatings of the condenser.

This constant potential-difference was maintained by the use of a variable resistance obtained by connecting the two coatings of the condenser to two parallel plates of tin, on one of which a thin layer of uranium fluoride, for the purpose of producing the necessary ionization of the air between the plates, was uniformly sifted. Owing to the difference of potential between these plates there will be a gradual leak of the residual charge across the air-gap between them, and if the effective area of the uranium surface can be adjusted the value of this current can be controlled as required. The effective area of the air conductor was regulated by a movable shutter which was made to slide in between the two parallel plates, thus curtailing the active area of the uranium fluoride surface and thereby the cross section of our conductor.

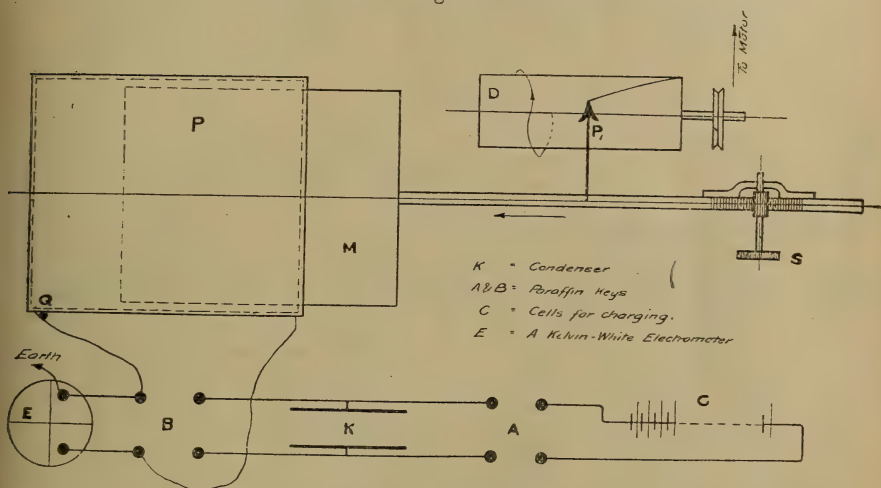
The mode of carrying out an experiment was to allow the difference in potential between the plates of the condenser to rise through the development of residual charge, to a certain selected value, and then to prevent further increase by connecting them through the variable ionized air-resistance. To preserve this constancy in potential the shutter must be continuously pushed in so as to diminish the current at the proper rate, namely, that at which the residual charge is developed by the unstraining dielectric.

The position of the shutter at any moment while being pushed in so as to keep the deflexion of the electrometer constant, could be readily found: this is a measure of the current or the rate at which the charge comes out of the condenser when a constant potential-difference is maintained between its coatings.

A "hand" method was first tried for closing the shutter. The apparatus used is indicated in fig. 1. K a condenser of mica plates was first charged by a series of cells C for a definite interval of time ($1\frac{1}{2}$ mins.), discharged through the key B, and its two coatings were then connected to the elec-

trometer E, one quadrant of which was earthed. P and Q are the two parallel plates, the uranium fluoride being spread on the bottom one Q. M the movable shutter was initially drawn right back, thus exposing the whole of the uranium fluoride surface.

Fig. 1.



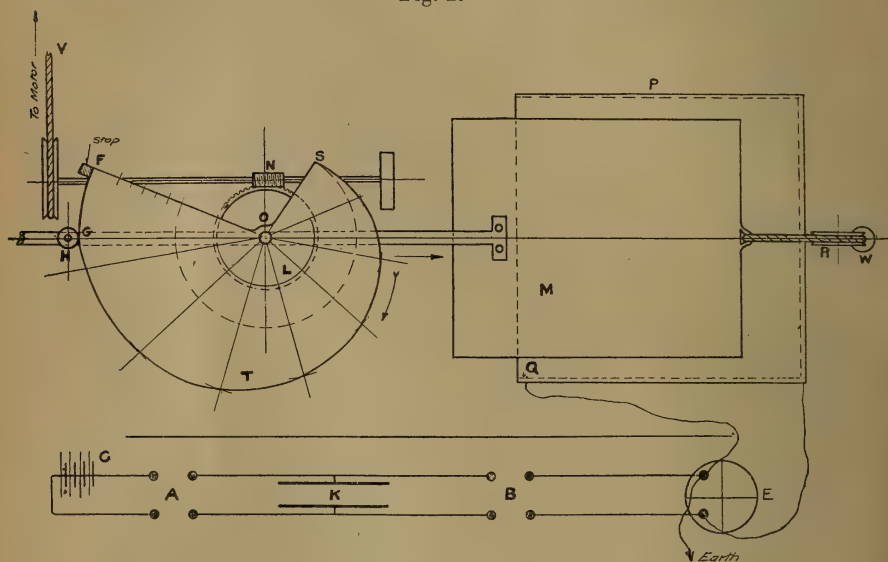
On connecting K to E the deflexion increased at first, but owing to the leak introduced the turning point was soon reached; at this instant the shutter M was moved in by means of the wheel S, which engaged in the milling of a brass strip connected to M. The motion of the shutter was recorded by a pointer P (attached to the brass strip), which traced a path over a blackened surface of paper on a cylindrical drum D which revolved at a known speed by means of a motor, to which it was connected by a system of gearing-down countershafting.

In this way the amount of active area exposed at each moment was recorded. This was proportional to the conductivity, and consequently to the current developed at each moment by the recovering dielectric.

In controlling the motion of the shutter by hand one is very apt to overshoot the mark, due no doubt to air remaining ionized for a short interval after the shutter has covered the ionizing surface, this making the apparatus slow in responding to any alteration in area. If S is turned too quickly, the leak is being cut too rapidly and the deflexion on the scale increases, so that S has to be stopped for a short interval in order that the deflexion may return to its former value. On account of this the curve traced out by P on the drum was

unduly irregular. After many trials made with this apparatus it was decided, for the above reasons, to move the shutter mechanically. The best of the tracings obtained by the previous experiments was utilized for constructing a cam for this purpose. The cam, which was cut out of wood, was made to close the shutter by a uniform rotation about its axis. Fig. 2 illustrates the way in which this was done.

Fig. 2.



H is a small brass wheel fixed to a brass strip which passes underneath the cam T and is joined to the shutter M; a cord fixed to the other end of the shutter passes over a pulley R and carries a weight W which keeps the wheel H tightly pressed against the cam. As the cam rotates H is pulled forward and the shutter moves in at a varying speed.

The uniform rotation of the cam was obtained by means of the toothed wheel L, which engaged in an endless worm N cut in a steel shaft, the other end of which carried a pulley-wheel V which was driven by a small motor highly geared down. The portion FG on the cam entails no forward motion of H, and is to allow the speed of the motor to become constant after starting. The method of carrying out an experiment was as follows:—

The condenser was charged for $1\frac{1}{2}$ minutes to 20 volts, discharged through the key B, and the coatings then connected to the quadrants and the parallel plates P, Q. The

cam was timed so that the point G on it was just opposite H when the coatings of the condenser were connected to P, Q.

The desired result was that the deflexion of the electrometer remained constant while the shutter was being moved in. How nearly this was attained will be seen from the figs. 3, 4, 5 (Pl. XV.), in which the deflexions are plotted against time, the data being given in Tables I., II., III. In each figure the top curve was obtained by allowing the Residual Charge

TABLE I.

$\frac{1}{2}$ Partition exposed.

<i>Time</i> in Minutes.	<i>Deflexions of Electrometer.</i>		
	Shutter kept closed.	Shutter moved in by cam.	Shutter kept open.
$\frac{1}{4}$	15.3	14	13.4
$\frac{1}{2}$	17.5	14.8	13.9
$\frac{3}{4}$	18.5	15	13.6
1	19	15	13.2
$1\frac{1}{4}$	19.3	15	12.6
$1\frac{1}{2}$	19.5	15	11.8
$1\frac{3}{4}$	19.6	14.9	11.1
2	19.5	14.8	10.4
$2\frac{1}{4}$	19.4	↑	9.7
$2\frac{1}{2}$	19.3	↓	9.0
$2\frac{3}{4}$	19.2		8.4
3	19.1		7.8

TABLE II.

2 Partitions exposed.

<i>Time</i> in Minutes.	<i>Deflexions of Electrometer.</i>		
	Shutter kept closed.	Shutter moved in by cam.	Shutter kept open.
$\frac{1}{4}$	13.5	15.3	14.9
$\frac{1}{2}$	18.0	15.5	14.3
$\frac{3}{4}$	18.5	15.6	12.8
1	18.85	15.6	11.1
$1\frac{1}{4}$	18.95	15.6	9.4
$1\frac{1}{2}$	18.95	15.65	7.8
$1\frac{3}{4}$	18.82	15.7	6.4
2	18.65	↑	5.1
$2\frac{1}{4}$	18.50		4.1
$2\frac{1}{2}$	18.3	↓	3.1
$2\frac{3}{4}$	18.1		2.3
3	18.0		1.5

TABLE III.

4 Partitions exposed.

Time in Minutes.	Deflexions of Electrometer.		
	Shutter kept closed.	Shutter moved in by cam.	Shutter kept open.
$\frac{1}{4}$	14.5	12.1	10.3
$\frac{1}{2}$	15.9	12.1	8.05
$\frac{3}{4}$	16.5	12.05	6.0
1	16.8	12.1	4.0
$1\frac{1}{4}$	16.9	12.3	2.5
$1\frac{1}{2}$	16.92	12.3	1.45
$1\frac{3}{4}$	16.88	↑	.60
2	16.80		.01
$2\frac{1}{4}$	16.7	↑	↑
$2\frac{1}{2}$	16.5		↑
$2\frac{3}{4}$	16.35		↑
3	16.2	↓	↓

to come out of the condenser in the ordinary unrestrained way, no correction being made for surface leakage. The middle curve was obtained with the cam in action, and the bottom curve with the plate Q fully exposed the whole time, thus showing how the Residual Charge immediately begins to leak across the gap between the two parallel plates P, Q.

The plate Q, upon which the uranium fluoride was spread, was divided into six partitions which could be separately covered with metal sheets, thus varying the surface over which the leak took place.

Fig. 5 was obtained with $\frac{1}{2}$ partition exposed.

Fig. 6 „ „ 2 partitions exposed.

Fig. 7 „ „ 4 partitions exposed.

From the foregoing curves it was seen that the shape of the cam was approximately correct as it maintained the deflexion of the electrometer constant.

Now since the difference of potential between the coatings of the condenser was constant, the rate the Residual Charge was coming out of the condenser at any moment (and passing across the space between the two parallel plates) was proportional to the area of the bottom plate exposed. This, in turn, was proportional to $(L_t - OS)$, where L_t is the length of the radius vector of the cam which at any moment passes through H, and OS is the length of the radius vector when

the shutter is closed and therefore no current passing across the air-gap.

If, then, we plot the values of $(L_t - OS)$ against time, we obtain the rate at which the Residual Charge came out of the condenser under a constant difference of potential.

In all the experiments, however, surface-leakage was present. Its value in terms of the uranium-fluoride leak was always found after the main experiment was finished. Owing to this surface-leakage a certain quantity (which admits of easy calculation) must be added to the value of $(L_t - OS)$ in order to give the rate of appearance of Residual Charge.

This has been done, and the corrected values of $(L_t - OS)$ which represent current have been plotted against time in figs. 6, 7, 8, where ordinates represent current and abscissæ time. The data corresponding to these figures will be found in Tables IV., V., and VI.

TABLE IV.

$\frac{1}{2}$ Partition exposed.

Time. 1 unit = 7.5 seconds.	Current (in arbitrary units).	
	Observed.	Calculated from equation $C = \frac{d}{t+b} - a$.
0	6.79	6.79
1	5.29	5.35
2	4.19	4.36
3	3.47	3.64
4	3.09	3.10
5	2.79	2.66
6	2.47	2.31
7	2.09	2.05
8	1.84	1.78
9	1.61	1.57
10	1.49	1.40
11	1.29	1.23
12	1.09	1.10
13	.89	.97
14	.79	.81
15	.65	.77
16	.49	.68

Values of constants in the above equation:—

$$a = 1.01, \quad b = 4.44, \quad d = 34.63.$$

TABLE V.—2 Partitions exposed.

Time. 1 unit=6·5 seconds.	Current (in arbitrary units).	
	Observed.	Calculated from equation $C = \frac{d}{t+b} - a.$
0	6·48	6·48
1	4·98	5·04
2	3·88	4·06
3	3·16	3·33
4	2·78	2·78
5	2·48	2·34
6	2·16	1·99
7	1·78	1·70
8	1·53	1·46
9	1·30	1·25
10	1·18	1·07
11	·98	·91
12	·78	·78
13	·58	·65
14	·48	·55
15	·34	·45
16	·18	·36

Values of constants in the above equation :—

$$a=1·33, \quad b=4·44, \quad d=34·68.$$

TABLE VI.—4 Partitions exposed.

Time. 1 unit=5 seconds.	Current (in arbitrary units).	
	Observed.	Calculated from equation $C = \frac{d}{t+b} - a.$
0	6·42	6·42
1	4·92	4·98
2	3·82	3·99
3	3·10	3·27
4	2·72	2·72
5	2·42	2·29
6	2·10	1·94
7	1·72	1·68
8	1·47	1·41
9	1·24	1·20
10	1·12	1·03
11	·92	·86
12	·72	·73
13	·52	·60
14	·42	·44
15	·28	·40
16	·12	·31

Values of constants in the above equation :—

$$a=1·38, \quad b=4·44, \quad d=34·63.$$

Fig. 6 corresponds to the case of $\frac{1}{2}$ partition exposed.

Fig. 7 " " " 2 partitions exposed.

Fig. 8 " " " 4 partitions exposed.

The amount of surface-leakage can be seen from Table VII.

TABLE VII.

Partitions exposed.	Ratio: Surface Leakage Uranium Leakage	Percentage.
$\frac{1}{2}$	$\frac{.0137}{.175}$	7.8
2	$\frac{.015}{.52}$	2.9
4	$\frac{.0118}{.646}$	1.8

The corrected values of the current when plotted against time were found not to agree with an equation of the exponential type, with which they should, were the actions going on in an unstraining dielectric in agreement with the theory of heterogeneous structure as given by Maxwell.

An equation of the form

$$C = \frac{d}{t+b} - a$$

was found to fit the experimental curves, figs. 6, 7, & 8 well. In these figures the continuous lines are obtained from the above equation, and the experimental values appear as circles.

Galvanometer Method.

The condenser used in these experiments was of about 5 microfarads. It was found to afford a very large residual charge. Celluloid sheets, in conjunction with layers of paper, laid on either side, were employed in its construction. This gave a dielectric of avowedly heterogeneous structure, and therefore might very well be expected to act in accordance with the exponential law; nevertheless it will be seen, from the character of the curves obtained with it, that this is not the case.

The method of experiment was to charge the condenser up to about 20 volts, discharge, and connect with a low-resistance D'Arsonval galvanometer which was practically

dead-beat. A sufficient current was then obtained, which was read at frequent intervals until too small to be observed. Two persons were required for this—one to note when the spot of light passed the divisions on the scale, the other to take the time of doing so.

It was always found possible to fit a hyperbola to the observations so obtained. One of the series (Table VIII.) of observations is shown plotted in fig. 9. The curve

$$C = \frac{2304.4}{t + 8.94}$$

was found to suit these observations, and is there shown. The experimental points are seen to fit the curve in a most satisfactory way.

TABLE VIII.

C.	<i>t</i> .	C.	<i>t</i> .
258	0	25	79
208	3	23	87
158	7	21	97
108	13	18	115
78	21	16	134
58	30	14	155
48	38	12	184
43	43	11	203
38	49	10	222
33	58	9	251
28	69		

In these experiments, since the condenser was practically short-circuited, there was comparatively speaking no difference in potential between the plates throughout the recovery; so that the dielectric recovered at its maximum or normal rate.

These and the experiments in the first part of the paper are in accordance with the analogy of recovery of elastic solids from overstrain. The quantity of electricity recovered up to any time *t* is given by an expression of the type

$$Q = a \log (t + b),$$

which is the expression found by Rankine and others to fit the recovery from overstrain in elastic bodies.

L. *Contributions to the Theory of Solutions.* By
L. VEGARD (Physical Institute, The University, Kristiania)*.

§ 1.

On the Change of Concentration in a Binary Solution, which is in a field of force where the force is proportional to the ponderable mass.

IN the thermodynamic treatment of mixtures of substances dissolved in each other, it is always understood that the effect of Gravity need not be considered. In other words, a solution is supposed homogeneous, provided that equilibrium has set in. It follows, then, that the functions become homogeneous with regard to the components of the mixture.

If, however, the solution is submitted to the influence of a force that is proportional to the ponderable mass of the individual components—exposed to the influence of Gravity for instance—the homogeneity will be lost, since the pressure varies from place to place in the fluid. Besides, it is *a priori* probable that the proportions of mixture also change within the solution, when the latter is in equilibrium. For if there were no inclination on the part of the substances to dissolve in each other, Gravity would cause the components to accumulate on the top of each other, each according to its own specific density. This effect of Gravity is now counteracted by the inclination towards solution; an equilibrium must ensue, in which the proportions of mixture change from place to place, so that the Centre of Gravity of the whole lies lower than if the system were homogeneous, but higher than if the substances had not dissolved in each other. In a solution exposed to the influence of Gravity, the thermodynamic functions respecting the components will be homogeneous no longer. In the neighbourhood of any point they can, however, be considered homogeneous.

Let us suppose that we have, in the following, a binary solution, that is to say, that we only have two components in the fluid system. Let us further suppose that the field to which the system is exposed is dependent on a potential U , which may be given as a function of coordinates in a rectangular coordinate system. U as well as its first partial derivatives are considered to be continuous in the space occupied by the system.

Let the two components have molecular masses M_1 and M_2 .

* Communicated by the Author. Placed before *Kristiania Videnskabselskab* on the 12th of October, 1906.

If in a small volume element of the solution there are n_1 molecules of the first and n_2 of the second component, the concentration will be defined by the equation

$$c = \frac{M_2 n_2}{M_1 n_1}.$$

The task we now set before us, is to find c as a function of the coordinates when equilibrium has set in, and the temperature is the same all over the system.

To keep the system in equilibrium, two conditions must be present—one mechanical and one thermodynamic. If the pressure at a point in the fluid is given by p , it gives the following mechanical condition for equilibrium:

$$dp = -\rho \left(\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz \right). \quad . \quad . \quad (1)$$

ρ is the density of the solution at the point considered. To maintain equilibrium it is absolutely necessary for p to be a function of the coordinates, *i. e.*, dp must be an exact differential. Consequently:

$$\frac{\partial \left(\rho \frac{\partial U}{\partial x} \right)}{\partial y} = \frac{\partial \left(\rho \frac{\partial U}{\partial y} \right)}{\partial x}$$

$$\frac{\partial \left(\rho \frac{\partial U}{\partial x} \right)}{\partial z} = \frac{\partial \left(\rho \frac{\partial U}{\partial z} \right)}{\partial x}.$$

On differentiation and rearrangement;

$$\frac{\frac{\partial \rho}{\partial x}}{\frac{\partial U}{\partial x}} = \frac{\frac{\partial \rho}{\partial y}}{\frac{\partial U}{\partial y}} = \frac{\frac{\partial \rho}{\partial z}}{\frac{\partial U}{\partial z}} = \frac{\frac{\partial \rho}{\partial n}}{-K},$$

where K denotes the resultant of the force intensity at the point, and $\frac{\partial \rho}{\partial n}$ the change of density per unit of length in the direction of the force.

As the density is a function of the coordinates,

$$d\rho = \frac{\partial \rho}{\partial x} dx + \frac{\partial \rho}{\partial y} dy + \frac{\partial \rho}{\partial z} dz,$$

or by use of the above equation

$$d\rho = -\frac{\partial \rho}{\partial n} \frac{1}{K} dU.$$

Furthermore, the concentration can be stated as a function of pressure, temperature, and density. If the temperature is always kept constant,

$$dc = \frac{\partial c}{\partial \rho} d\rho + \frac{\partial c}{\partial p} dp.$$

If the values of $d\rho$ and dp are substituted,

$$dc = -\left(\frac{\partial c}{\partial \rho} \frac{\partial \rho}{\partial n} \frac{1}{K} + \frac{\partial c}{\partial p} \rho\right) \left(\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz\right).$$

We must also be able to express it thus :

$$dc = \frac{\partial c}{\partial x} dx + \frac{\partial c}{\partial y} dy + \frac{\partial c}{\partial z} dz. \quad . \quad . \quad . \quad (2a)$$

And since both the terms for dc must be identical, and x, y, z are independent variables,

$$\frac{\frac{\partial c}{\partial x}}{\frac{\partial U}{\partial x}} = \frac{\frac{\partial c}{\partial y}}{\frac{\partial U}{\partial y}} = \frac{\frac{\partial c}{\partial z}}{\frac{\partial U}{\partial z}} = \frac{\frac{\partial c}{\partial n}}{-K}.$$

If these equations are inserted in (2a)

$$dc = -\frac{1}{K} \frac{\partial c}{\partial n} dU. \quad . \quad . \quad . \quad . \quad (2b)$$

From this equation it appears that $dc=0$ when $dU=0$. From this we conclude, that in the state of equilibrium the concentration must be constant along a potential surface and must have the greatest slope in the direction of the force. U and K are given quantities. The only thing we now have

to determine is $\frac{\partial c}{\partial n}$; or, as we might call it, the concentration gradient. This must be determined by help of the thermodynamic equilibrium condition, which we shall briefly mention.

If at constant temperature a system suffers a change from a state 0 to a state 1, the amount of heat that the system disengages during the change can, as is well known, be expressed as follows :

$$Q = T(S_0 - S_1) + TP.$$

Here T means the absolute temperature, S the entropy, $T(S_0 - S_1)$ the amount of heat that the systems would have disengaged if the process had been carried out reversibly.

According to the second law of thermodynamics P is a quantity that, for all processes which can arise in the system of their own accord, can never be negative.

If our system is considered in the neighbourhood of the point of equilibrium the *vis viva* can be left out of consideration, and in accordance with the First Law of Thermodynamics

$$Q = I_0 - I_1 + A,$$

the heat being measured in work units. I is the internal energy, and A the external work done by the forces of the system.

By a combination of both these equations:

$$TP = (I_0 - TS_0) - (I_1 - TS_1) + A.$$

Here I and S are functions of the variables of the system. We put

$$I - TS = \psi,$$

where ψ is the so-called thermodynamic potential. If the system suffers an infinitely slight change :

$$T\delta P = -\delta\psi + \delta A.$$

The condition which is necessary, in order that a system left to itself at constant temperature, shall be in equilibrium, is that $\delta P = 0$ for every small change that the system may suffer*. If the system depends on r independent variables $\alpha_1, \alpha_2 \dots \alpha_r$, the system can suffer r independent variations. The equilibrium condition may therefore be expressed as follows :

$$\left. \begin{aligned} (\delta\psi)_{\alpha_1} &= (\delta A)_{\alpha_1} \\ (\delta\psi)_{\alpha_2} &= (\delta A)_{\alpha_2} \\ &\vdots \\ (\delta\psi)_{\alpha_r} &= (\delta A)_{\alpha_r} \end{aligned} \right\} \dots \dots \dots (3')$$

Here $(\delta\psi)_{\alpha_r}$ and $(\delta A)_{\alpha_r}$ mean the changes respectively of ψ and A for the same variation of α_r , when the other variables are kept invariable.

The change that must come into consideration here is a relative displacement of both components in a volume element. This variation will generally cause a small volume change. The pressure on the element will, however, not be changed by it. If the external pressure is kept constant during the

* Duhem, *Mécanique Chimique*, tome i, livre i. p. 89.

change, the exterior work will consist of two parts, and we have

$$\delta A = -p\delta v + \delta a,$$

p is the pressure, v the volume, and δa the work done by the force in the field, on account of the variation. The equilibrium condition will in our case then be

$$\delta\psi + p\delta v = \delta a;$$

and as p is constant,

$$\delta(\psi + pv) = \delta a,$$

$(\psi + pv)$ is the exterior thermodynamic potential or the thermodynamic potential under constant pressure. If this function is indicated by ω , we get

$$\delta\omega = \delta a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We will now consider a small volume element dx, dy, dz , containing n_1 molecules of the first component, and n_2 molecules of the second component. We select the X axis parallel to the force at the point considered. Let us suppose that the thermodynamic potential for the element—neglecting quantities of a higher order—has the same value as it would have had if the system had been homogeneous, and pressure and concentration had had the same value as is to be found in the centre of the element. ω then becomes for the element a homogeneous function of the first order; and according to a well-known theorem of Euler

$$\omega(M_1n_1, M_2n_2, p, T) = M_1n_1 \frac{\partial \omega}{\partial (M_1n_1)} + M_2n_2 \frac{\partial \omega}{\partial (M_2n_2)}. \quad (4a)$$

The two partial derivatives are again homogeneous functions of the 0th degree of the components. We put

$$\frac{\partial \omega}{\partial (M_1n_1)} = f_1(c, p, T)$$

$$\frac{\partial \omega}{\partial (M_2n_2)} = f_2(c, p, T).$$

If we once more make use of Euler's theorem the result will be

$$M_1n_1 \frac{\partial^2 \omega}{\partial (M_1n_1)^2} + M_2n_2 \frac{\partial^2 \omega}{\partial (M_1n_1) \partial (M_2n_2)} = 0.$$

If $f_1(c, p, T)$ and $f_2(c, p, T)$ are introduced, we get the following very important combination :

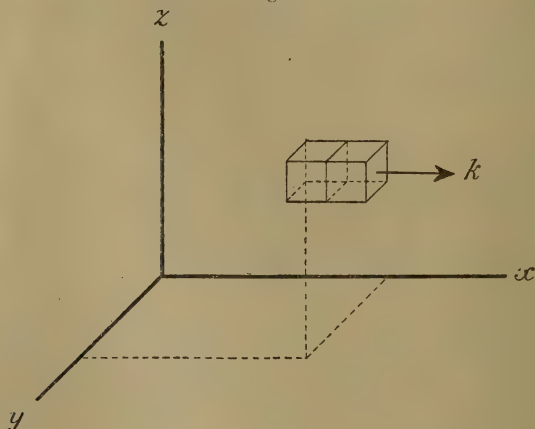
$$\frac{\partial f_1(c, p, T)}{\partial c} + c \frac{\partial f_2(c, p, T)}{\partial c} = 0^* . . . (5)$$

If f_1 and f_2 are introduced in equation (4a) we get

$$\omega = M_1 n_1 f_1(c, p, T) + M_2 n_2 f_2(c, p, T) . . . (4b)$$

If we now add to x a small quantity $\Delta x = dx$ we get a new element of the same capacity as the original (see fig. 1).

Fig. 1.



On the average the pressure will be a different one in this element, and besides the mixture proportion will also be different. Supposing the numbers of molecules of the new element were $n_1 + dn_1$ and $n_2 + dn_2$, this will involve the following change in the concentration

$$dc = c \left(-\frac{dn_1}{n_1} + \frac{dn_2}{n_2} \right) (6)$$

If we leave out quantities of the second and higher orders, we get for the thermodynamic potential of the second element

$$\begin{aligned} \omega' = \omega + M_1 f_1 dn_1 + M_2 f_2 dn_2 + M_1 n_1 \left(\frac{\partial f_1}{\partial c} + c \frac{\partial f_2}{\partial c} \right) dc \\ + M_1 n_1 \left(\frac{\partial f_1}{\partial p} + c \frac{\partial f_2}{\partial p} \right) dp. \end{aligned}$$

* Duhem, *loc. cit.* tome iii. livre vi. p. 5.

Here dn_1 , dn_2 , dc , dp , mean the average changes of the four quantities n_1 , n_2 , c , p from one element to the other; or rather the changes which these quantities undergo when we go from the centre of the first element to the centre of the second. Then we can write

$$dn_1 = \frac{\partial n_1}{\partial x} dx, \quad dn_2 = \frac{\partial n_2}{\partial x} dx,$$

$$dc = \frac{\partial c}{\partial x} dx, \quad dp = \frac{\partial p}{\partial x} dx,$$

and when we take equations (1) and (5) into consideration :

$$\omega' = \omega + \left[M_1 f_1 \frac{\partial n_1}{\partial x} + M_2 f_2 \frac{\partial n_2}{\partial x} + M_1 n_1 \left(\frac{\partial f_1}{\partial p} + c \frac{\partial f_2}{\partial p} \right) \rho K \right] dx. \quad (7a)$$

It must now be noticed that n_1 and n_2 do not change independently of each other; for there must always be sufficient substance of both kinds in each element to keep the volume element full of fluid.

Supposing that the molecular volumes of the components are indicated at the concentration c by v_1 and v_2 respectively, the volume of the first element will then be

$$dx \cdot dy \cdot dz = v_1 n_1 + v_2 n_2.$$

In the other the concentration is different; and the pressure also. Let us, however, suppose that the fluid is incompressible, and that we leave out of consideration the change of volume with pressure. The molecular volumes of both components of the second element will then be

$$\left(v_1 + \frac{\partial v_1}{\partial c} dc \right) \quad \text{and} \quad \left(v_2 + \frac{\partial v_2}{\partial c} dc \right)$$

and the volume of the second element

$$\left(v_1 + \frac{\partial v_1}{\partial c} dc \right) (n_1 + dn_1) + \left(v_2 + \frac{\partial v_2}{\partial c} dc \right) (n_2 + dn_2)$$

$$= dx \cdot dy \cdot dz = v_1 n_1 + v_2 n_2.$$

For brevity's sake let us write

$$k_1 = v_1 - \left(\frac{\partial v_1}{\partial c} + \frac{M_1}{M_2} c \frac{\partial v_2}{\partial c} \right) c$$

$$k_2 = v_2 + \left(\frac{\partial v_1}{\partial c} + \frac{M_1}{M_2} c \frac{\partial v_2}{\partial c} \right) \frac{M_2}{M_1}.$$

Then we get

$$k_1 dn_1 + k_2 dn_2 = 0.$$

Or as dn_1 and dn_2 are variations corresponding to the same variation dx :

$$k_1 \frac{\partial n_1}{\partial x} + k_2 \frac{\partial n_2}{\partial x} = 0. \quad (8a)$$

If now $\frac{\partial n_1}{\partial x}$ is eliminated between equations (7a) and (8a) we get

$$\omega' = \omega + \left[\left(M_1 f_1 - \frac{k_1}{k_2} M_2 f_2 \right) \frac{\partial n}{\partial x} + M_1 n_1 \left(\frac{\partial f_1}{\partial p} + c \frac{\partial f_2}{\partial p} \right) \rho K \right] dx. \quad (7b)$$

The thermodynamic potential for both elements taken together will be

$$\omega' + \omega = 2\omega + \left[\left(M_1 f_1 - \frac{k_1}{k_2} M_2 f_2 \right) \frac{\partial n}{\partial x} + M_1 n_1 \left(\frac{\partial f_1}{\partial p} + c \frac{\partial f_2}{\partial p} \right) \rho K \right] dx.$$

In order to find the condition for equilibrium, we must carry out a small variation, which consists in the masses $M_1 \epsilon n_1$ and $M_2 \epsilon n_2$ being carried from the first to the second element. These variations ϵn_1 and ϵn_2 are not, however, independent of each other but are connected by the equation

$$v_1 \epsilon n_1 + v_2 \epsilon n_2 = 0. \quad (8b)$$

This equation (8b) can with complete mathematical exactness only be applied on the supposition that the molecular volumes do not vary with the concentration. As, however, we shall see later on, the result must also, on account of the continuity of nature, approximately be applicable to such solutions for which the concentration is quite small; and this is really the case with most solutions. And furthermore, the actual change of concentration in the state of equilibrium is a very small quantity.

The thermodynamic potential after the variation of our two elements, can now be found by calculating the potential of each element separately.

The thermodynamic potential after the variation will then be for the first element,

$$\omega_1 = \omega - \left(M f_1 - \frac{v_1}{v_2} M_2 f_2 \right) \epsilon n_1;$$

for the second,

$$\begin{aligned}\omega_1' = & \omega + \left[\left(M_1 f_1 - \frac{k_1}{k_2} M_2 f_2 \right) \frac{\partial n_1}{\partial x} + M_1 n_1 \left(\frac{\partial f_1}{\partial p} + c \frac{\partial f_2}{\partial p} \right) \rho K \right] dx \\ & + \left(M_1 f_1 - \frac{v_1}{v_2} M_2 f_2 \right) \epsilon n_1 + \left[\left(M_1 \frac{\partial f_1}{\partial c} - \frac{k_1}{k_2} M_2 \frac{\partial f_2}{\partial c} \right) \frac{\partial n}{\partial x} \epsilon c \right. \\ & \left. + \rho \cdot K \cdot \left(M_1 \frac{\partial f_1}{\partial p} - \frac{v_1}{v_2} M_2 \frac{\partial f_2}{\partial p} \right) \epsilon n_1 + M_1 n_1 \cdot \rho \cdot K \left(\frac{\partial^2 f_1}{\partial c \partial p} + c \frac{\partial^2 f_2}{\partial c \partial p} \right) \epsilon c \right] dx.\end{aligned}$$

On a superficial view, one would have thought that also ρ , as a function of the concentration, would have varied here. We must, however, remember that ρ is the average density between the centres of the elements; and by this relative exchange of masses nothing will be carried away from the whole double element. The average density keeps its value unaltered, and change of pressure between the centres is not influenced by the variation.

The above term for ω_1' can be considerably simplified, since

$$\frac{\partial^2 f_1}{\partial c \partial p} + \frac{\partial^2 f_2}{\partial c \partial p} c = \frac{\partial}{\partial p} \left(\frac{\partial f_1}{\partial c} + c \frac{\partial f_2}{\partial c} \right) = 0.$$

As will be seen later,

$$\left. \begin{aligned} k_1 = v_1, & \quad k_2 = v_2 \\ M_1 \frac{\partial f_1}{\partial p} - M_2 \frac{v_1}{v_2} \frac{\partial f_2}{\partial p} = 0 \end{aligned} \right\} \dots \dots \dots (9)$$

The term for ω_1' will then appear as follows :

$$\omega_1' = \omega' + \left(M_1 f_1 - \frac{v_1}{v_2} M_2 f_2 \right) \epsilon n_1 + \left(M_1 \frac{\partial f_1}{\partial c} - \frac{v_1}{v_2} \frac{\partial f_2}{\partial c} M_2 \right) \frac{\partial n_1}{\partial x} \epsilon c dx.$$

By the aid of equations (5), (8a), (8b), and (9) we get :

$$\begin{aligned} dc &= -c \left(\frac{1}{n_1} - \frac{v_1}{v_2} \frac{1}{n_2} \right) dn_1 \\ \epsilon c &= -c \left(\frac{1}{n_1} - \frac{v_1}{v_2} \frac{1}{n_2} \right) \epsilon n_1,\end{aligned}$$

and consequently

$$\frac{\partial n_1}{\partial x} \epsilon c = \frac{\partial c}{\partial x} \epsilon n_1.$$

If this is introduced into the equation for ω_1' , we get at last for the whole variation of the thermodynamic potential :

$$\delta \omega = (\omega_1 + \omega_1') - (\omega + \omega') = \left(M_1 \frac{\partial f_1}{\partial c} - \frac{v_1}{v_2} M_2 \frac{\partial f_2}{\partial c} \right) \frac{\partial c}{\partial x} \epsilon n_1 dx. \quad (10a)$$

We have still to find the work done by the force during the variation. On the average the masses have been moved a space dx , in the direction of the force, and accordingly

$$\delta a = K(M_1 \epsilon n_1 + M_2 \epsilon n_2) dx;$$

or by the aid of equation (8b) :

$$\delta a = K \left(M_1 - \frac{v_1}{v_2} M_2 \right) \epsilon n_1 dx. \quad . \quad . \quad (10b)$$

If the values found for $\delta \omega$ and δa (equations (10a) and (10b)) are introduced into equation (3), we get as the condition for equilibrium :

$$\left(\frac{M_1}{v_1} \frac{\partial f_1(c, p, T)}{\partial c} - \frac{M_2}{v_2} \frac{\partial f_2(c, p, T)}{\partial c} \right) \frac{\partial c}{\partial x} = K \left(\frac{M_1}{v_1} - \frac{M_2}{v_2} \right).$$

If we now eliminate $\frac{\partial f_2}{\partial c}$ by the aid of equation (5) and write—in view of equation (2)— n instead of x ,

$$\frac{\partial f_1(c, p, T)}{\partial c} \left(\frac{M_1}{v_1} + \frac{1}{c} \frac{M_2}{v_2} \right) \frac{\partial c}{\partial n} = K \left(\frac{M_1}{v_1} - \frac{M_2}{v_2} \right). \quad (11a)$$

From this equation it appears that when the solution is in equilibrium, *the concentration gradient is at every point proportional to the force.*

In equation (11a) we will try to introduce quantities that are more spontaneously accessible, and begin by finding other values for the ratios $\frac{M_1}{v_1}$ and $\frac{M_2}{v_2}$. These quantities mean the molecular masses divided by the molecular volumes. To make clearer the meaning of these quantities, we may suppose that we have a very big quantity of the solution at the concentration c . If to the fluid is added a mass M_1 of the first or M_2 of the second component, the volume will be increased by the quantities v_1 and v_2 respectively.

Instead of an infinite, we consider a finite quantity and add infinitely small masses $M_1 dn_1$ and $M_2 dn_2$ to the solution. By doing that the volumes will be increased by dv_1 and dv_2 ; and then

$$\frac{M_1}{v_1} = \frac{M_1 dn_1}{dv_1} \quad \text{and} \quad \frac{M_2}{v_2} = \frac{M_2 dn_2}{dv_2}.$$

If the whole volume of the solution is V ,

$$\rho V = M_1 n_1 + M_2 n_2.$$

If a mass $M_1 dn_1$ is added to the solution, we get

$$\rho dv_1 + V d\rho = M_1 dn_1$$

and

$$\rho dv_1 + \frac{M_1 n_1 + M_2 n_2}{\rho} \frac{\partial \rho}{\partial c} \frac{\partial c}{\partial n_1} dn_1 = M_1 dn_1.$$

By a very simple calculation we get from this

$$\frac{M_1}{v_1} = \frac{\rho^2}{\rho + c(1+c)} \frac{\partial \rho}{\partial c} \quad . \quad . \quad . \quad (12 a)$$

If the second component is treated in a similar manner,

$$\frac{M_2}{v_2} = \frac{\rho^2}{\rho - (1+c)} \frac{\partial \rho}{\partial c} \quad . \quad . \quad . \quad (12 b)$$

By help of equations (12 a) and (12 b) we can easily prove the following term :

$$\frac{\partial v_1}{\partial c} + c \frac{M_1}{M_2} \frac{\partial v_2}{\partial c} = 0,$$

out of which immediately follow the equations

$$k_1 = v_1 \quad \text{and} \quad k_2 = v_2.$$

According to a well known property of thermodynamic potential,

$$\frac{\partial \omega}{\partial p} = V.$$

If we now introduce the values of ω (equation 4 b) and V , and obtain the partial derivatives with respect to $M_1 n_1$ and $M_2 n_2$, we get

$$\frac{\partial f_1}{\partial p} = \frac{\rho + c(1+c)}{\rho^2} \frac{\partial \rho}{\partial c} = \frac{v_1}{M_1},$$

$$\frac{\partial f_2}{\partial p} = \frac{\rho - (1+c)}{\rho^2} \frac{\partial \rho}{\partial c} = \frac{v_2}{M_2}.$$

From these equations :

$$\frac{M_1}{v_1} \frac{\partial f_1}{\partial p} - \frac{M_2}{v_2} \frac{\partial f_2}{\partial p} = 0.$$

The correctness of the equation (9) is thereby proved.

When the terms found for $\frac{M_1}{v_1}$ and $\frac{M_2}{v_2}$ are introduced into equation (11 a) a very simple calculation gives

$$\rho \frac{\partial f_1(c, p, T)}{\partial c} \frac{\partial c}{\partial n} = -c(1+c) \frac{\partial \rho}{\partial c} \cdot K. \quad (11 b)$$

In this way it appears that we can obtain a determination of the concentration gradient by help of the thermodynamic condition of equilibrium. The result is therefore independent of every hypothesis as to the internal constitution of the solution. The formula cannot, however, be considered exact in a mathematical sense. In the course of the development we have taken it for granted that the fluid is incompressible, and that the molecule volumes do not change greatly with the concentration ; which, however, very nearly corresponds to real cases.

In our term for the concentration gradient, we have still to consider the function $f_1(c, p, T)$. Hitherto we have only mentioned that it is a function characteristic for both substances and closely connected with the external thermodynamic potential. In the work by Duhem above mentioned * the function is found for solutions in which the concentration is small, *i. e.* for so-called Dilute Solutions.

Furthermore, van Laar †, starting out from van der Waals's equation of state, has established a more general equation which is approximately applicable also to concentrated solutions.

Instead of $f_1(c, p, T)$ we shall, however, introduce a quantity which is more spontaneously accessible to our conception—namely, the Osmotic Pressure. Not that the solution as such can, properly speaking, be ascribed to an osmotic pressure ; but let us suppose that solution and solvent are in equilibrium on the two sides of a semipermeable membrane. The difference of hydrostatic pressures on the two sides of the membrane—the osmotic pressure π —is now connected with the function f_1 in the following simple manner :

$$f_1(0, p, T) - f_1(c, p, T) = \frac{1}{\rho_0} \pi. \quad (13)$$

ρ_0 is the *density* of the pure solvent since we have, throughout the development, calculated with mass units.

* Duhem, *Mécanique Chimique*, tome iii. livre vi. p. 44.

† See J. J. van Laar, *Sechs Vorträge über das thermodynamische Potential*, 1906, p. 85.

p is the pressure on the same side of the membrane as the solution. In order to apply these equations to our particular case, we must remember that we have to place the solution under this pressure. If equation (13) is differentiated with regard to c , we get

$$-\frac{\partial f_1(c, p, T)}{\partial c} = \frac{1}{\rho_0} \left(\frac{\partial \pi}{\partial c} \right)_p.$$

$\left(\frac{\partial \pi}{\partial c} \right)_p$ means the change of osmotic pressure per unit of concentration change, when the temperature and the pressure in the solution are kept constant. If this is introduced into equation (11 b), the result is

$$\left(\frac{\partial \pi}{\partial c} \right)_p \frac{\partial c}{\partial n} = \frac{\rho_0}{\rho} c(1+c) \frac{\partial \rho}{\partial c} K. \quad (11 c)$$

It must be noticed that this equation holds good for all concentrations.

If the value of $\frac{\partial c}{\partial n}$ is introduced into equation (2):

$$dc = -\frac{\rho_0 c(1+c)}{\rho} \left(\frac{\partial \pi}{\partial c} \right)_p \frac{\partial \rho}{\partial c} dU. \quad (14 a)$$

The quantities $\left(\frac{\partial \pi}{\partial c} \right)_p$, $\frac{\partial \rho}{\partial c}$, and ρ are certain continuous functions of c , p , and T for the fluid mixture considered. As we have considered the fluid as incompressible, and as $\left(\frac{\partial \pi}{\partial c} \right)_p$ is approximately independent of the pressure, we may consider the three quantities as functions of c and T . Term (14 a) is then an exact differential; and by integration, we get

$$\int \frac{\rho \left(\frac{\partial \pi}{\partial c} \right)_p}{\rho_0 c(1+c) \frac{\partial \rho}{\partial c}} dc = U. \quad (14 b)$$

The expression to the left, when integrated, gives a function of the concentration, which however contains the temperature as a parameter but not the pressure. By equation (14 b) the concentration is determined as a function of the coordinates.

We shall now consider some special cases :

(1) The solution is exposed to the influence of Gravity. Then we have

$$U = gh + \text{constant};$$

g is the acceleration of gravity, h the height above a fixed horizontal plane. This being introduced into equation (14 a),

$$dc = - \frac{\rho_0 c(1+c)}{\rho} \frac{\partial \rho}{\left(\frac{\partial \pi}{\partial c}\right)_p} g dh. \quad . \quad . \quad (11 d)$$

(2) The solution is placed in a vessel that turns round a vertical axis with constant velocity. It is supposed that the fluid follows the motion of the vessel. It is desired to find the potential U_1 referred to a coordinate system, where the Z-axis coincides with the axis of rotation and the two other axes follow the motion. Then

$$dU = g dz - \gamma^2 (x dx + y dy),$$

and consequently

$$dc = \frac{\rho_0 c(1+c)}{\rho} \frac{\partial \rho}{\left(\frac{\partial \pi}{\partial c}\right)_p} [\gamma^2 (x dx + y dy) - g dz]. \quad . \quad (11 e)$$

The force-intensity at the considered point is expressed by the following term :

$$K = \sqrt{g^2 + \gamma^4 (x^2 + y^2)} = \sqrt{g^2 + \gamma^4 R^2};$$

γ is the angular velocity and R the distance from the point to the axis of rotation.

As the concentration gradient is proportional to the force, we see that when c and $\frac{\partial \rho}{\partial c}$ are not very small, we might make R and γ sufficiently great to obtain a considerable change of concentration from place to place. In other words, we might expect to be able to separate the solutions for which $\frac{\partial \rho}{\partial c}$ has a considerable value, as milk is separated from cream in a separator. The separation will, however, never be complete. Besides it would be very difficult to carry out the experiment practically. In order that the separation might be made visible, the rotation would have to be continued for a long time—since the rate of diffusion is very small—and the motion would have to be perfectly even. Furthermore, temperature differences would have to be avoided, as otherwise currents would arise in the fluid.

Nevertheless Van Calcar and De Bruyn * have succeeded in showing that solutions really can be concentrated by centrifugalisation.

From the term for the concentration gradient we see, that—independent of the intensity of the force—it is equal to zero, when $\frac{\partial \rho}{\partial c}$ is equal to zero at the concentration

present. In such a solution dc will consequently be equal to zero, and the concentration will be the same throughout the solution. We shall suppose that we only have to do with the influence of gravity, and we shall notice the expression

(11d) more minutely. If $\frac{\partial \rho}{\partial c}$ is positive, the density will increase with the concentration, and the concentration will decrease with the height. When $\frac{\partial \rho}{\partial c}$ is negative, the con-

centration increases upwards. From this it follows, as we have already pointed out at the beginning of this article, that in both cases the centre of gravity of the solution will be lowered by the action of the force.

For each solution we can consider the density as a function of the concentration, provided that the temperature and pressure are kept constant. This function can be represented as is well known, by putting ρ proportional to the distance from the point to a fixed line A—B, and c is represented by the ratios of the distances from the point to two lines perpendicular to A—B (fig. 2). Each solution gives a curve.

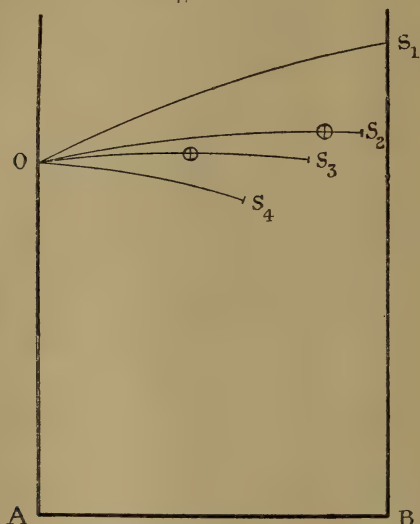
At a point where $\frac{\partial \rho}{\partial c}$ is equal to zero, the tangent of the curve is parallel to A—B and the density has for this concentration an intermediate maximum (or minimum). If ρ is a rational non-linear function of c , it would always be possible to find one or more values of the concentration, for which $\frac{\partial \rho}{\partial c}$ is equal to zero.

If, however, these values are to have a physical interpretation they must be positive and real, and also be within the domain of solubility. In fact we shall find, that to most solutions there will be no mixture proportion for which the concentration is constant throughout the solution. As the contraction of the solution is quite slight, we can take it for granted that the solution for which the gradient vanishes must consist of components whose densities differ very little from each other.

* *Rec. Trav. Chem. Leiden*, vol. xxiii. pp. 219–223 (1903).

These proportions have been represented in the adjacent fig. 2, where four substances S_1 , S_2 , S_3 , and S_4 are supposed to be dissolved in the same solvent. The density curves will

Fig. 2.



either rise always as OS_1 or fall always as OS_4 . Only in the case of such curves which are nearly horizontal may we expect to find maximum points.

Finally we shall make a comparison between solutions and gases in this respect. If the absolute density is indicated by σ , we have in the case of gases

$$\frac{\partial \sigma}{\partial h} = -\frac{M_1' g}{RT} \sigma, \quad \dots \quad (15 a)$$

where R is the gas constant for a molecular mass M_1' of the gas.

In order to get a term for the solution that might be compared with this, we must introduce another definition of the concentration, and mean now by the concentration the mass of the dissolved substances per volume unit of solution. If the concentration defined in this way is indicated by C , we get the equations of transformation

$$C \left(\frac{1}{c} + 1 \right) = \rho;$$

$$\left(\frac{\partial \pi}{\partial c} \right)_p = \left(\frac{\partial \pi}{\partial C} \right)_p \cdot C_c';$$

$$\frac{\partial c}{\partial h} = \frac{\partial C}{\partial h} c_c';$$

and remembering that $C'_c = \frac{1}{c_c}$, by introduction into (11 d):

$$\left(\frac{\partial \pi}{\partial C}\right)_p \frac{\partial C}{\partial h} = -\frac{\rho_0}{\rho} c(1+c) \frac{\partial \rho}{\partial c} g.$$

We will not transform the expression to the left. If the solution is not very concentrated, and the vapour-pressure of the second component may be disregarded in comparison with that of the solvent, we have approximately

$$\left(\frac{\partial \pi}{\partial C}\right)_p = \frac{1}{M_1} RT.$$

R has the same value as before. If this is introduced,

$$\frac{\partial C}{\partial h} = -\frac{\rho_0}{\rho} \frac{M_1}{RT} c(1+c) \frac{\partial \rho}{\partial c} g. \quad (15 b)$$

A comparison between the equations (15 a) and (15 b) immediately shows us that there is a fundamental difference between solutions and gases with regard to the change of concentration with the height. In the case of solutions

$\frac{\partial C}{\partial h}$ can, for instance, be equal to zero without the concentration itself being equal to zero, and this may just as well happen at the greatest concentrations. In the case of gases, however, the concentration-gradient will never be equal to zero, but will approach this value on the density decreasing *ad infinitum*. It might be said, however, that solutions and gases approach conformity, at unlimited dilution, since $\frac{\partial C}{\partial h}$ as well as $\frac{\partial \sigma}{\partial h}$ then approach zero; but that similarity disappears in the case of concentrated solutions.

In the following table $\frac{\partial C}{\partial h}$ has, according to equation (14 b) been calculated for solutions of Cane Sugar and Potassium Hydrate at some concentrations*. The numbers refer to the absolute system of units. Then $\frac{\partial C}{\partial h}$ is of the dimension

$$[\text{Mass} \cdot \text{Length}^{-4}] = |\text{gr. cm.}^{-4}|.$$

* The density measurements are taken from Landolt and Börnstein.

c	Cane-Sugar Solution.		KOH Solution.	
	$\frac{\partial \rho}{\partial c}$	$\frac{\partial C}{\partial h}$	$\frac{\partial \rho}{\partial c}$	$i \frac{\partial C}{\partial h}$
1/19	0.380	$2.87 \cdot 10^{-7}$	0.720	$0.87 \cdot 10^{-7}$
1/9	0.337	5.57 „	0.720	1.87 „
3/17	0.311	8.49 „	0.650	2.72 „
1/4	0.286	11.50 „	0.700	4.23 „
1/3	0.260	14.55 „	0.590	4.85 „
1	0.138	31.20 „	0.320	9.45 „

With regard to the calculation in the case of Potassium Hydrate, we must remark that such a solution is usually considered to be dissociated. The solution is then not binary in the strict sense of the word. If, however, the electrical action between the ions is very great compared with the influence of gravity, we can here suppose that the pairs of ions cannot be separated by the action of gravity, to any appreciable extent.

We can, however, not apply equation (15 *b*) without further consideration. But equation (11 *c*) still remains true, and we can write :—

$$\left(\frac{\partial \pi}{\partial C}\right) = i \frac{1}{M_1} R \cdot T,$$

where i is the dissociation constant introduced by Van't Hoff, and which is generally dependent on the concentration.

Consequently

$$i \frac{\partial C}{\partial h} = -\frac{\rho_0}{\rho} \frac{M_1}{R \cdot T} c(1+c) \frac{\partial \rho}{\partial c} g.$$

The calculation has been carried out for the potassium hydrate solution according to this formula.

§ 2.

On the Variation of the Osmotic Pressure with the Hydrostatic Pressure.

Osmotic pressure has already been mentioned in § 1; as also what is to be understood by it. When defined in this manner, the osmotic pressure will not only depend on temperature and concentration, but also upon the pressure to which the fluid is exposed on one side of the membrane. In passing through the membrane from the solution into the solvent, the hydrostatic pressure will fall from p to p_0 , and

the osmotic pressure is then defined by the equation,

$$\pi = p - p_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

at the same time

$$\pi = \phi(c, p, T), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

where ϕ is a single-valued, continuous function of c , p , and T . When p is introduced from equation (1), we get

$$\pi = \phi(c, \pi + p_0, T).$$

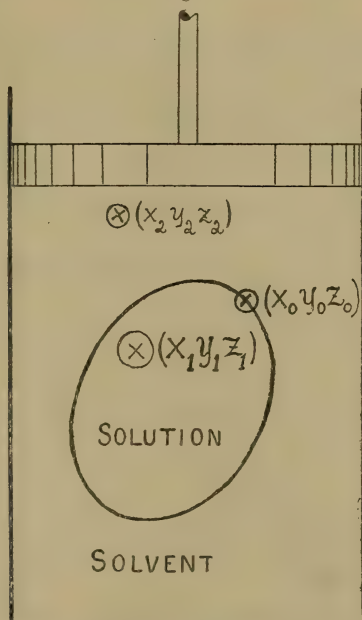
This on solving for π gives

$$\pi = \phi_0(c, p_0, T). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

Consequently it is immaterial, whether we consider π as a function of p or of p_0 .

We shall now try to find the influence of the external pressure upon the osmotic pressure. For this purpose we find the partial derivatives of π with regard to p or p_0 —that is to say, the quantities $\frac{\partial \pi}{\partial p}$ and $\frac{\partial \pi}{\partial p_0}$. Let us suppose, that we have dissolved a ponderable substance in a ponderable

Fig. 3.



fluid. The solution is separated from the solvent by a semi-permeable membrane, that forms a closed surface which we can suppose given by the equation

$$\psi(x, y, z) = 0. \quad (\text{Fig. 3.})$$

* See Duhem, *Mécanique Chimique*, tome iii. livre vi.

Let us furthermore suppose that the membrane is exceedingly thin, and that it can, at the same time, stand a great pressure; conditions which cannot actually be realized in practice, but can be approximately, by fixing the membrane in a porcelain cell.

We consider the system after an ideal state of equilibrium has set in. As long as we are in one of the fluids, the general condition for mechanical equilibrium must be fulfilled. According to equation (1) § 1, we get:—

(1) For the neighbourhood of a point (x_1, y_1, z_1) of the solution:

$$d\rho = -\rho \left(\frac{\partial U(x_1 y_1 z_1)}{\partial x} dx + \frac{\partial U(x_1 y_1 z_1)}{\partial y} dy + \frac{\partial U(x_1 y_1 z_1)}{\partial z} dz \right);$$

(2) For a point (x_2, y_2, z_2) of the solvent:

$$d\rho_0 = -\rho_0 \left(\frac{\partial U(x_2 y_2 z_2)}{\partial x} dx + \frac{\partial U(x_2 y_2 z_2)}{\partial y} dy + \frac{\partial U(x_2 y_2 z_2)}{\partial z} dz \right).$$

U means as before, the potential for the field to which the system is submitted. We suppose that U , as well as its first partial derivatives with regard to x, y, z , are continuous functions in the space occupied by the system.

Now suppose the points $(x_1 y_1 z_1)$ and $(x_2 y_2 z_2)$ approach the point $(x_0 y_0 z_0)$, situated on the surface $\psi=0$; the total variation of osmotic pressure along the surface will be

$$\left. \begin{aligned} d\pi = d\rho - d\rho_0 = -(\rho - \rho_0) \left(\frac{\partial U(x_0 y_0 z_0)}{\partial x} dx + \frac{\partial U(x_0 y_0 z_0)}{\partial y} dy + \frac{\partial U(x_0 y_0 z_0)}{\partial z} dz \right) \\ \frac{\partial \psi(x_0 y_0 z_0)}{\partial x} dx + \frac{\partial \psi(x_0 y_0 z_0)}{\partial y} dy + \frac{\partial \psi(x_0 y_0 z_0)}{\partial z} dz = 0 \end{aligned} \right\} \quad (3a)$$

Besides, according to (2a) we have π as a function of T, c , and p . We suppose the temperature to be the same throughout the system, while p and c vary with the coordinates. Consequently we get

$$d\pi = \left(\frac{\partial \pi}{\partial c} \right)_p dc + \frac{\partial \pi}{\partial p} dp.$$

If the values for dc and dp are introduced from equations

(1) and (2) § 1, we get

$$d\pi = - \left[\left(\frac{\partial \pi}{\partial c} \right)_p \frac{\partial c}{\partial n} \frac{1}{K} + \frac{\partial \pi}{\partial p} \rho \right] \left(\frac{\partial U(x_0 y_0 z_0)}{\partial x} dx + \frac{\partial U(x_0 y_0 z_0)}{\partial y} dy + \frac{\partial U(x_0 y_0 z_0)}{\partial z} dz \right) \quad (3b)$$

$$\frac{\partial \psi(x_0 y_0 z_0)}{\partial x} dx + \frac{\partial \psi(x_0 y_0 z_0)}{\partial y} dy + \frac{\partial \psi(x_0 y_0 z_0)}{\partial z} dz = 0.$$

As both expressions for the total differential of π along the membrane must be identical, we get as the necessary condition for it :

$$\rho - \rho_0 = \left(\frac{\partial \pi}{\partial c} \right)_p \frac{\partial c}{\partial n} \frac{1}{K} + \rho \frac{\partial \pi}{\partial p}. \quad (4a)$$

If we choose p_0 instead of p as variable, we get in a similar manner

$$\rho - \rho_0 = \left(\frac{\partial \pi}{\partial c} \right)_{p_0} \frac{\partial c}{\partial n} \frac{1}{K} + \rho_0 \frac{\partial \pi}{\partial p_0}. \quad (4b)$$

By means of equations (4a) and (4b) there has been developed a simple connexion between the concentration gradient and the dependence of the Osmotic Pressure upon the External Pressure, a connexion which makes it possible to calculate one of these quantities when the other is known. In the development of this formula nothing has been assumed that restricts the general applicability of the solution found ; the fluid may also even be compressible. The formula will hold good, provided always that the fluid fulfils the equilibrium condition (1) § 1, characteristic of an ideal fluid.

We will now proceed to apply the formula for the calculation of $\frac{\partial \pi}{\partial p}$, as we have already found a value for $\frac{\partial c}{\partial n}$ in the first section. By the aid of equation (11c) § 1 we get

$$\frac{\partial \pi}{\partial p} = \frac{\rho - \rho_0}{\rho} - \frac{\rho_0}{\rho^2} c(1+c) \frac{\partial \rho}{\partial c} \quad (5)$$

The term that we in this way have found for the dependence of Osmotic Pressure upon External Pressure, is now fully in accordance with the equations found by Planck and Duhem from another point of view.

Planck * has expressed the law in the following terms :

$$v_0 dp_0 = v dp.$$

v_0 is the volume of 1 gr. solvent, v is the increase of volume suffered by a very great quantity of the solution, when 1 gr. of solvent is added to it. If equation (1) is taken into consideration, we get

$$\frac{\partial \pi}{\partial p} = 1 - \frac{v}{v_0} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In order to find the proportion $\frac{v}{v_0}$ we will take a limited instead of an unlimited quantity, and add to it an exceedingly small quantity of the solvent. We have as before :

$$V\rho = M_1 n_1 + M_2 n_2.$$

If we add a quantity $M_1 dn_1 = \rho_0 dv_0$ of the first component we get :

$$\frac{dv}{dv_0} = \frac{v}{v_0} = \frac{\rho_0}{\rho} \left(1 + \frac{1}{\rho} c(1+c) \frac{\partial \rho}{\partial c} \right) \quad . \quad . \quad . \quad (7)$$

From equations (6) and (7) we almost immediately get equation (5).

Duhem † has developed the following formula :

$$\frac{P_1 - P}{\Pi_1' - \Pi} = \frac{U_1(T) - v(c, T) + c(1+c) \frac{\partial v(c, T)}{\partial c}}{v(c, T) - c(1+c) \frac{\partial v(c, T)}{\partial c}}.$$

The term to the left corresponds fully to what we have called $\frac{\partial \pi}{\partial p}$. Furthermore, we have :

$$U_1(T) = \frac{1}{\rho_0} \quad \text{and} \quad v(c, T) = \frac{1}{\rho}.$$

If this is introduced, we get :

$$\frac{\partial \pi}{\partial p} = \frac{\frac{\rho}{\rho_0} - \left(1 + \frac{1}{\rho} c(1+c) \frac{\partial \rho}{\partial c} \right)}{1 + \frac{1}{\rho} c(1+c) \frac{\partial \rho}{\partial c}},$$

* Planck, *Zeitschrift für physik. Chemie*, xliii. p. 584.

† *Mécanique Chimique*, tome iv. livre vi. p. 65.

and with the help of equation (7) :

$$\frac{\partial \pi}{\partial p} = \frac{1 - \frac{v}{v_0}}{\frac{v}{v_0}}.$$

But $\left(1 - \frac{v}{v_0}\right)$ is a very small quantity in comparison with the unit. We can therefore, when quantities of second and higher order are left out of consideration, immediately introduce one in the dominator and get the formula of Planck.

As already mentioned the formula (4a) holds good independently of the compressibility as also of the molecular volume's variation with the concentration.

In Planck's formula the contraction is also taken into consideration. From equations (4a), (6), and (7), we can now reversibly deduce the expression found in § 1 for the concentration gradient.

The dependence of Osmotic Pressure upon External Pressure has also been theoretically treated by O. E. Schiötz *. He comes to the result, that the change of pressure on the two sides of the membrane is connected by the following equation :

$$\rho dp = \rho_0 dp_0,$$

and consequently

$$\frac{\partial \pi}{\partial p} = \frac{\rho - \rho_0}{\rho}.$$

This value for $\frac{\partial \pi}{\partial p}$ is obviously appreciably different from the one found above. The reason for the discrepancy must be sought in the fact, that Schiötz in developing this formula, has not taken under consideration the fact that the concentration varies with the height. This, however, must be the case, as we have seen. From our formula it appears that when the solution fulfils the condition $\frac{\partial c}{\partial n} = 0$, the formula developed by Schiötz will hold good for this special solution.

We shall finally make a few comments concerning the variation of Osmotic Pressure with Concentration. In order to examine the influence of the concentration, we can suppose

* O. E. Schiötz, "Ueber die Abhängigkeit des osmot. Druckes und der Dampfspannung von dem Drucke" (*Boltzmann-Festschrift*, 1904, p. 618).

that it changes while the temperature is maintained constant; but still the change of concentration can occur under two different circumstances.

Either, the pressure can be kept constant in the solution; then we get the relation indicated by $\left(\frac{\partial \pi}{\partial c}\right)_p$. Or we can keep the pressure on the solvent constant—in practice the more general case—and we get $\left(\frac{\partial \pi}{\partial c}\right)_{p_0}$.

We can now prove that these two quantities must generally differ from each other. If equations (4a) and (4b) are subtracted,

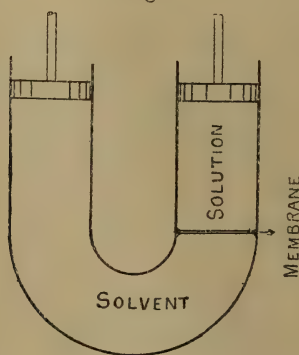
$$\left[\left(\frac{\partial \pi}{\partial c}\right)_p - \left(\frac{\partial \pi}{\partial c}\right)_{p_0}\right] \frac{1}{K} \frac{\partial c}{\partial n} = \rho_0 \frac{\partial \pi}{\partial p_0} - \rho \frac{\partial \pi}{\partial p}.$$

Let us now suppose that $\frac{\partial c}{\partial n}$ has a finite value, which is generally the case. The necessary and sufficient condition for $\left(\frac{\partial \pi}{\partial c}\right)_p$ being equal to $\left(\frac{\partial \pi}{\partial c}\right)_{p_0}$ is that

$$\rho_0 \frac{\partial \pi}{\partial p_0} = \rho \frac{\partial \pi}{\partial p}. \quad (8)$$

Generally, however, this equation is not fulfilled. To prove this, let us suppose that we increase the pressure on the two sides by just so much, that equilibrium is not disturbed.

Fig. 4.



We increase the pressure without moving along the membrane. Such a change of pressure can be carried out, for instance, by means of the arrangement represented in fig. 4.

On this supposition

$$d\pi = dp - dp_0 = \frac{\partial \pi}{\partial p} dp - \frac{\partial \pi}{\partial p_0} dp_0,$$

and consequently,

$$\frac{\partial \pi}{\partial p_0} = \frac{\partial \pi}{\partial p} \left(1 + \frac{\partial \pi}{\partial p_0} \right).$$

In order that equation (8) also be fulfilled, we must have

$$\frac{\rho}{\rho_0} \frac{\partial \pi}{\partial p'} = \frac{\partial \pi}{\partial p} \left(1 + \frac{\rho}{\rho_0} \frac{\partial \pi}{\partial p'} \right).$$

That this equation be fulfilled, we must either have

$$\frac{\partial \pi}{\partial p} = 0$$

or

$$\frac{\partial \pi}{\partial p'} = \frac{\rho - \rho_0}{\rho}.$$

The first of these equations implies that the osmotic pressure is independent of external pressure, which will not be the case generally.

The second equation can according to (4a) only be fulfilled when $\frac{\partial c}{\partial n}$ is equal to zero; but this is in contradiction to our supposition. If, however, we had to do with such a special case, we should get

$$\left(\frac{\partial \pi}{\partial c} \right)_p - \left(\frac{\partial \pi}{\partial c} \right)_{p_0} = 0.$$

This is apparently an indeterminate expression, which is, however, determined by the limiting value

$$\text{K. Lim.} \left(\frac{\rho_0 \frac{\partial \pi}{\partial p_0} - \rho \frac{\partial \pi}{\partial p}}{\frac{\partial c}{\partial n}} \right)_{\frac{\partial c}{\partial n} = 0}.$$

We thus come to the result, so very important to the theory of Osmotic Pressure, that the variation of this pressure with the concentration will be different, according as the pressure during the variation is kept constant on the solution or on the solvent.

LI. *The Effect of Change of Temperature on Spontaneous Ionization.* By NORMAN CAMPBELL, *Fellow of Trinity College, Cambridge* *.

1. **I**N a recent paper † I described some curious changes in the spontaneous ionization of a vessel which is being either heated or cooled. Since the changes appeared to have no connexion with the problem of the radioactivity of ordinary materials, which was at that time under consideration, further investigation of them was postponed. At intervals during the last two and a half years a large number of additional observations have been made: the effects can now be described with some completeness, and a tentative explanation of them offered. The changes observed are of great complexity and seem to be of little intrinsic interest, but, since other workers may be troubled by them, it is desirable to record the following notes.

Experimental Arrangements.

2. The dimensions of the tin vessel used in most of the experiments is shown in fig. 1. The walls are maintained at a potential (about 500 volts) high enough to send a saturation current through the gas to the electrode which is connected to a Wilson tilted electroscope and initially earthed. The vessel stands on blocks of slate and is usually heated by a small gas flame placed underneath it. Since it is of the utmost importance that the insulation through which the electrode passes should be kept cool, the tube carrying that insulation is surrounded by a water-jacket through which a slow stream of water passes. This device was found to simplify considerably the effects observed.

3. The investigation has been hampered by the difficulty of obtaining consistent results. Indeed the attempt to obtain quantitative consistency has been abandoned and a general constancy in the nature of the changes observed accepted as the most satisfactory condition possible. The almost invariable sequence of changes when the vessel is first heated and then cooled may be summarized as follows.

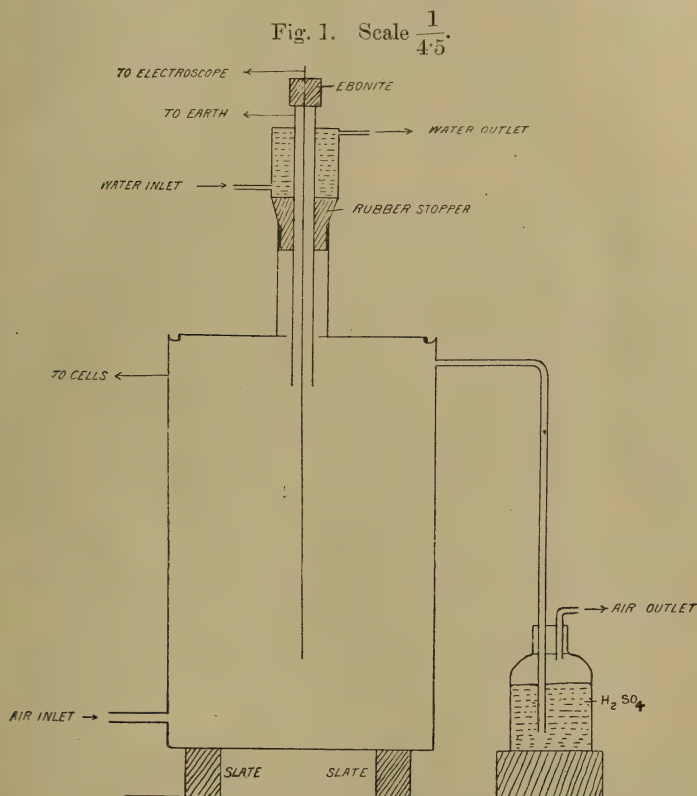
Nature of the Changes.

4. When the source of heat is applied to the vessel the leak through the latter increases temporarily but falls to its former value in a period of the order of a minute: this period is

* Communicated by the Author.

† Phil. Mag. April 1905, p. 545.

much shorter than that required for the temperature of the vessel to reach a steady state. The normal value of the leak is maintained as long as the heating is continued. When the source of heat is removed a temporary decrease of the



leak ensues lasting for a period of the same order as that of the increase, but, I think, a little longer. The variations of the magnitude of the changes in different experiments makes the estimation of this period very difficult, but it may be roughly indicated as 15 seconds for the increase and 30 seconds for the decrease. After the initial decrease the leak returns to its normal value and remains constant, though the temperature of the vessel may continue to fall perceptibly for half an hour.

(The statement that the leak through the vessel when hot is equal to that through the vessel when cold is not strictly

accurate: the "hot" leak is about 5 per cent. smaller than the "cold" leak. Since the pressure of the gas in the vessel is kept constant, the difference may be attributed to diminution of its density when heated.)

5. Such measurements as are possible indicate that the effect of heating is to cause a definite quantity of electricity of the same sign as the charge on the walls to cross from the walls to the electrode: and that the effect of cooling is to cause an equal quantity of electricity to pass in the opposite direction. The evidence for this statement will now be given.

6. The leak was measured by observing the time t required for the insulated electrode to reach a known potential V : t is about three minutes and V about 26 volt. When the vessel was cold individual readings varied about 8 per cent., and the mean of a long series was constant to 2 per cent.: when the vessel was hot these variations were doubled. When the effect of heating is to be observed, the electrode is insulated as usual and the flame immediately placed below the vessel: the time t' required for the electrode to reach the potential V is less than t . Similarly during the first cooling the time t'' is greater than t . Let C be the capacity* of the electrode system, Q the extra quantity of electricity which passes during the initial heating, and Q' that which passes during the initial cooling. Then if t_1 is the time required for the leak when the vessel is hot, q the steady current when the vessel is cold, q_1 when the vessel is hot,

$$CV = qt = qt' + Q \quad \text{and} \quad CV = q_1 t_1 = q_1 t'' + Q'$$

$$\therefore \frac{Q}{C} = V \frac{t - t'}{t} \quad (1) \quad \text{and} \quad \frac{Q'}{C} = V \frac{t_1 - t''}{t_1} \quad (2).$$

7. The following table gives the values of Q and Q' for several series of experiments performed on the same vessel at different times. The series have been specially selected, for the inconsistency between different observations is such that it is useless to draw quantitative conclusions except from those experiments which were thought from the general behaviour of the apparatus to be especially reliable. It will be seen that $Q = -Q'$ approximately and is independent of V : accordingly the statement in the last paragraph is substantiated.

* Direct observations showed that the alteration of the capacity with the heating was not appreciable.

TABLE I.

V.	t .	t' .	$Q/C \times 10^3$.	t_1 .	t'' .	$-Q'/C \times 10^3$.
·261	203	157	59	209	243	47
·261	207	155	65	210	259	59
·261	199	145	58	202	257	71
·261	175	129	69	176	213	55
·261	171	140	47	181	230	71
·572	471	403	83	492	561	80
·572	472	438	41	492	552	70
·572	469	420	67	483	535	62
·572	520	479	45	540	601	65
·572	513	467	51	532	583	54

8. The equality of Q and $-Q'$ may be tested more directly. The electrode is insulated, the flame applied for a minute and then removed: after the lapse of another minute all the heating changes are over. If $Q = -Q'$ the total leak will be the same as if no heating had taken place. However, such observations showed that Q' was a little greater numerically than Q . The difference between the two results may arise from the fact, that in the second of the two methods the initial cooling does not take place from a condition of thermal equilibrium.

Conditions which determine the changes.

9. The remainder of the observations were directed to determining the factors which influence the magnitude of Q , which may be taken as a measure of the intensity of the changes.

Q is independent of the direction of the current through the vessel and of the magnitude of the P.D. between the walls and the electrodes, so long as this is sufficient to send a saturation current through the gas. (The degree of saturation was such that an increase of the P.D. from 500 volts, the value usually employed, to 1000 volts produced an increase of 5 per cent. in the leak.) If the P.D. is considerably less than that required for saturation the magnitude of the effect diminishes, until, when the vessel is heated with the walls and the electrode at the same potential, no charge is communicated to the latter.

10. Q does not vary greatly with the material of the walls of the vessel, except in so far as these influence the ionization in the vessel (see § 12): experiments were made with zinc, tin, copper, and lead walls. It is also nearly

independent of the gas in the vessel: air, hydrogen, carbon dioxide, and coal-gas were employed.

11. Several methods of heating the vessel were tried—a small luminous gas-jet burnt from a fine glass tube, a Bunsen burner, a steam-jet, a metal box through which steam was passed, an oil-bath and a coil of wire wrapped round the vessel and carrying a current. The magnitude of the changes do not depend directly on the temperature of the source of heat, for all the above sources could be made to give similar effects: it seems rather to be determined by the rate at which heat is communicated to the vessel. No actual measurements of this quantity were taken, since they would have required great experimental elaboration and did not seem likely, in the light of the only explanations of the matter that could be devised, to lead to any valuable results. In any given vessel the magnitude of the changes could not be increased beyond a certain limit whatever the source of heat employed. But by employing less powerful sources the changes might be diminished without limit. Thus in the case of the vessel described in § 2, the value of Q was approximately the same whether a luminous jet 2 cms. long or a large Bunsen burner was used; but if a jet only 1 cm. was used, Q was too small to be measured with any certainty. With the big vessel used in the experiments described in the former paper, no source of heat less powerful than a Bunsen burner produced the maximum effect.

If the vessel is turned upside down and heated on the top, the changes are very much smaller. Indeed, it was only by playing a large blowpipe-flame on the top of the vessel, at the risk of melting the joints, that changes were obtained that could be detected with certainty.

It should be remarked that, since all methods of heating produce the changes, they cannot be due to the increase of the ionization in the air outside the vessel owing to the presence of the flame gases.

The effect of cooling the top of the vessel by placing on it ice or liquid air was also tried. When the cooling agent was applied a temporary *increase* in the leak was noted, and when it was removed a temporary *decrease*. Hence it seems that the sign of the change is not determined by the direction of the change of temperature, but that an increase in the leak is the accompaniment of a disturbance of the normal thermal equilibrium, and a decrease of a return to that condition.

12. Q increases with the ionization in the vessel, but not so rapidly as the ionization. When the latter was increased twenty-fold by placing uranium oxide in the vessel, Q was

raised to about twice its normal value. Similar effects were noticed when the air was ionized by Röntgen rays or by β -rays from radium placed outside the vessel.

13. The magnitude of Q increases, both absolutely and relatively to the normal leak q , with an increase in the volume of the vessel and with an increase of the ratio of the heated surface to the whole surface of the vessel. In the very large flat vessel (a cylinder of 1 metre diameter and 19 cms. height) used in the former experiments, Q' may be so large that the direction of the apparent current during the initial cooling is reversed and there is a leak against the electric field (which amounted to some 50 volts per cm.). The reversal of the leak was so surprising that every effort was made to discover its cause in some peculiarity of the experimental arrangements. However, it has been observed in two different vessels with no material (except the air inside them) common to both, in which the leak was measured by different methods.

14. Q is increased by the presence of dust in the air; but I have never been able to reduce its value below a certain limit by blowing in air through a plug of cotton-wool two feet long. Moreover, if dusty air is used, the effects obtained on heating up the vessel a second time will be less intense than those observed at the first heating; but if dust-free air is used, alternate heating and cooling for days will not stop the change. No difference could be observed between air saturated with moisture and air dried by a long tube of calcium chloride, except that the former spoilt the insulation.

15. A few experiments were made in which the air was kept at constant volume instead of at constant pressure. The changes observed were of the same nature as before.

16. The above summary by no means includes all the observations that have been taken, but includes all that I have been able to reduce to any sort of order. On several occasions a wholly new set of heat-changes has appeared; but when an attempt was made to investigate them later, they could not be reproduced.

Explanation of the Changes.

17. Only one explanation can be suggested which seems at all competent to account for the effects which have just been described. It may be supposed that the air of the vessel contains charged particles much heavier than the ordinary ions, possessing, in fact, a mass so large that the gravitational forces acting on them are larger than the electrical forces due to the field in the vessel. When the vessel is cold

these particles collect on the walls, or on the floor. If a source of heat is applied, convection currents are set up which carry the particles, bearing a charge of the same sign as that of the walls, into the body of the gas and into the neighbourhood of the electrode. In approaching the electrode they will induce on it a charge of such a sign that it appears to increase the leak due to the field; but since they move but slowly under the field, they will not increase the steady current appreciably. When the source of heat is removed, the convection currents become less violent and the particles sink again to the floor of the vessel; in leaving the neighbourhood of the electrode, they induce on it a charge of such a sign that it appears as a decrease of the steady leak. If the number of the particles is sufficiently great, the induced electrification may be great enough to cause the appearance of a temporary reversal of the current.

18. If this be indeed the true explanation, the heavy particles must have masses of an altogether different order to those of the ions. If X is the electric intensity (seldom more than $\frac{1}{4}$ el. st. unit in these experiments), e the charge, and m the mass of a particle such that its weight is equal to the electrical force on it,

$$Xe = mg, \quad \text{or} \quad e/m = \frac{981}{\frac{1}{4}} = 3.9 \times 10^3.$$

e/m for an oxygen molecule is about 2×13^{13} in the same units: hence the particles must contain about 5×10^9 molecules, the number which is contained in a sphere of air of 6×10^{-4} cms. radius. If the effects did not increase with the ionization in the vessel, these particles might be safely viewed as fragments of the metal walls; but, as it is, it seems probable that they must consist, at least in part, of aggregations of the molecules of the gas.

19. Three attempts have been made to test the truth of this hypothesis.

(1) The air in the big vessel was stirred up by working a fan in the vessel by means of a shaft which passed through a stuffing-box in the side. If the heat-changes are due to convection currents, they should be reproduced by the action of the fan. It was thought that small changes of the nature expected were observed. In a series of 50 observations, the mean value of t' (just after starting the fan) was 5 per cent. less, and the mean value of t'' (just after stopping the fan) was 4 per cent. greater than the mean value of t . It was only to be expected that the effects should be small; for the currents produced by the fan differ markedly from

those produced by heating in the feature which is most important for the present experiments, namely, the velocity of the current near the walls. On the other hand, it should be pointed out that the difficulty of getting the air in so large a vessel free from dust reduces the conclusiveness of the experiment.

20. (2) In the vessel shown in fig. 1 was placed a similarly shaped vessel of iron wire gauze with a mesh of $\cdot 15$ cm. The gauze was supported on slate blocks, and was about $1\cdot 5$ cms. distant from the walls of the vessel at each point; it could be maintained at any desired potential by means of a wire passing through the lid of the vessel. The gauze acted as an almost perfect electrostatic shield but permitted the passage of ions; when the walls and the gauze were at the same potential of ± 500 volts, the leak was 30 per cent. greater than when the vessel was earthed and the gauze at ± 500 volts.

When the vessel was heated with the walls and the gauze at the same potential, the changes observed were exactly similar to those when the gauze was absent except that the value of Q was reduced by about 25 per cent. But when the gauze was at ± 500 volts and the vessel earthed or at a potential differing still more from that of the gauze, the sign of the changes was reversed: the leak on initial heating was *less*, and that during initial cooling *greater* than the normal. The value of Q is only about one-third of that observed in the absence of the gauze, and Q' is notably less than Q .

21. This difference is in entire accordance with the explanation offered. Since the superficial area of the walls of the tin vessel is much greater than that of the wire in the walls of the gauze vessel, most of the particles will come from the former and their charge will be determined by the charge on the former. Since the P.D. between the walls and the electrode is less than that between the gauze and the electrode (it is the latter which determines the leak), the approach of the charged particles to the electrode will produce changes equivalent to a decrease of the leak. It will be seen that there must be some value of the potential of the walls for which the sign of the effects changes and their magnitude becomes zero. The value could only be estimated roughly; but it appears that the effects change sign when the P.D. between the electrode and the walls is about 100 volts less numerically than that between the electrode and the gauze.

22. (3) It was hoped that some light might be thrown on the subject by investigating the leak through the vessel immediately after the field was put on. A great many observations were taken, but the results were so inconsistent

and unreliable that no conclusions of value could be drawn from them.

23. The matter cannot be regarded as definitely settled ; but after trying almost every experiment, reasonable or unreasonable, that I can devise, I think that little more light can be thrown upon it by direct investigation.

My best thanks are due to Prof. Thomson for the innumerable helpful suggestions with which he has favoured me.

Summary.

24. The saturation current through the gas in a vessel which was alternately heated and cooled has been investigated. It was found that the current increased during the initial heating and decreased during the initial cooling.

The circumstances which determine the magnitude of these changes have been investigated.

It is suggested that the changes may be due to the induction of electrification on the electrode by very massive charged particles carried about by convection currents.

Cavendish Laboratory,
Feb. 1906.

LII. *Some Applications of the Theory of Electrons to the Theory of Phosphorescence.* By JOSEPH DE KOWALSKI, Dr. phil., Professor of Experimental Physics, University of Fribourg, Switzerland*.

I. **T**HE experiments performed by various scientists such as Wiedemann, Leonard, Nichols, and Merritt, and others, demonstrate with increasing certitude that the difference between phosphorescence and fluorescence is only apparent, and rather of a quantitative than of a qualitative nature.

In the presentation of the theory which follows, I have tried to correlate these phenomena and to find an explanation which accounts for them both on the same basis.

I intend to present only the general ideas of the theory and to indicate its utility in explaining the fundamental facts in this domain. I will leave aside all details as I propose to study them apart.

I admit with Prof. J. J. Thomson that a system of corpuscles,

* Communicated by the Author.

This note is based on the ideas set forth by Prof. J. J. Thomson in 'Nature,' March 22, 1906. A summary thereof has been presented in a note to the Comptes Rendus de l'Académie des Sciences, Paris.

belonging to an atom, begins to produce light as soon as its internal energy exceeds a certain minimum, which Prof. J. J. Thomson calls the critical value. This value may be reached either by raising the temperature (which method, however, is not very efficient) or by increasing the number of electrons of the system of corpuscles.

If the luminosity of a system of corpuscles is brought about by increasing the number of electrons, this system will remain luminous as long as its energy does not fall beneath the critical value. This diminution may arise from the following causes:—(1) Diminution of the energy by radiation. (2) Expulsion of the electrons of the system. (3) By the formation of a new configuration of the electrons in the interior of the system.

I suppose the existence in every phosphorescent or fluorescent body of two kinds of systems of corpuscles, endowed with different properties, and intimately united. To denote these systems I will use the terms *electronogenous* system and *luminophorous* system.

An electronogenous system has the power of shooting out electrons when acted upon by light. Nearly all bodies contain electronogenous systems, as is proved by experiments made on the secondary rays produced on the surface of bodies. Moreover, the experiments of Lenard, Elster and Geitel, Ramsay and Spencer, Sagnac, and others demonstrate that the quantity of electrons emitted under the action of light, varies considerably with different bodies and with the quality of the light employed. The velocity of the electrons shot out varies likewise. We are thus forced to admit that the systems of corpuscles are electronogenous in different degrees.

A luminophorous system is a system of corpuscles whose internal energy is very near the critical value. Hence the absorption by it of even a relatively small quantity of electrons is sufficient to make it luminous. In certain cases it will be able to absorb a quantity of electrons such that its internal energy considerably surpasses the critical value of Prof. J. J. Thomson. It is certain that a luminophorous system will absorb those electrons that traverse it only if the velocity of these latter bears a certain relation to its own internal energy and structure.

Every luminophorous system may, to a certain extent, be also electronogenous.

Every atom may include one or several luminophorous systems, but every luminophorous system produces only one kind of light having its own characteristic spectrum.

According to this hypothesis the phenomena of phosphorescence and fluorescence are explained as follows:—*In strongly electronogenous systems light produces the expulsion of electrons in considerable quantities. Now, as these electronogenous systems are intimately bound up with the luminophorous systems, the secondary rays are able to penetrate into these latter provided their velocity be in proper accord. The critical value of the internal energy of the luminophorous systems will soon be surpassed and light will result*.*

In the case of fluorescent bodies these electrons are only temporarily absorbed. The luminophorous system becomes luminous whilst it is being traversed by the electrons. In the case of phosphorescent bodies the electrons are absorbed by the luminophorous system and this latter remains luminous until it has shot out a quantity of electrons and radiated energy, to the extent of causing its internal energy to fall beneath the critical value.

The fundamental principle of my theory, namely, the coexistence of two kinds of systems of corpuscles in phosphorescent and fluorescent bodies, is amply confirmed by experimental facts.

II.—In the matter of phosphorescence we have the works of Lenard, Klatt, Urbain, de Visser, and Waentig, which prove that in order to produce phosphorescence, two different bodies, which they call the solvent and the active metal, must be brought into contact. The active metal plays the rôle of the luminophorous system. Now as the sulphides employed possess, to a high degree, the power of producing secondary rays under the influence of light, it is very probable that they contain the electronogenous systems.

As regards fluorescent bodies, the interesting work of Kauffmann on the constitution of organic fluorescent bodies, likewise confirm my manner of viewing the matter. The molecular groupings which this scientist calls fluorogenous and luminophorous, correspond to my electronogenous and luminophorous systems.

III.—My theory permits me to establish a relation, amply confirmed by experiment, between the intensity of phosphorescence and the time.

Let E_0 be the initial energy of a luminophorous system, before it is acted upon by secondary rays. Under the influence of light the electronogenous system which surrounds the luminophorous will cause these latter to receive and absorb

* Quite recently Prof. Lenard called attention to the importance of secondary rays for phosphorescence, but in a quite different manner from that in which I am doing it here.

a certain quantity of electrons, if the velocity of these latter permits it.

As the luminophorous system is to a certain extent electronogenous, it will, when acted upon by light, send back electrons into the surrounding space. If the quantity of electrons absorbed is greater than the quantity emitted, the value E_m , the critical value of the luminophorous system, will be attained after a certain time, and the luminophorous system will produce its own particular light.

The luminophorous system is not able to absorb electrons indefinitely. When the condition of equilibrium is destroyed the system will send back a number of electrons, the magnitude of the number depending upon the energy the system received from the electrons which came to it. Thus a stationary condition is established. The internal energy corresponding to the stationary condition will be equal to $E_0 + C$. The value of the constant C depends upon the quality and quantity of the secondary rays that traverse the luminophorous system, as also on the surrounding medium and on the temperature.

Let us consider the moment at which the production of secondary rays ceases, and at this moment let us put $t=0$.

From this moment on the internal energy E of the luminophorous system will decrease and we may admit, in a first approximation,

$$-\frac{dE}{dt} = \alpha E - E_0,$$

whence

$$E = E_0 + C_1 e^{-\alpha t}.$$

The values of C_1 and of α are functions of the properties of the luminophorous system, of the surrounding medium, and of the temperature.

The intensity of the phosphorescence may be expressed, in a first approximation, as

$$I = c(E - E_m),$$

whence we get

$$I = A e^{-\alpha t} - B,$$

$$A = C_1 c,$$

$$B = c(E_m - E_0).$$

This formula, which gives us a relation between the intensity of the phosphorescence and the time, is in agreement with the recent experiments of Nichols and Merritt on sidotblende.

IV.—As I said, the constants A and B are functions of the temperature. A and B diminish with an increase of temperature. Consequently it follows that there will be a certain

temperature at which the initial intensity of the phosphorescence will be a maximum, and another temperature, which need not necessarily be the same as the first, at which the decrease in intensity will be slowest. These conclusions are confirmed by the experiments of Lenard and Klatt.

An upper limit of temperature for the existence of the phenomenon of phosphorescence may likewise be explained by analogous considerations. A sudden elevation of the temperature of a phosphorescent body excited at a rather low temperature, will diminish the constant B and hence the intensity of the light ought to increase; still another fact confirmed by experience.

The spectrum of extinction discovered by E. Becquerel and studied by H. Becquerel, would be explained, according to my theory, by the following considerations:—Experiments on the ionization produced by bodies exposed to light, show us that certain regions of the spectrum possess the power of exciting secondary rays in a higher degree than other regions. If then the luminophorous system of the phosphorescent body increases its electronogenous properties in the infra-red region of the spectrum, the excitation of the phosphorescent body by the spectrum will accelerate the emission of the electrons of the luminophorous system. Thus its energy will rapidly fall beneath the critical value.

V.—According to my theory, the phosphorescence produced by the cathode rays, discovered by Sir William Crookes, would be a rather complex phenomenon. It would be due in part to the electrons which come directly from the cathode rays and traverse the luminophorous system, and in part to the electrons which are emitted by the secondary rays. A course of reasoning analogous to that given by Prof. J. J. Thomson to explain the two spectra of argon, permits me likewise to explain certain peculiarities in the experiments, particularly the differences in the phosphorescent spectra of europium when it is diluted with lime, and with gadolinia.

It is quite evident that an atom which is able to contain several distinct luminophorous systems may present different spectra when treated with the same solvent.

It is impossible for me to mention in this note all the phenomena of phosphorescence and fluorescence which confirm my theory. I do not think that it is complete, but I believe it presents a good working hypothesis that might give an impulse to investigations along these lines and serve as a guide in new experimental researches.

Physical Institute,
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LIII. *Some new Experiments with the Flame Tube.*

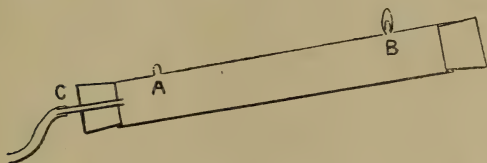
By U. BEHN*.

THE first experiments with the flame-tube were made to show the change of air-pressure with height and for some meteorological demonstrations †. Most elegantly it has been used by Mr. Rubens ‡, and by Messrs. Rubens and Krigar-Menzel § in acoustics.

In order to obtain the maximum sensitiveness of the flame-tube, which was not fully appreciated even by Mr. Rubens ‡§, the friction in the tube must be as small as possible. Hence it has been pointed out as of first importance, that the tube and the outlet holes should be wide (*e. g.*, tube not less than 5 cm. diameter and the holes .5 to 1 cm. diam.). If these conditions are not fulfilled, as recently occurred ||, the sensitiveness of the tube will of course be spoiled.

The simple method of using the flame-tube to show the change of the pressure of the atmosphere with the height may be again described here. A metal tube of suitable length is closed at the ends. At C (fig. 1) is an opening,

Fig. 1.



connected by a flexible tube to the gas mains. A and B are two holes of equal diameter (instead of having only two holes, there may be a number spaced equidistantly). Gas is allowed to pass into the tube (turned over to prevent

* Communicated by the Author.

† U. Behn, *Z. S. f. d. phys. u. chem. Unterricht*, xvi. p. 129 (1903).

‡ *Ber. d. Deutsch. Physic. Ges.* p. 351 (1904).

§ *Ann. d. Physik*, xvii. p. 149 (1905).

Regarding the theory compare Lord Rayleigh's paper "On the Circulation of Air observed in Kundt's Tubes and on some allied Acoustical Problems." *Phil. Trans.* clxxv. p. 1 (1883).

|| P. Steindel, *Z. S. f. d. phys. u. chem. Unterr.* 19.

In acoustics the best results are obtained with notes of high pitch. Mr. Rubens chose, sacrificing part of the sensitiveness, the holes relatively small to extend the application to notes of some hundreds oscillation.

escape from the holes) and is then lit at the openings. The flow is adjusted so that there is a small luminous tip to the flames, the tube being meanwhile supported in a horizontal position.

If the tube is now slightly tilted, so that one end, say A, is higher than the other, it will be found that the flame at the higher end of the tube becomes comparatively large and that at the lower end comparatively small. The same effect is obtained, whichever end of the tube is raised, so it cannot be caused by inequality in the size of the holes, or by friction in the tube. (At A the gas-pressures inside and outside the tube are sensibly equal, as is shown by the small size of the flame. Above A the pressure decreases more quickly in the dense air than in the light coal-gas. Hence at B the pressure inside the tube exceeds the pressure outside, and so the flame is large.) If one end lies only three millimetres higher than the other, a difference in the size of the flames will be produced, which, if the air is undisturbed, may be plainly seen from all over a large lecture-hall (15–20 m.). In this way a difference in pressure of about ($\cdot 0012$ – $\cdot 0006$) $\cdot 03$ gram weight per sq. cm., or $\cdot 000,000,2$ atmosphere* is made visible.

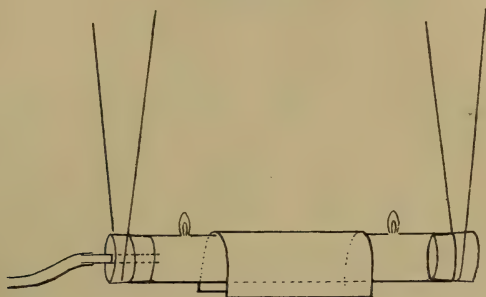
A new field of application for the flame-tube lies in its use for indicating small motions, or rather small accelerations. A slight movement of the tube causes changes of pressure in gas, due to the latter's inertia, and these changes of pressure in the tube are made visible by the difference in the sizes of the flames. For example, the beats of a seconds pendulum, though they be very small, may be made visible to the naked eye. For this purpose the flame-tube is suspended like a pendulum. The tube for these experiments need only be about $\cdot 5$ m. long† and one flame-hole at either end is sufficient. On account of the experiments described further on, the tube should be as light as possible. For the experiment in question, one ought to increase its weight by putting a $\cap$ -shaped piece of lead sheeting on it (fig. 2). Oscillations of one cm. amplitude will suffice to make the jumping of the flames visible from all over a lecture theatre.

* Approximately; the difference of temperature of coal-gas and surrounding air being not very large.

† In experiments, for which, on the contrary, a very long flame-tube is required, it is sometimes advisable to build the tube up by a number of pieces put loosely together. In this way one may avoid the bending of the tube, which of course is stronger heated by the flames from the upper side.

If one wants to show the kind of motion during the oscillations of a pendulum, one preferably suspends the

Fig. 2.



flame-tube by longer threads, and uses larger amplitudes. Then it is easily seen, that the changes in the sizes of the flames do not take place, when the pendulum is at the point of turning back, but when it is near the position of rest, where the acceleration changes sign.

If another pendulum is hung on to the swinging flame-tube (or *vice versa*) and the length of both pendulums suitably adjusted, the changes of the flames show how the simple swinging motion is altered by the addition of the octave, twelfth, etc.

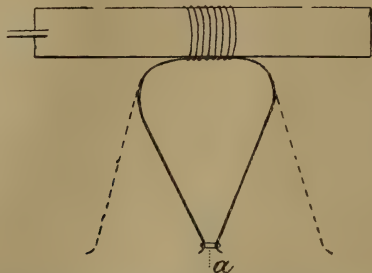
As one can see from the foregoing experiment the flame-tube pendulum might prove of use in detecting recoils. Thus a disk (of cardboard) whose centre of gravity may be displaced from the geometric centre by fastening a small weight to it near the circumference, is fastened to the flame-tube. Rotating the disk (taking care not to swing the flame-tube by the impulse) the flames will remain undisturbed until the weight is brought into play, when the oscillatory reaction becomes noticeable even though the weight may amount to less than one-hundredth of the total weight.

Similarly the reactionary motion caused by a very small pendulum may be shown. A pendulum consisting of a shot suspended by a thread will be sufficient.

It can also be shown by means of the flame-tube pendulum, that a system originally at rest cannot alter the position of its centre of gravity, unless acted on by an external force (*e. g.*, case of the recoil of a gun). For instance, a steel band, the shape of a large tuning-fork, is fastened to the

flame-tube pendulum as shown in fig. 3 and bent by tying the two ends at (a) together. The thread joining the two

Fig. 3.



ends at (a) is burnt, and the spring thus set in motion. If the band is geometrically and physically perfectly symmetrical, the flames remain at rest*.

Another case, where inertia plays a rôle, we have in the so-called centrifugal force. If the flame-tube lying on the table is turned about one end as centre, the flame at the other end becomes large through the centrifugal force causing an increase of the internal pressure at that end of the tube. This effect is shown even if the motion is very slow.

The experiments described above do not of course exhaust the uses of the flame-tube, but merely serve as examples.

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LIV. *The Decomposition of Water-Vapour in contact with Hot Platinum Wire.* By ALFRED HOLT, Junr.†

RATHER more than a year ago the author made several attempts to estimate the extent of the dissociation of water-vapour at high temperatures by heating a platinum wire (by means of an electric current) in contact with it at low pressures. The results were consistent among themselves, but at high temperatures were decidedly lower than those previously obtained by Nernst and Wartenberg (*Göttinger Nachr.* 1905, p. 35). The work was not published, owing to the circumstance that some doubt arose as to whether the numbers were a true measure of the dissociation of the vapour at the temperature of the wire.

* A cork pushed in between the prongs at (a) serves well enough for holding the axle for the disk used in the last experiment.

† Communicated by the Author.

Although it was recognized that a gas heated by contact with a fine platinum wire must be very rapidly cooled, yet it was feared that as the cooling took place there might be an appreciable amount of recombination in the immediate neighbourhood of the hot wire, owing to the catalytic action of the platinum vapour thrown off from the incandescent metal.

The amount of the dissociation would obviously be reduced by this source of error, and the magnitude of the discrepancy between it and the observed values would become greater as the temperature of the wire, and the consequent amount of metallic vapour thrown off it, increased.

It was not evident at the time when these determinations were made, that it could be independently ascertained whether the error in question influenced the results to an appreciable extent. More recently, however, while performing a series of experiments for a different purpose, the extreme rapidity with which the separation of oxygen and hydrogen took place in pure water-vapour at about 15 mm. pressure was noticed ; and the idea occurred that, by taking cognizance of this fact, an experiment could be devised to test whether any combination of the electrolytic gas took place in the immediate neighbourhood of the wire.

The principle of the proposed experiment was that the water-vapour and electrolytic gas should be withdrawn both from the surface of the wire and at some distance away from it. If no recombination occurred during cooling, then the oxygen and hydrogen should be present in both samples in combining proportions ; but in the event of a partial reunion, the sample of gas taken from the immediate neighbourhood of the wire should contain an excess of oxygen, whereas that taken simultaneously from another portion of the water-vapour should be correspondingly rich in hydrogen. For, when a wire is heated in water-vapour, electrolytic gas is produced at its surface. Diffusion would at once take place, and if no recombination occurred, the oxygen and hydrogen would become uniformly distributed in combining proportions throughout the vapour, since any diffusion back of the hydrogen from the cooler towards the hotter layers of vapour, would be exactly compensated by the direct outward diffusion from the hot wire.

If, however, recombination took place in a zone at some little distance from the surface of the wire, its effect would be to diminish the amount of electrolytic gas in the cooler and more distant portions of the water vapour ; so that

outward diffusion would be continually going on, with the result that the gas which had the greater rate of diffusion would be found in excess outside the zone of recombination, while that with the slower rate would tend to concentrate round the wire.

Since these experiments on the dissociation of water-vapour were performed, Langmuir (Journ. Amer. Chem. Soc. 1906, p. 1357) has carried out a very similar investigation, and has obtained results in close agreement with those of Nernst and Wartenberg. He does not seem to have considered that any appreciable source of error arose from this recombination in the immediate neighbourhood of the wire, and this view is confirmed by the experiments described in this paper; but he is of opinion that the highest temperature is reached at which it is feasible to determine the amount of dissociation by this method when the wire begins to disintegrate, and deposits finely divided metal as a film on the inner surface of the glass vessel employed. Some amount of recombination of the electrolytic gas may take place in contact with this film, and so cause considerable errors in the determinations.

In his experiments, however, the apparatus employed was only 0.9 cm. diameter, and was heated during each experiment to at least 100° C. by the steam passing through it, whilst I used a vessel 20 cms. diameter, the walls of which hardly became heated during the short time occupied by each determination. Under these conditions it is less probable that there would be any large error from this cause, though it is possible that it may be an explanation of the lowness of the numbers at high temperatures. The catalytic action of this spray and deposit is being further examined, and it is impossible at present to say how far it really influences the results.

The values obtained for the amount of dissociation of water-vapour are slightly higher than those of Langmuir and Nernst and Wartenberg at low temperatures, but they steadily become smaller as the temperature rises, till at 2000° absolute they are only about $\frac{1}{4}$ of the quantities obtained by the other authors.

The cause of this marked discrepancy of results is being further investigated.

Experimental.

The apparatus employed for the determination of the amount of dissociation consisted of a large glass bulb of rather more than 4 litres capacity, in the centre of which a pure platinum wire about 5 cms. long was hung.

The globe was completely evacuated, and then filled with water-vapour by opening a tap beneath it which communicated with a smaller bulb containing pure distilled water.

When the large globe was completely filled with vapour, the wire was heated to the required temperature, which was kept as constant as possible till equilibrium had been established. The current which heated the wire was then cut off, and the whole contents of the bulb pumped off through a freezing-mixture in order to condense the water-vapour. The electrolytic gas was collected, and estimated by explosion, when it was always found to contain a minute excess of hydrogen. This excess appeared to be due to the absorption of a trace of oxygen by the mercury in the fall-tube of the pump.

The temperature of the wire was determined from its resistance by the following formula:—

$$t - \theta = \delta \left\{ \left(\frac{t}{100} \right)^2 - \frac{t}{100} \right\},$$

$$\theta = \frac{R_t - R_0}{R_{100} - R_0} 100.$$

1.54 was the value adopted for δ .

Experiments were made with two different wires, in order to see whether they gave concordant results.

In the following tables the percentage decomposition of water-vapour at the various temperatures is given.

The values at the pressure of the water-vapour were determined experimentally, while those at 760 mms. pressure were calculated from them.

Wire No. I.

Temperature absolute.	Per cent. decomposition at the pressure of water-vapour.	Calculated per cent. decomposition at 760 mms. pressure.
1220° C.	0.0038	0.00108
1540 „	0.0573	0.015
1640 „	0.0884	0.0246
1700 „	0.118	0.0317
1760 „	0.153	0.04

Wire No. II.

Temperature absolute.	Per cent. decomposition at the pressure of water-vapour.	Calculated per cent. decomposition at 760 mms. pressure.
1350° C.	0.00644	0.00175
1500 „	0.034	0.00887
1560 „	0.0593	0.0158
1630 „	0.0704	0.01842
1770 „	0.148	0.0388
1890 „	0.203	0.053
2030 „	0.264	0.068

The values obtained with the two wires are very fairly concordant, and would undoubtedly seem to represent the amount of decomposition taking place under the conditions of the experiments.

The apparatus used for the experiments on the separation of the oxygen and hydrogen through partial recombination by the platinum spray, is shown in fig. 1.

A current of water-vapour entered at A and was decomposed by the hot platinum wire Pt (which had a small weight W attached to it) while passing up the narrow portion of the apparatus, so that a condition of equilibrium between the vapour and electrolytic gas was reached in the compartment D. The oxygen and hydrogen were withdrawn through B and C, and as the entrance into E from D was only about 2 mms. wide and had the hot wire passing through it, it was expected that the gas collected through B would largely consist of that which immediately surrounded the wire in D, while the portion pumped off through C would contain the gas from the cooler parts of D.

Fig. 1.



As already pointed out, one would expect in the event of a partial reunion of the electrolytic gas by the platinum spray, to find an excess of oxygen in B and hydrogen in C.

The details of the experiments are given below, where it will be noticed that there is no separation of this kind ; on the contrary, there is a minute excess of oxygen in C and hydrogen in B. One is, therefore, bound to conclude that any recombination which takes place through this cause has no appreciable influence on the results.

Total volume of Gas collected in B.	Excess of Hydrogen in B.	Total volume of Gas collected in C.	Excess of Oxygen in C.
6.0 c.c.	0.25 c.c.	4.0 c.c.	0.12 c.c.
12.0 c.c.	0.37 c.c.	9.0 c.c.	0.195 c.c.
4.0 c.c.	trace	2.5 c.c.	trace

The University, Manchester.

LV. *On the Influence of Stress on the Electrical Conductivity of Metals.* By W. ELLIS WILLIAMS, B.Sc., Research Fellow of the University of Wales*.

THE effect of hydrostatic pressure in altering the electrical resistance of metals was discovered by Chwolson † in 1880, and his results were soon afterwards confirmed by Tomlinson ‡. In both cases, however, the range of pressure employed was too small to enable accurate measurements to be made. The question has, in recent years, been taken up by Lussana § and also by E. Lissell ||.

The results obtained by Lissell were :—That the resistance of all the pure metals investigated is diminished by hydrostatic pressure, and that the diminution is nearly but not quite proportional to the pressure. He found further that with alloys the decrease of resistance is much smaller than with the pure metals, and that with some alloys the resistance is increased by pressure. He paid special attention to the case of *manganin*, in which he found the increase to be, within the limits of his experiments, accurately proportional

* Communicated by Prof. E. Taylor Jones, D.Sc.

† Chwolson, *Bull. Petersburg*, 1880.

‡ Tomlinson, *Phil. Trans.* 1883.

§ Lussana, *Nuovo Cimento*, (4) x. p. 73, (3) v. p. 305, 1903.

|| Lissell, *Diss. Upsala*, 1902.

to the pressure, and he proposed to make use of this property in the measurement of hydrostatic pressure.

The results obtained by Lussana are very different from these. He also found that the resistance was diminished by pressure, but that the diminution was very far from being proportional to the pressure, the rate of diminution with pressure falling off rapidly as the pressure increased.

The actual values of the decrease are also different from those given by Lissell, and for manganin Lussana found that the resistance was diminished by about $5 \cdot 10^{-7}$ of its value per atmosphere, whereas according to Lissell it is increased by $23 \cdot 10^{-7}$ per atmosphere.

In view of these conflicting results it was suggested to me by Prof. Röntgen that I should attempt to investigate the question, and the experiments described below were undertaken for that purpose. I also took the opportunity of investigating the influence of stress on the resistance of bismuth, and obtained some interesting results which are described below.

Apparatus and Method of Experimenting.

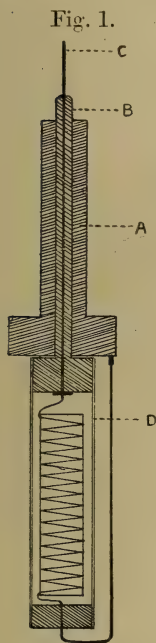
The hydrostatic pressure was obtained by means of a Schaffner and Budenberg screw-pump furnished with a Bourdon gauge reading up to 1500 atmospheres, made by the same firm. The gauge was calibrated by comparison with a standard manometer whose behaviour had been carefully investigated by Dr. Wagner*.

The pump was connected to a steel cylinder 4 cms. $\times$ 80 cms. inside dimensions. Into the open end of the cylinder was fitted a brass piece, which served to carry the terminals of the wire whose resistance was to be measured, and which could be screwed down on a leather collar so as to form a watertight cover to the cylinder.

In order to diminish as much as possible the rise of temperature which takes place on compression, water, which is advantageous both on account of its small coefficient of expansion and its high specific heat, was used in the pump and cylinder. Hence it was necessary that the wire and its connexions should be carefully insulated. The apparatus for this purpose is shown in fig. 1. A is the brass piece forming the cover of the pressure cylinder, through which passes an ebonite rod B, serving to insulate the copper wire C, to which one end of the wire under investigation is soldered. A piece of the ebonite rod about 2 cms. long and 2.5 cms. wide projects into the cylinder, and a leather collar between

* Wagner, Wied. *Ann.* p. 906 (1904).

the shoulder of the rod and the brass piece makes the connexion water-tight. Over this rod a rubber tube, D, is drawn, completely enveloping the terminal and the wire connected to it. The other terminal passes out through an ebonite block closing the lower end of the rubber tube, and is soldered to a wire coming from the brass piece A. The



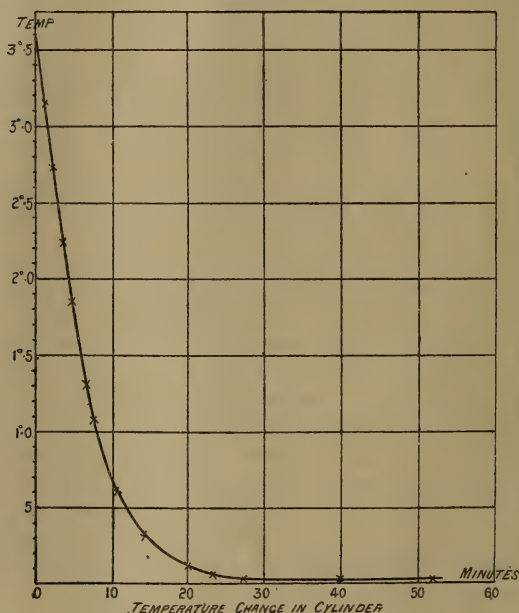
rubber tube is filled with machine oil, in which the wire is therefore completely immersed. In the first experiments made with manganin wire, the wire was wound on an ebonite cylinder placed inside the rubber tube mentioned above. It was found, however, that this introduced a source of error into the results, shown by a small bend near the origin in the curve representing the relation between change of resistance and pressure. A certain amount of tension is necessarily used in winding the wire on the cylinder, and this causes a small increase in its resistance. On application of pressure the ebonite cylinder is compressed and the wire is freed from tension, causing a corresponding alteration in the resistance. In all further experiments great care was taken to wind the wire loosely on a frame without the least tension. The change of resistance was measured by means of a Wheatstone's bridge made up of rheostats without a slide-wire. Owing to the smallness of the effect measured, it was found better to observe the deflexions of the galvanometer, keeping the resistances in

the arms of the bridge constant. A resistance-box having $\frac{1}{100}$ -ohm coils was placed in the arm of the bridge adjacent to the wire under investigation, and the galvanometer-scale was calibrated by altering the resistance in this box and observing the deflexions of the galvanometer. As the resistances in the other two arms were equal, this gives directly the change of resistance in the wire corresponding to a given deflexion of the galvanometer. The sensitiveness of the galvanometer was $1 \cdot 10^{-8}$ ampere per division, and with 100 ohms in each arm of the bridge a change of $\cdot 1$ per cent. in the resistance of the wire gave a deflexion of 200 divisions. The chief difficulty in making the measurements arises from the heat of compression in the cylinder. In order to obtain a constant temperature, the cylinder was immersed in an iron vessel filled with water kept continually agitated by means of a small fan worked by an electric motor. The capacity of the

vessel was a little more than a cubic metre, and its rate of change of temperature, due to fluctuations in the temperature of the room, never exceeded $\frac{1}{100}^{\circ}\text{C.}$ per hour. By measuring the rate of change before and after each experiment, the temperature could be judged with considerable certainty to $\frac{1}{300}^{\circ}\text{C.}$

A number of preliminary experiments were made in order to ascertain the change of temperature occurring on the application of pressure and the time taken by the cylinder to attain the temperature of the bath. The curve (fig. 2) shows

Fig. 2.



the effect of the sudden release of a pressure of 400 atmospheres. The fall of temperature is about $3^{\circ}.5$, and the time taken to attain the temperature of the bath is from 40 to 60 minutes. The temperature in the cylinder was in this case measured by means of a copper-constantan thermojunction, immersed in a rubber tube full of oil precisely similar to that used with the wire coils.

RESULTS.

Manganin wire.—The manganin wire used was .1 mm. diameter, double silk-covered, and was arranged as described above.

The following readings were obtained :—

Pressure in Atmospheres.	Change of Resistance, $\frac{dR}{R} \cdot 10^4$.	Change per Atmosphere, $\frac{dR}{R} \cdot 10^7$.
10.5	.23	22.5
31	.66	21.3
91	2.00	21.7
192	4.12	21.3
292	6.39	23.0
392	8.59	22.5
492	10.82	22.1
592	13.13	22.1

The resistance is increased by pressure, and the increase is within the limits of experimental error directly proportional to the pressure, the mean value being :

$$\frac{dR}{R} = 22.2 \cdot 10^{-7} \text{ per atmosphere.}$$

This is the actual change of resistance observed. To obtain the change of specific resistance it is necessary to correct for the change of dimensions of the wire due to its compressibility. As the compressibility of manganin is not known, it is impossible to calculate the value of this correction. A rough approximation may, however, be obtained by assuming the compressibility to be the same as that of copper, viz. $.83 \cdot 10^{-6}$ per atm., and with this value the change of specific resistance is

$$18.2 \cdot 10^{-7} \text{ per atmosphere.}$$

Lead Wire.—The arrangements were precisely the same as those employed for manganin wire. The dimensions of the lead wire were : diameter .37 mm., length 1.5 metres. Resistance 3.28 ohms.

The readings obtained were:—

Pressure in Atmospheres.	Change of Resistance, $\frac{dR}{R} \cdot 10^4$.	Change per Atmosphere, $\frac{dR}{R} \cdot 10^7$.
32	4.56	142
92	13.3	156
192	26.8	139
292	42.27	144
392	56.0	142
482	69.2	143
596	85.8	144
692	100.0	143

The resistance is diminished by pressure, the diminution being proportional to the pressure, and its mean value being :

$$\frac{dR}{R} = 143 \cdot 10^{-7} \text{ per atmosphere.}$$

Aluminium wire.—The aluminium wire employed was .1 mm. diameter, and was annealed by passing an electric current through it. The ends of the aluminium wire were soldered to copper leads by means of an alloy of aluminium and zinc. All the other arrangements were as before.

The readings obtained were :—

Pressure in Atmospheres.	Change of Resistance, $\frac{dR}{R} \cdot 10^4$.	Change per Atmosphere, $\frac{dR}{R} \cdot 10^7$.
92	3.64	39.66
192.5	7.83	40.8
292	11.45	39.16
387	15.03	38.82
482	18.13	37.5
592	23.09	38.8

The resistance of aluminium is thus diminished by pressure, the mean value of the diminution being :

$$\frac{dR}{R} = 38.8 \cdot 10^{-7} \text{ per atmosphere.}$$

Bismuth.—The influence of hydrostatic pressure and also of longitudinal tension on the resistance of bismuth wire was investigated. A Lenard's bismuth spiral (Hartmann and

Braun) was used in the experiments. One of the mica plates which protect the spiral was removed, the other being left to support the wire. The spiral was placed in the rubber tube as used in the previous experiments. The resistance of the spiral was 26 ohms and the diameter of the wire .25 mm. The readings obtained were :—

Pressure in Atmospheres.	Change of Resistance, $\frac{dR}{R} \cdot 10^4$.	Change per Atmosphere, $\frac{dR}{R} \cdot 10^7$.
31	6.3	200
92	18.6	201
292	57.7	197
492	95.7	194

The resistance of bismuth is increased by pressure, the mean value of the increase being :

Fig. 3.

$$\frac{dR}{R} = 196 \cdot 10^{-7} \text{ per atmosphere.}$$

Bismuth is the only one of the pure metals so far investigated whose resistance is increased by pressure.

Influence of Tension on the Resistance of Bismuth.

The Lenard's spiral used in the previous experiment was unwound and two pieces of wire about 10 cms. long were obtained. These were soldered to two copper terminals of the form shown in fig. 3. The two wires were then placed in a wide glass tube filled with oil and suspended vertically as shown. The upper terminal of one wire was hung from the arm of a balance, and the wire was stretched by hanging weights from the other arm. The two wires were arranged in adjacent arms of a Wheatstone's bridge, the two upper terminals dipping into a mercury-cup connected to the galvanometer. The whole arrangement was covered with a thick cloth to protect it from air-currents, and when this was done a fairly steady zero was obtained.

The following readings were taken :—

Tension in grammes / mm. ²	Change of Resistance, $\frac{dR}{R} \cdot 10^4$.	Change per gramme / mm. ²
1·88	·102	·054
18·8	·99	·053
37·6	1·95	·052
94·05	5·08	·054
131·6	7·05	·053
188·8	10·11	·0537
282·1	15·16	·0537
376·0	19·95	·0530
470·2	24·9	·0529
564·3	29·8	·0528

The resistance of bismuth is *decreased* by tension, the mean value of the decrease being :—

$$\frac{dR}{R} = \cdot 0535 \cdot 10^{-4} \text{ per } \frac{\text{gramme}}{\text{mm.}^2}.$$

Summary of Results.

<i>Metal.</i>	<i>Pressure.</i> Change of Resistance, $\frac{dR}{R} \cdot 10^4$ per atmosphere.	<i>Tension.</i> $\frac{dR}{R} \cdot 10^4$ per gramme / mm. ²
Lead	— 143	
Aluminium	— 38·5	
Bismuth	+ 197	— ·535
Manganin	+ 22·2	

Of the substances investigated in the present experiments, two, manganin and lead, have been investigated both by Lissell and Lussana. The present results are in very good agreement with those obtained by Lissell, showing considerable differences from those of Lussana, thus :—

Manganin.

Lussana	$dR/R = -4.65 \cdot 10^{-7} p + 2.02 \cdot 10^{-10} p^2.$
Lissell	$dR/R = +23 \cdot 10^{-7} p.$
W.E.W.	$dR/R = +22.2 \cdot 10^{-7} p.$

Lead.

Lussana	$dR/R = -197 \cdot 10^{-7} p + 47.6 \cdot 10^{-10} p^2.$
Lissell	$dR/R = -144 \cdot 10^{-7} p + 2.4 \cdot 10^{-10} p^2.$
W.E.W.	$dR/R = -143 \cdot 10^{-7} p.$

p denotes pressure in atmospheres and a plus sign denotes increase of resistance. The term in p^2 obtained by Lissell is too small to be detected in the present experiments, his experiments having reached to 3000 atmospheres. At 700 atmospheres it would only amount to 1 per cent. of the total change. The agreement between my results and those of Lissell is therefore as close as could be expected from the accuracy obtainable in these experiments.

The above experiments were carried out in the Physical Laboratory of the University of Munich under the direction of Professor Röntgen, to whom I desire to express my best thanks for much helpful advice given in the course of the work.

University College, Bangor, 1907.

LVI. *Notices respecting New Books.*

Five-Figure Mathematical Tables for School and Laboratory Purposes. By A. DU PRÉ DENNING. London: Longmans, Green, & Co. 1906.

THIS is a thin book, quarto size, with 4 pages of logarithms, 6 pages of trigonometrical functions with their logarithms, 1 page of squares and 1 page of cubes, and 2 pages of constants and formulæ. The chief novelty is in the arrangement of the logarithms and antilogarithms. Neither table is complete in the usual way. The antilogarithmic table runs from logarithm .00000 to logarithm .60900, that is up to number 4.0644; and the logarithmic table begins with number 4 and goes up to number 9.99. In other words, to keep the differences as small as possible and to economise space, half the table is built on the antilogarithmic plan, and half on the logarithmic plan. The same is true for the log. and antilog. reciprocals. The obvious inconvenience of this method is that, in looking up the logarithms of a series of numbers, the worker will have to be constantly on the alert when

passing from numbers with high initial digits to numbers with low initial digits. Time must be wasted, and it is doubtful if the somewhat greater accuracy in the fifth figure when working with differences at either end will make up for this. For schoolboys especially the plan is distinctly inappropriate. To be of real service in schools mathematical tables must be as simple as possible, with no traps for the unwary; and for this reason also the using of the same trigonometrical tables downwards for the sines and tangents and upwards for the cosines and cotangents is not to be commended. For use by a practised hand in laboratory or office the book will no doubt be found of value. The table of squares and cubes which may be used with care for finding square roots and cube roots cannot fail to be of great service. We have noticed one mistake in the logarithm of 476. The introduction of the logarithms of reciprocals is a feature of merit; but it is a pity that some tables are printed down the page and others across, compelling the worker to turn the book through a right angle when using different tables in succession. In this respect also convenience has been sacrificed for economy's sake. The book is a model of clear printing.

The Science Year Book. Edited by Major B. F. S. BADEN-POWELL.
London: King, Sell & Olding. 1907.

A FEW minor changes have been introduced into the present issue of this year-book. Instead of a series of brief independent articles on progress in various branches of science, a single summary is given. All the tables which have appeared in previous editions are retained, and a diary, with a full page to each day, and a number of extra pages at the end for various entries, is provided. The diary part of the book consists of excellent paper tastefully ruled. We notice that in the biographical section Prof. Minchin is described as still on the staff of the R.I.E. College, although this college has ceased to exist. An attractive frontispiece, consisting of a portrait of Lord Rayleigh, adorns the work.

The Physical Laboratories of the University of Manchester. A record of 25 years' work. Prepared in Commemoration of the 25th Anniversary of the Election of Dr. ARTHUR SCHUSTER, F.R.S., to a Professorship in the Owens College, by his old Students and Assistants. Manchester: at the University Press. 1906. Pp. 142.

ALTHOUGH this volume is primarily of interest to the *alumni* of the University of Manchester and the personal friends of Dr. Schuster, some matter of general interest to teachers of physics will be found in the detailed description of the physical laboratories and the methods of instruction there adopted. An excellent portrait of Prof. Schuster forms a fitting frontispiece to this tribute to his 25 years' professorial work at Manchester.

The Teaching of Elementary Mechanics. Discussion at Johannesburg at the Meeting of the British Association, 1905. *Edited by JOHN PERRY.* London: Macmillan & Co. 1906.

HAVING read this book, the predominant feeling in our minds is best expressed in the wish uttered by Sir David Gill: that he had been born fifty years later and could have been taught mechanics by more rational methods. The trouble we had over "g," over the pound of matter and pound force and pound of inertia! There were no experiments to help us. Sir Robert Ball's 'Experimental Mechanics' was indeed available, but few then made use of it. We who write believe in the experimental method of teaching above all things; only we should not classify the levers into orders and confuse and burden the mind with details. But inertia, acceleration, weight and force, kinetic energy, should in this way be introduced. "Begin with the things themselves" as the President (Professor Forsyth) observes. After that, whether or not a man acquires, for instance, an adequate mental impression of what inertia is, depends a great deal more on the man himself than upon his teachers. I do not believe doubtful definitions will make a clear head go wrong or influence a thoughtful student injuriously. In most cases the student gains by having to form his own mental picture and not receiving too much in the way of carefully prepared definitions. The mere fact that there can be so much difference of expert opinion over (*e. g.*) the utility of the term "poundal," suggests that all minds are not alike in these matters. If so, a certain degree of liberty should certainly be left to the student: "poundal" may be useful to one student and not to another.

But we must not depart from our duty as reviewer of a valuable book, which may be read with pleasure and profit by teachers and thinkers. Thanks are due to Prof. Perry for leading this useful discussion. Our remarks above are by no means intended to decry its usefulness. There should, of course, be agreement among teachers when once the best methods of teaching are arrived at, and such a discussion as we have just read is the surest way of arriving at a knowledge of the best methods of teaching.

Hermann von Helmholtz. By LEO KOENIGSBERGER. Translated by FRANCES A. WELBY. With a Preface by Lord KELVIN. Oxford: at the Clarendon Press. 1906.

THE English translation of Professor Leo Koenigsberger's *Life of Helmholtz* is certain to be welcomed by English readers. The work of translation has been excellently done. There is added an interesting preface by Lord Kelvin.

We have already referred at some length to the original German edition of this book, and it is hardly requisite to refer in detail to the contents of the English version, more especially in view of the commendation which the German work received on every side. In the biography are admirably blended and interwoven the episodes of life in its more material aspect and the mental or

creative life of the almost inspired thinker. Both are well treated: the first with sympathy, and the last with real scientific insight.

The portraits are a most interesting feature, and are from the German edition. Every thing about the work is attractive, and few English scientific men will care to be without this memorial of one of the greatest discoverers of the generation which is passing away.

A History of Chemical Theories and Laws. By M. M. PATTISON MUIR, M.A. New York: John Wiley & Sons. London: Chapman & Hall, Ltd. 1907. Pp. xx + 555.

HOWEVER convenient and, in some cases, necessary it may be, for purposes of elementary teaching, to neglect the past history of a science, and to group the results achieved by it in a manner calculated to smooth the path of the beginner and to present the various aspects of the subject to him in the most natural order of logical sequence, there can be no doubt that no serious student can properly estimate the relative importance of the various units which go to make up a harmonious scientific theory without some study of the frequently slow and painful steps by which our present store of knowledge has been acquired. In chemistry—more, perhaps, than in any other science—the progress of truth has frequently been interfered with by more or less blind gropings in the dark, by fanciful and utterly unjustifiable interpretations of results, and by dogmatic insistence on errors which had become traditional and difficult of uprooting. All advanced students of chemistry will feel deeply grateful to Mr. Pattison Muir for the extremely interesting sketch of the evolution of chemical theory contained in the book under review. It is not a history of chemistry, nor did the author ever intend it to be such; it is an attempt “to set forth . . . the main lines along which the science of chemistry has advanced to its present position.” It is thus a history of ideas and theories, rather than of chemical facts and their discoverers—a history of the attempts to furnish answers to the two fundamental questions with which chemistry has to deal, viz., “what is a chemically homogeneous substance?” and “what happens when chemically homogeneous substances interact?” In accordance with this plan the book is divided into two parts, dealing respectively with the attempts made to answer the above questions. Owing to the restrictions which the author has voluntarily imposed on himself, he is able to deal with some aspects of the subject much more fully than would otherwise have been possible, and as a result the treatment has gained in richness and relief. In connexion with the earlier history of chemical theories, we may perhaps suggest that, for the sake of those readers who have not previously studied the history of chemistry, it would be advisable to indicate more clearly the modern meanings or equivalents of the terms used by the early writers, as this would somewhat facilitate the reading of certain chapters of the book. This has, to a certain extent, been done by the author, but not quite so fully as would appear to be desirable.

Cours de Physique. Par H. BOUASSE. Première Partie.
Mécanique Physique. Paris : Ch. Delagrave. Pp. 236.

No intelligent study of physical problems is possible without a thorough understanding of the fundamental principles of dynamics, and any student who intends taking the study of physics seriously will do well not to shrink from the severe mental discipline by the exercise of which alone a really thorough insight into the principles and methods of dynamics can be obtained. The author of the book under review, which forms the first part of a treatise on physics, has therefore done well to devote to dynamical principles the comparatively large amount of space which the importance of the subject fully warrants. The work is not intended for the beginner. A knowledge of elementary dynamics, and of differentiation, integration, and co-ordinate geometry is assumed on the part of the reader. The subjects dealt with include the general principles of dynamics, the general properties of fluids, capillarity, fluid motion, viscosity, the transmission of waves through a compressible fluid, hydrodynamics, the deformation of perfectly elastic solids, the elements of the theory of elasticity for isotropic solids, permanent deformations, friction, the equilibrium of strings, vibrations of strings, resonance, and vibrations of rods. The treatment is excellent in every way, being characterized by clearness and brevity, and due stress being always laid on the purely physical aspect of each of the problems dealt with. We can warmly recommend to all advanced students this very concise and elegantly written guide to a subject of fundamental importance in physical science.

LVII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 288.]

November 21st, 1906.—Sir Archibald Geikie, D.C.L., Sc.D.,
 Sec.R.S., President, in the Chair.

THE following communications were read:—

1. 'The Kimeridge Clay and Corallian Rocks of the Neighbourhood of Brill (Buckinghamshire).' By Arthur Morley Davies, A.R.C.S., B.Sc., F.G.S.

The paper contains two principal divisions:—

(1) An account of the section of Rid's Hill, Brill. At the summit are Lower Portlandian sands, not well exposed: below these comes a thickness of about 50 feet of Upper Kimeridge Clay, with only one small exposure of grey shale containing black lignite

and a few fossils. At the foot of the hill, a large brickfield shows the following sequence :—

	Feet
8. Grey, creamy-weathering clay (zone of <i>Exogyra virgula</i>) ...	4
7. Grey, creamy-weathering clays with crushed shells, darker and slightly selenitic towards the base	about 17½
6, 5, & 4. Two bands of creamy limestone, with marly clay between (<i>Pictonia</i> cf. <i>Cymodoce</i> , <i>Pholadomya æqualis</i>)	4
3. Dark grey, brown-weathering clay, with occasional phosphatic nodules	8
2. Black shaly clay, weathering lilac-grey with yellow stains, highly selenitic (<i>Belemnites abbreviatus</i> , <i>B. nitidus</i> , <i>Ostrea deltoidea</i> , <i>Gryphaea dilatata</i> , <i>Trigonia Voltzii</i> , etc.)	14
1. Band of grey limestone (<i>Perisphinctes decipiens</i> , <i>Trigonia Juddiana</i>) at the base.	

There have also been dug up, probably from the selenitic clay (No. 2), doggers composed largely of *Serpula tetragona*, Sow., and *Arctica* (*Cyprina*) *cyreniformis* (?), Blake, yielding many other fossils, including *Cardioceras alternans* (von Buch) and *Belemnites nitidus*, Dollf. The possibility of the selenitic clay being Amphill Clay is discussed, the conclusion being that only Lower Kimeridge Clay is exposed in this section.

(2) The rock of Studley and Arngrove, described by Phillips as an argillaceous chert, is shown to be mainly composed of the globate spicules of the tetractinellid sponge *Rhaxella*. It has a much more limited extension eastward than the geological map would suggest; but, on the other hand, it is found to form part of the outcrop of normal Corallian rocks from Waterperry by Stanton St. John to Holton Wood. Its exact relation to the Oxford Clay is described, and an increased list of fossils is recorded from it.

Palæontological notes are given on certain species of Lamellibranchia and Annelida, chiefly from the Lower Kimeridge Clay.

2. 'On the Skull and greater portion of the Skeleton of *Goniopholis crassidens* from the Wealden Shales of Atherfield (Isle of Wight).' By Reginald Walter Hooley, F.G.S.

December 5th.—Sir Archibald Geikie, D.C.L., Sc.D., Sec.R.S.,
President, in the Chair.

The following communications were read :—

1. 'On the Geological Conditions which have contributed to the Success of the Artesian Boring for Water at Lincoln.' By Prof. Edward Hull, M.A., LL.D., F.R.S., F.G.S.

This boring has its source of supply in strata which rise to the west, but to the east dip down towards the North Sea. There exists no information as to whether the eastern border of the water-bearing formation thins out against a concealed ridge of Palæozoic rocks. The water-yielding stratum is reddish, soft, porous, sand-rock, reached at a depth of 1561 feet and penetrated to a depth of 474 feet. About one million gallons of water rise to the surface daily. The sand-rock belongs to the New Red Sandstone, which crops out from York to Nottingham with a breadth of about 8 miles. The hydraulic pressure at the bottom of the

boring is that due to about 2035 feet, and the friction of the water in percolating the rock accounts for the fact that the water can be pumped down during the day but rises again in the night. The formations penetrated are the following:—Alluvium and Lower Lias 641 feet, Rhætic Beds 52 feet, Red Marl and Lower Keuper Sandstone 868 feet, Bunter Sandstone 454 feet. The quantity of water drawn from the New Red Sandstone, at and below the outcrop defined, amounts to not less than 20 million gallons, and the total available quantity of water percolating into the Sandstone amounts to about 300 millions.

2. 'Notes on the Raised Beaches of Taltal (Northern Chile).' By Oswald Hardey Evans, F.G.S.

The town of Taltal is situated, partly on the dry bed of a broad river, and partly on a gently-inclined plain that fringes the bays of the coastal ranges far to the northward, and runs up the valleys to a considerable altitude and distance from the coast. The material of this plain consists of sands and well-rounded gravel derived from the rocks of the adjacent hills, mingled with shells and some isolated boulders of considerable size. The formation is impregnated with salt, and there protrude through it curiously-weathered remnants of former stacks and islets. The plain rises in terraces, the highest of which are somewhat obscure, and sometimes portions of these higher terraces are preserved in the stacks and islets. A second coastal shelf also occurs, marked by a line of shallow caverns, some excavated in igneous rock. Some, at least, of the shell-accumulations associated with the plain contain pottery, and are associated with Indian kitchen-middens, but the beds of shells in the gravel, containing occasionally whale-bones, give satisfactory evidence of the marine origin of the terraces. Some of these shells are replaced by crystallized brine, and calcium sulphate occurs in some sections. Profound ravines (*quebradas*) occur in the massive rocks bordering the plain, although the climate is now so dry that rain-erosion is practically non-existent.

December 19th.—Sir Archibald Geikie, D.C.L., Sc.D., Sec.R.S.,
President, in the Chair.

The following communications were read:—

1. 'The Post-Cretaceous Stratigraphy of Southern Nigeria.' By John Parkinson, B.A., F.G.S.

In this paper, which is a first attempt to outline the sequence of the later deposits of Southern Nigeria (now including the Colony of Lagos), a series of beds are described from four localities—three from the western side of the Niger, and one around Calabar near the Kameruns frontier. The alluvium of the river-beds and the lower terraces are referred to, and the succeeding sediments grouped under three heads.

The youngest of these, termed the Benin Sands, are of wide distribution, and are found extensively developed in all four areas. Their almost universal appearance near the coast, and the height, occasionally 300 feet, to which their denuded tops are now raised

above sea-level, indicate an important submergence and subsequent elevation in comparatively-late times, while from the latter much of the present physiography appears to date. The Benin Sands have been seen in contact with both the older groups of sediments, but unfortunately the work has not proceeded far enough yet to show conclusively the relationships of the latter.

The older of the two, the Lignite Series, occurs typically near Asaba on the Niger, although indications of a similar deposit have been met with near Moroko, south of Abeokuta; the younger, the Ijebu Series, has so far been found only in that district of the Lagos Province. These beds are extensively impregnated with bitumen. The suggested difference of age depends on the lesser dip and degree of consolidation of the Ijebu Series. In them a few lamellibranchs have been found, of no value as regards the determination of the horizon, and some plant-remains; in the Lignite Series only undeterminable plant-remains. The Benin Sands are apparently unfossiliferous. The information regarding the Ijebu Beds has been derived almost entirely from boreholes.

2. 'The Geology of the Oban Hills (Southern Nigeria).' By John Parkinson, B.A., F.G.S.

The country described in this paper comprises some 1800 square miles of the Eastern Province of Southern Nigeria, adjacent to the Kameruns frontier. The rocks are crystalline, principally gneisses and schists with later granites, pegmatites, and basaltic dykes, surrounded on the north, west, and south by Cretaceous sediments. For purposes of description the series is divided under nine headings, according to locality and petrographical character; and it is concluded that, neglecting the basaltic dykes, two broad groups may be distinguished—the one characterized by the presence, the other by the absence, of foliation. In the former the foliation tends to be lost, giving a passage between types which petrographically are acid orthogneisses and granites.

Observation of the banded gneisses, which are typical of several districts, leads to the conclusion that the acid magma is intrusive into mica and hornblende-schists, and that the banded gneisses are composite rocks produced by injection. The orthogneisses themselves exhibit many variations; and study in the field shows that the series consists of several members, produced by the differentiation probably of one magma, which have reached their present positions at different times: that is, they differ slightly in age. At Uwet was found a group of phyllites and grits altered into garnet, andalusite, and staurolite-schists, and hornfels, with the development of much biotite, by an intrusive gneiss; and it is considered as probable, though not capable at the time of absolute proof, that the sillimanite-gneisses of the Ekankpa ford, a few miles away, are the result of an extension of the same belt of contact-metamorphism, a suggestion which possibly might truly include the schists of the entire district.

Typical specimens of the granites, pegmatites, and dykes are described, and sketch-maps given.

3. 'The Crystalline Rocks of the Kukuruku Hills (Central Province of Southern Nigeria).' By John Parkinson, B.A., F.G.S.

In this paper a short account is given of the crystalline rocks found in the Central Province of Southern Nigeria, between the Station of Ifon (north of Benin City) and the Northern Nigerian frontier. The rocks fall under two heads: (a) a group of gneisses, and (b) a group of schists.

The former is considered as intrusive in the latter, and consists of varieties of biotite-gneisses, often well-banded, but this group is not infrequently represented by several rocks readily separable by mineralogical peculiarities and by intrusion.

By loss of foliation the gneisses tend to pass into granites, and the series is accompanied by pegmatites.

The second group is distinguished by the wide distribution of quartz-schists, with which occur quartz-mica-schists, mica-schists (locally containing sillimanite), and hæmatite-schists. A partial analysis of a specimen of the last-named showed 67·77 per cent. of iron-peroxide.

The district is compared with the Oban Hills of the Eastern Province.

January 9th, 1907.—Sir Archibald Geikie, D.C.L., Sc.D., Sec. R.S.,
President, in the Chair.

The following communications were read:—

1. 'On the Cretaceous Formation of Bahia (Brazil) and on the Vertebrate Fossils contained therein.' By Joseph Mawson, F.G.S., and Dr. Arthur Smith Woodward, F.R.S., F.L.S., F.G.S.

2. 'On a New Dinosaurian Reptile from the Trias of Lossiemouth, Elgin.' By Arthur Smith Woodward, LL.D., F.R.S., F.L.S., F.G.S.

January 23rd.—Sir Archibald Geikie, D.C.L., Sc.D., Sec.R.S.,
President, in the Chair.

The following communication was read:—

'On the Geology of the Zambezi Basin around the Batoka Gorge (Rhodesia).' By George William Lamplugh, F.R.S., F.G.S. With Petrographical Notes by Herbert Henry Thomas, M.A., B.Sc., F.G.S.

This paper contains an account of the physiographical and geological structure of the hitherto undescribed country bordering the Batoka Gorge, which was investigated by the author in 1905, under the auspices of the British Association.

The physical features of the country fall into two well-marked divisions:—(1) The unbroken plateau, a portion of the great central basin-plain of South Africa. (2) The area of rejuvenated drainage, a wedge-shaped tract having its apex at the Victoria Falls, due to the lowering of the trunk-drainage by the erosion of the deeply-cut gorge.

The following rock-systems are described:—

5. Kalahari Sands, Chalcedonic Quartzite, and other surface-deposits.
4. Flaggy Sandstone of Boomka.
3. Batoka Basalts.
2. Wankie Sandstones and Coal-Measures (= Matobola Beds and ? Escarpment Grits of Molyneux).
1. Fundamental Complex of Metamorphic and Igneous Rocks.

(1) The Fundamental Complex was studied *in situ* only at one point, near the Wankie Coal-mine. The position of its outcrop north of the Zambezi is inferred from the character of the river-borne detritus in some of the stream-beds.

(2) The Wankie Series consists of massive sandstones and pebbly grits, containing boulders near the base, with intermediate flaggy sandstones and shales with coal-seams. This series is abruptly truncated on the south-east side, along the Deka River, by a great fault which brings in the Batoka Basalts; but reasons are given for supposing that beds of similar character reappear at the northern margin of the Basalts, north of the Zambezi, beyond the district examined.

The Deka Fault is described in some detail, and its important influence upon the structure of the country is discussed. It is suggested that the 'Sijarira Quartzites' of Molyneux, of the country farther to the north-east, may be only the indurated and contorted sandstones that accompany this great fault.

(3) The Batoka Basalts were found to extend unbrokenly from the Victoria Falls eastward to the mouth of the Deka, and southward to the edge of the Kalahari Desert; and it is shown that they also cover a very wide area to the southward and westward of the region traversed. They consist of a succession of massive lava-flows and flow-breccias; no intercalated sediments were found in the districts examined, nor was any eruptive centre discovered. Some curious structures observed in basalts are described; and it is remarked that the zigzagging character of the stream-gorges below the Victoria Falls is due to differential erosion along the strong joints and fault-planes in these rocks, under the alternating conditions of wet and dry seasons.

The age of the Basalts is discussed, and it is considered that they are probably Mesozoic, and may date back to the later stages of the period of volcanic activity represented by the Stormberg Beds of the more southerly part of the continent.

(4) Some limited patches of Flaggy Sandstone, etc., seen near the head of the Deka Basin, appear to represent sediments newer than the Basalts, and may be equivalent to part of the Forest-Sandstone of Mr. A. J. C. Molyneux and Mr. F. P. Mennell.

(5) The 'Kalahari Sands,' which cover large areas of the plateau, occur mainly in broad low swells or 'bults' of friable red sand, now tree-clad. These sweep down to the Zambezi River in the shallow valley above the Victoria Falls, and cannot have been accumulated under existing conditions. The author endorses Dr. S. Passarge's view that they indicate a former period of greater aridity than the present in the Central Basin.

Some irregular patches of surface-quartzite and limestone, due

to the percolation and evaporation of ground-water, are described and compared with the 'Botletle-Schichten' (Passarge) of the Kalahari, but the author considers that the antiquity of these beds is not so great as that assigned to the 'Botletle-Schichten.'

The paper concludes with a petrographical appendix by Mr. H. H. Thomas, giving the results of his microscopic examination of a series of rock-specimens collected by the author.

February 6th.—J. E. Marr, Sc.D., F.R.S., Vice-President,
in the Chair.

The following communications were read:—

1. 'Note on the Cervical Vertebra of *Zeuglodon* from the Barton Clay of Barton Cliff (Hampshire).' By Charles William Andrews, B.A., D.Sc., F.R.S., F.G.S.

2. 'The Origin and Age of the Plateaus around Torquay.' By Alfred John Jukes-Browne, B.A., F.G.S.

The existence of high-level plains or plateaus near Torquay has long been known, but since Pengelly's time little attention seems to have been paid to them. Pengelly believed that there were several such plains at different levels, and thought that the time of their production was not very remote. On examination, however, his evidence breaks down, and the author regards the plateaus as portions of one inclined plain.

The plateau-area has an extent of about 3 miles in length from north to south, and about 2 in width. Its highest level is found at the foot of the ridge which forms the watershed between Torbay and the estuary of the Teign. From the foot of this a platform stretches southward on each side of Barton at a level of from 350 to 340 feet; south of this is the flat summit of Lummaton Hill (also about 340 feet); and about a mile farther south are the flat-topped hills known as Yaddon and Daison Hills, both about 320 feet, and only separated by a narrow gorge cut by the little stream which flows from Barton to Torquay.

East of these hills we find a plateau-area extending continuously through St. Marychurch and Babbacombe, for a distance of a mile and a half, and an average width of half a mile, but cut off abruptly along the coast-line. The northern end of this plateau is about 330 feet above Ordnance-datum, but it slopes gradually southward, till on Walls Hill its highest part is only 267 feet.

On the west of Torquay the old plain has been so dissected by streams, and so lowered by detritive agencies, that it is almost destroyed. Remnants, however, remain in Waldon Hill (200 feet), Stentiford Hill (230 feet), and Daddy-Hole Plain (about 200 feet).

The central hills of Torquay, which are known as Warberry Hill and Lincombe Hill, rise to much greater heights (440 and 400 feet) and stand up as conspicuous eminences above the level of the surrounding plateau.

The evidence as to the age of the planation is next considered. It is proved to be post-Permian, by the fact that Permian breccia forms part of the plateau-surface at St. Marychurch. It is also probably post-Cretaceous, because Cretaceous planation is not likely

to have removed all the Permian. Its present dissected condition shows that it is older than the Pleistocene, and consequently an Eocene date would agree with local evidence.

The author accepts the Eocene age of the Haldon gravels and of the Bovey deposits, and adduces evidence for a connexion between the Haldon plateau and the Bovey basin on the one hand, and between the Aller basin and the Torquay plateau-area on the other. He regards the Bovey basin as a locally-deep depression, along a line of Miocene deflexion which crosses Devon from north-west to south-east, and believes the plateau-area near Torquay to be the flattened-out extension of this flexure. He thus considers the plateau to be part of the basal plain on which the Eocene deposits lie, and infers that these deposits passed directly out of the Aller basin on to this plateau, burying it to a depth of several hundred feet.

Thus it was on the surface of the Eocene deposits that the existing system of drainage was established.

The Brixham plateau, south of Torbay, presents similar features and appears to be of the same age, but in its case the drainage is northward from higher ground to the south. This plateau can be traced westward through Churston and Galmpton to Stoke Gabriel, where there is high ground both to the north and south of it. It is inferred that this plateau is the site of a second shallow flexure, the axis of which ran nearly west and east, so as to meet and merge into the other line of flexure outside the entrance of Torbay. Incidentally this would explain the formation of Torbay.

February 27th.—Sir Archibald Geikie, Sc.D., D.C.L., LL.D.,
Sec. R.S., President, in the Chair.

The following communications were read:—

1. 'On the Lower Ordovician Succession in Scandinavia.' By William George Fearnside, M.A., F.G.S.

The paper is a stratigraphical account of the *Dictyonema*-Shales, the *Ceratopyge*-Beds, the *Didymograptus*-Shales, and the *Orthoceras*-Limestone of Sweden and Southern Norway, and is based upon field-observations of Scandinavian type-localities made by the author during the summer of 1906. The beds are discussed under the following headings:—

(C) *Didymograptus*-Shales and *Orthocera*kalk,

(B) Glauconite-Shales and *Ceratopyge*kalk,

(A) *Dictyonema*- and *Bryograptus*-Shales,

which are found to be applicable to all the sections visited.

For purposes of ready correlation with the British Tremadoc rocks, the upper part of the *Acerocaris*-zone of the Alum-Shales is considered along with Division A, and the presence of trilobites of Ordovician type and related to those of the *Ceratopyge*kalk in that as well as in the *Dictyonema*-Beds is emphasized. It is also held that the branching graptolites of Division A show a progressive evolutionary divergence from an original *Dictyonema* through *Dictyograptus flabelliformis* to the true graptolites of Clonograptid or Bryograptid types, and through *D. norvegicus* to the so-called 'dendroid graptolites.' The gradual replacement of the western

type of Alum-Shales of this division by the sandy beds with *Obolus* in Northern Öland and the Baltic Provinces is discussed, and is considered as an early manifestation of the crustal instability which brought about the curious vicissitudes of the Glauconite-Shale period. Many of the phenomena at the base of Division B are suggestive of an unconformity, and in all Swedish areas except Skåne the succession from A to B is shown to be incomplete. In general, the base of Division B has much the aspect of the Cambridge Greensand, and the Glauconite-Shales as a whole mark a period during which the alternating processes of deposition and erosion were nicely balanced. A curious small-scale erosion-phenomenon occurring intermittently throughout Division B is redescribed under the name 'Korrosionsgrupper,' and along with the variable detailed succession of the beds is considered as a further proof of discontinuous sedimentation throughout the period. Division C of *Didymograptus*-Shales and their equivalent *Orthocera*-kalk is discussed only in outline. Both shales and limestone pass downwards continuously into the highest *Ceratopyge*-Beds, both in lithology and fauna, and the difference between the lowest *Orthoceras*-Limestone (Planilimbatenkalk) and the higher limestone of the *Ceratopyge*kalk is almost imperceptible.

This stratigraphical evidence is then considered in its bearing upon the question of the definition of the boundary between the Cambrian and the Ordovician Systems; and the author follows the Scandinavian authorities in considering that, so long as the *Dictyonema*-horizon is available, the evidence of sudden faunistic change within the series discussed is too slight to warrant a palaeontological separation of the systems at any other horizon. A comparison of the British Tremadoc and Arenig Series with these Scandinavian rocks concludes the paper; and it is maintained that the time has now arrived for British geologists to come into line with their Continental brethren, and to include the *Dictyonema*- and the overlying Tremadoc Beds as the lowest series of the Ordovician System.

2. 'The Occurrence of Pseudomorphous Pebbles of Pyrites at the Crown-Reef Mine (Witwatersrand).' By Cuthbert Baring Horwood, A.R.S.M., Assoc.M.Inst.C.E., F.G.S.

Reference is first made to the existence of calcite-'pebbles' in the Main Reef, which Mr. Julius Kuntz believes to be due to the replacement of quartz by calcite. Pellets of iron-bisulphide known as 'buckshot' occur at the Rietfontein 'A' Mine in the Buckshot Reef: they exhibit radiate fibrous structure, and are probably of concretionary origin. At the Crown-Reef Mine a few 'pebbles' of pyrites, some measuring as much as an inch in length, occur in a narrow band of conglomerate at the contact of the reef with a basic dyke. It is conjectured that the mineralizing solutions, which deposited the pyrites (together with, some if not, all of the associated gold) ascended along the fractures due to the intrusion of the dyke, and found an easy course along the small conglomerate-bed, where they replaced some of the quartz-pebbles with pyrites, being kept up by a band of shale underneath the conglomerate.

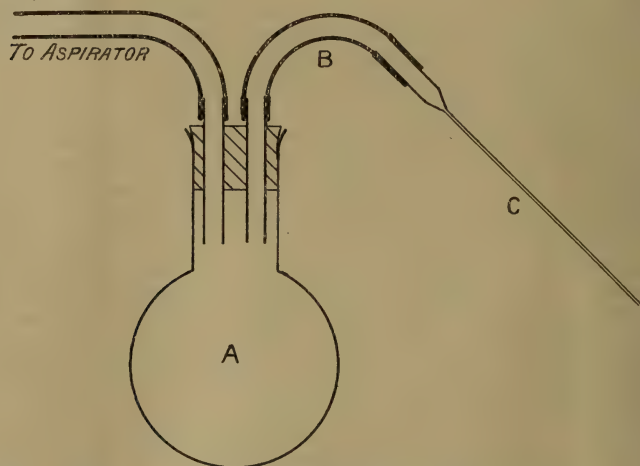
LVIII. *Intelligence and Miscellaneous Articles.*

ON A SIMPLE DEVICE FOR THE RECOVERY OF SPILLED MERCURY.

BY WILL. C. BAKER, M.A., LECTURER ON PHYSICS, SCHOOL OF MINING, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA.

OF the common laboratory accidents, few are more annoying than the spilling of mercury on the floor, especially if there be cracks between the boards into which the liquid may run. In such cases so much comfort has been experienced from the use of the simple device herein described, that it has been thought worth while to suggest it to others.

A small flask, A, is fitted with a rubber stopper through which pass two short glass tubes. One of these is connected, by a stout rubber tubing, to a water-pump, and the other leads by a short rubber tube, B, to a piece of glass tubing drawn out to a fine nozzle, C.



The action is obvious:—When the pump is working, if the nozzle be approached to a drop of mercury, the liquid is quickly sucked into the flask. It is generally best to raise the nozzle from time to time so as to let the mercury in the tube, B, run into the bottle, A.

As the glass tube, C, may be drawn out as fine as required, there is no difficulty in recovering mercury from cracks in the floor even less than a half a millimetre wide. Under these circumstances the nozzle often becomes stopped by dust particles and the flow of air ceases. Then it is best to use nozzles drawn out, ten or fifteen centimetres long, and when stopped by dust to break off four or five millimetres of the dust-choked end, thus leaving a free passage for the air.

Working in this way mercury may be quickly and easily recovered. The dust is separated by a filtration. This method possesses many obvious advantages over the usual way of collecting the mercury with a piece of zinc.

FIG. 1.

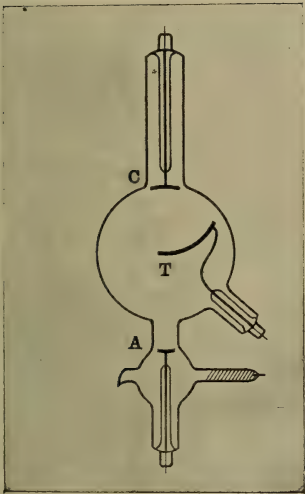
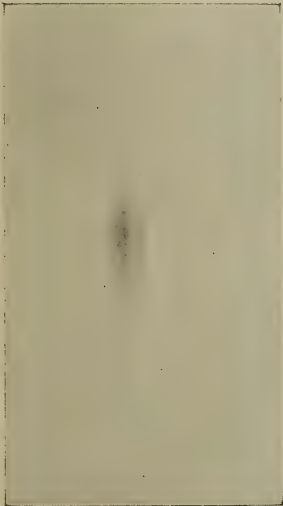


FIG. 2.



A

B

FIG. 3.

G

E

C



H

F

D

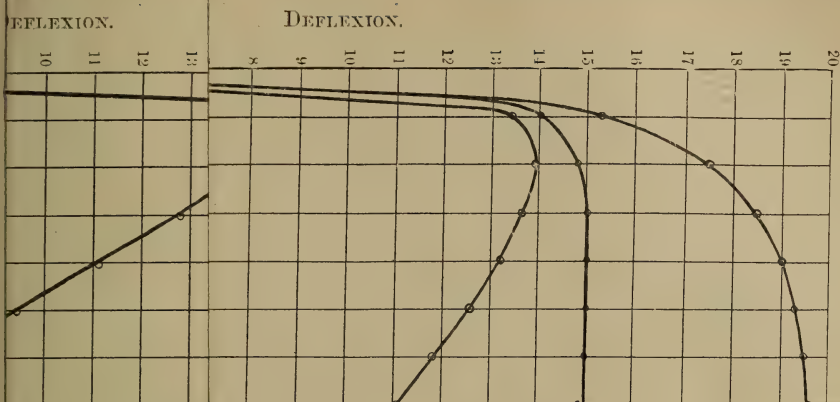


Fig 3

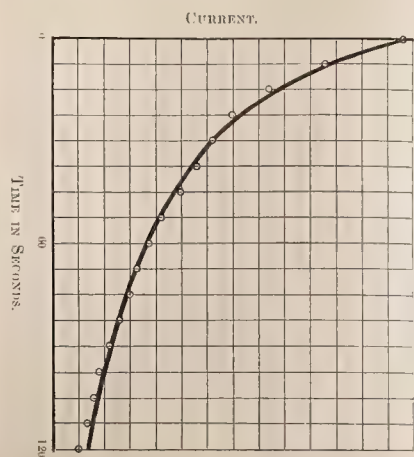
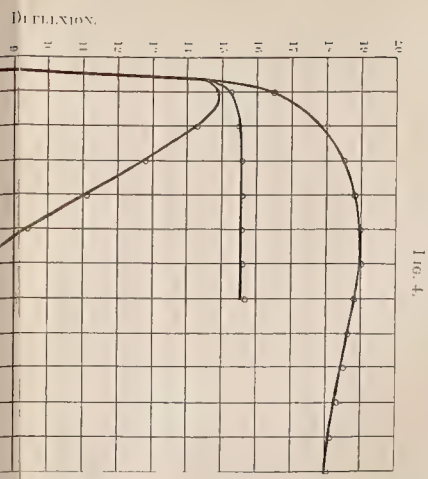
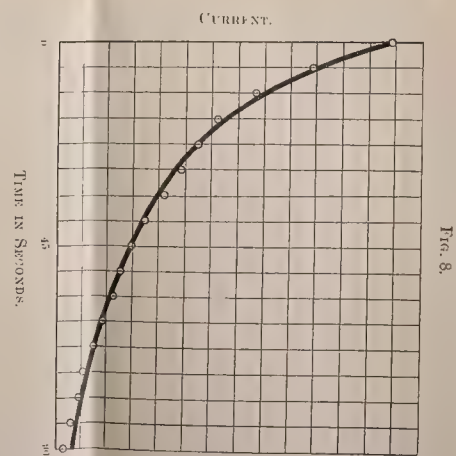
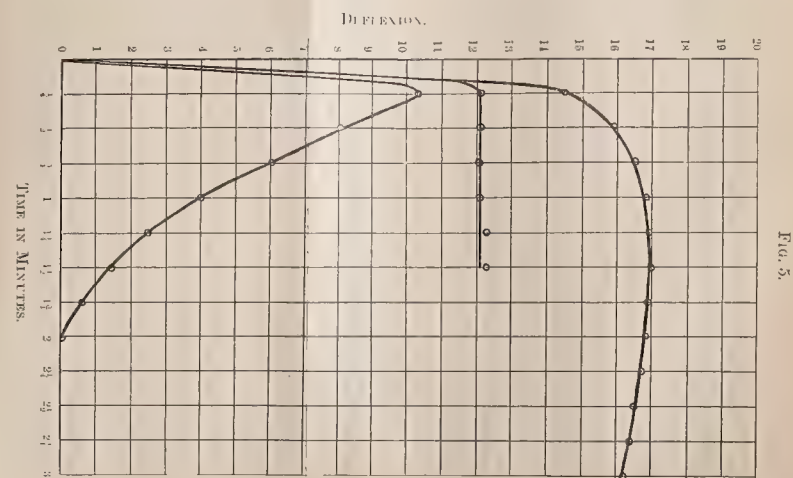
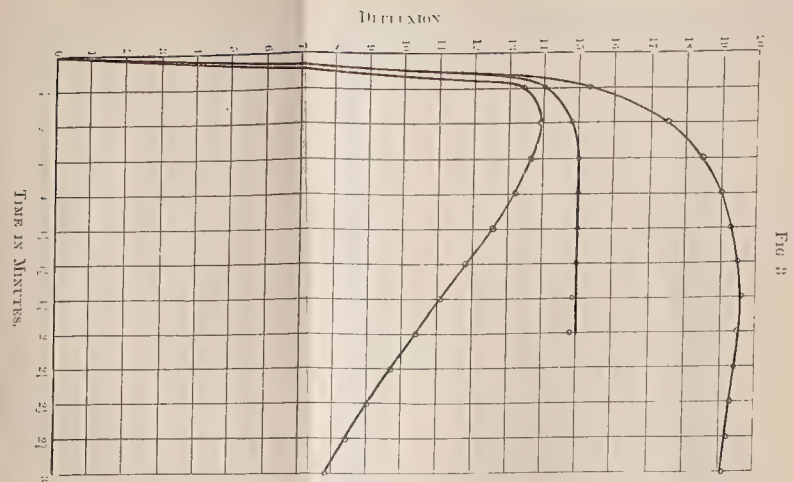


Fig. 7.

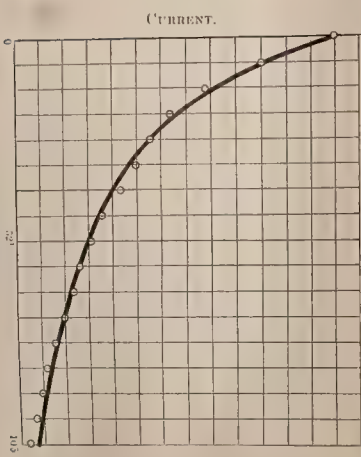
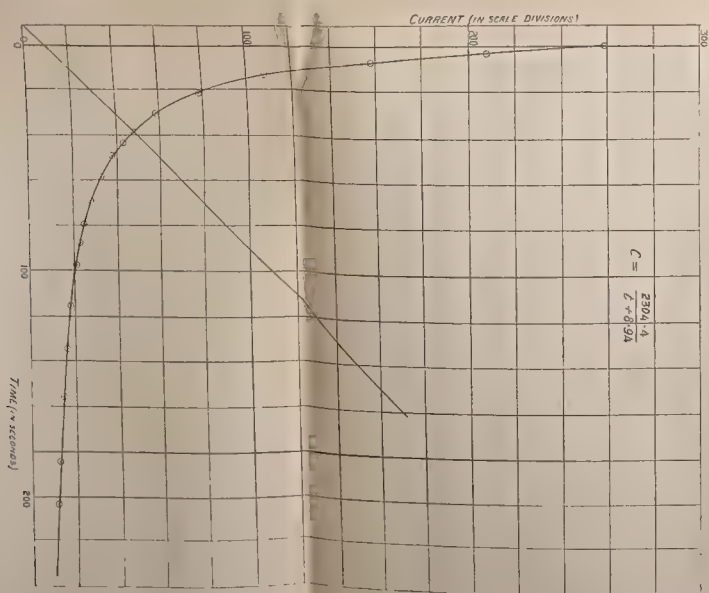
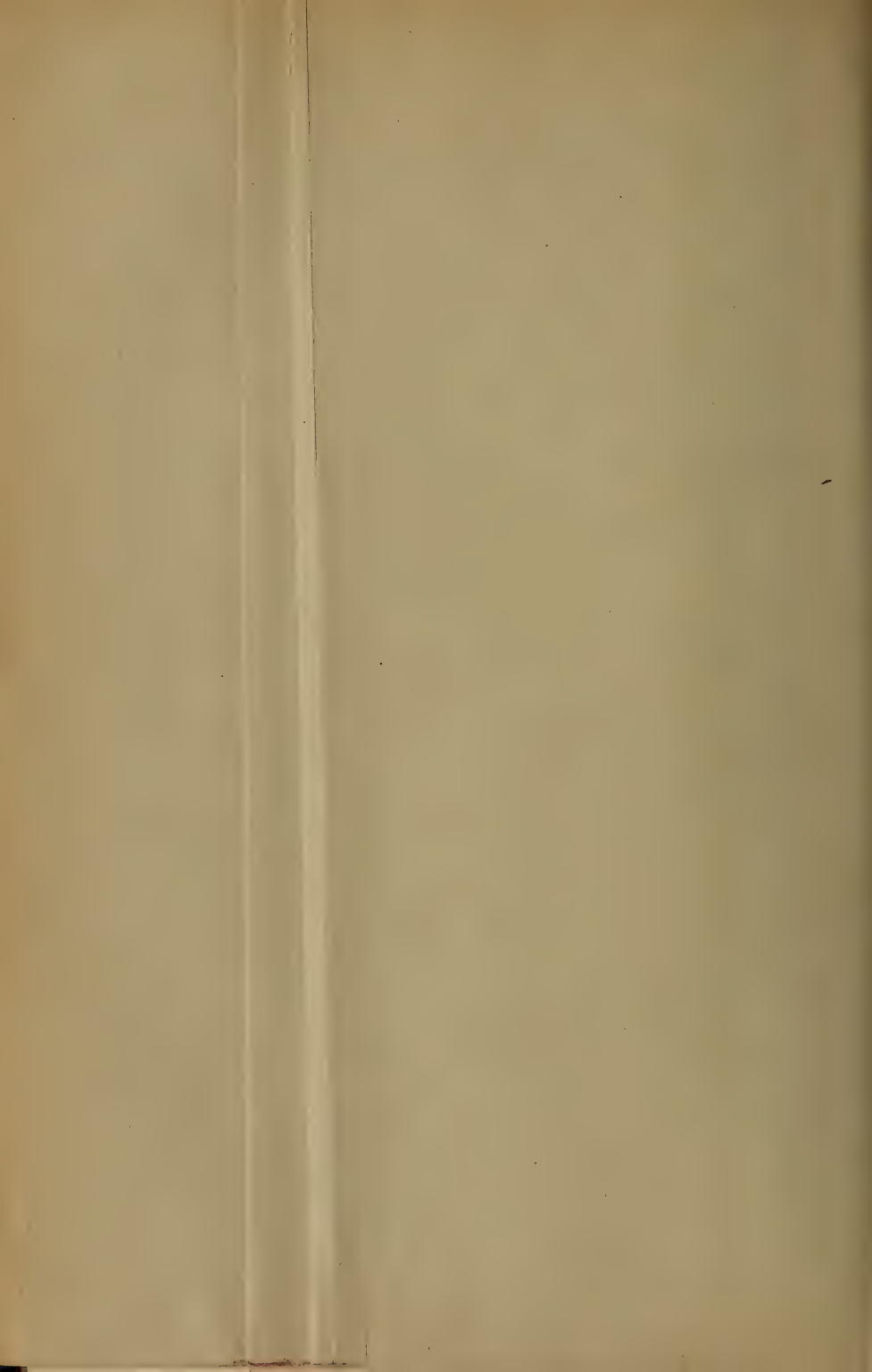


Fig. 9.





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AND
JOURNAL OF SCIENCE.

[SIXTH SERIES.]

JUNE 1907.



LIX. *On the Radiation from Moving Systems of Electrons, and on the Spectrum of Canal Rays.* By G. A. SCHOTT, B.A., B.Sc., University College of Wales, Aberystwyth*.

§ 1. **I**N a previous communication † I have examined that radiation from a stationary ring of electrons, which are in orbital motion in a suitable controlling field, and also from a system composed of a large number of similar independent rings. A ring of n electrons emits $6n$ waves of frequencies of the same order as those of spectrum lines; but of these waves at most 18 are of sufficient intensity, under ordinary conditions, to produce observable lines. Thus we cannot, by means of a single ring, account for a complete spectrum series or band; but we may be able to do so by means of a system of rings coupled together by their mutual actions. For this purpose the study of the single ring is necessary.

§ 2. In a series of important experiments J. Stark ‡ has shown that the canal-ray spectrum contains series lines, each line being split up into a narrow undisplaced line, due to nearly stationary ions, and a broad displaced line due to ions moving with velocities ranging from 10^7 to $5 \cdot 10^7$ cm./sec. The latter is separated by a dark space from the former, and has a

* Communicated by the Author.

† Schott, *Phil. Mag.* [6] vol. xlii. p. 189.

‡ J. Stark, '*Nature*,' vol. lxxiii. pp. 78, 389, 533; *Ann. der Physik*, vol. ccxxvi. p. 401.

maximum of intensity, which increases with the velocity, and that more rapidly for high-frequency lines. Mainly because of the dark space, he considers that the energy radiated in the displaced line is transformed from kinetic energy of translation, not by the agency of collisions, but by the direct action of radiation pressure. The energy radiated in the undisplaced line is attributable directly to collisions. He formulates the following question:—

“Given a system of electrons in orbital motion, so arranged that their radiation into the external medium is mutually compensated” (presumably by interference); “does the compensation subsist when the system is given a motion of translation through the æther?”

§ 3. The following theoretical investigation of §§ 4–24 supplies the answer to this question. For a uniform translation it is in the affirmative, in so far as no periodic waves are emitted in consequence of the motion; for non-uniform motions periodic waves are emitted when the changes in velocity and orientation of the system are rapid enough. It is further shown that the radiant energy cannot be due to radiation pressure, since the work done by that pressure is but a very small fraction of the energy radiated.

The rest of the paper is devoted to a brief discussion of Stark’s experimental results in the light of the theoretical conclusions arrived at in the first portion of the paper. [*vide* Note I.]

§ 4. Lorentz* has given a general method of deducing the electromagnetic field due to a system of electric charges, in motion relative to a set of axes, which are themselves moving with a uniform velocity less than that of light, from the field due to a second system in motion relative to a fixed set of axes.

Let $\mathbf{E}$ be the vector which represents the electric force, (X, Y, Z) its rectangular components in the actual system (Σ) , $\mathbf{H}$, (L, M, N) the magnetic force, $\mathbf{V}$, (u, v, w) the relative velocity, ρ the electric volume density.

Let accented letters denote the same quantities in the system of reference (Σ') .

Let U be the velocity of the moving axes, C the velocity of light, $\kappa = \sqrt{1 - U^2/C^2}$; and let Ox be taken in the direction of U .

The relations between the coordinates, time and velocities of the two systems are as follows:—

(x, y, z) are coordinates of a point relative to axes fixed in the æther, supposed at rest in Lorentz’s theory.

* Abrahamet Langévin, *Ions, Electrons, Corpuscles*, vol. i. p. 477.

Relative to the moving axes its coordinates at time t are $(x - Ut, y, z)$.

Write

$$x - Ut = \kappa x', \quad y = y', \quad z = z', \quad t = \frac{t'}{\kappa} + \frac{Ux'}{\kappa C^2};$$

and

$$u = \kappa^2 u', \quad v = \kappa v', \quad w = \kappa w'.$$

[*vide* Note II.]

For a point fixed relative to the æther we have, since $t' = \frac{t}{\kappa} - \frac{Ux}{\kappa C^2}$, $dt = \kappa dt'$; for a point fixed relative to the moving origin we have $dt = \frac{dt'}{\kappa}$; for in the first case (x, y, z) and in the second case (x', y', z') are unchanged.

We shall choose the system (Σ') so that corresponding elements of volume, and therefore also corresponding electrons, have equal charges: this assumption makes

$$\rho' = \kappa \rho, \quad \rho' dx' dy' dz' = \rho dx dy dz, \quad e' = e.$$

§ 5. The following relations hold between the electromagnetic fields of the two systems at corresponding places and times:—

$$\left. \begin{aligned} X = X', \quad \kappa Y = Y' + \frac{U}{C} N', \quad \kappa Z = Z' - \frac{U}{C} M' \\ L = L', \quad \kappa M = M' - \frac{U}{C} Z', \quad \kappa N = N' + \frac{U}{C} Y'. \end{aligned} \right\} \dots (1)$$

$\mathbf{E}'$, $\mathbf{H}'$ are given in terms of a scalar potential ϕ' and a vector potential $\mathbf{A}'$, (F' , G' , H') by the equations

$$\left. \begin{aligned} \mathbf{E}' = -\frac{1}{C} \frac{\partial \mathbf{A}'}{\partial t'} - \nabla \phi' + \frac{U}{C} \nabla F', \quad \mathbf{H}' = \text{curl } \mathbf{A}' \\ \text{div. } \mathbf{E}' = 4\pi \left(1 - \frac{Uu'}{C^2}\right) \rho', \quad \text{div. } \mathbf{H}' = 0, \end{aligned} \right\} \dots (2)$$

while ϕ' , $\mathbf{A}'$ are given by the equations

$$\left. \begin{aligned} \nabla^2 \phi' - \frac{1}{C^2} \frac{\partial^2 \phi'}{\partial t'^2} + 4\pi \rho' &= 0, \\ \nabla^2 \mathbf{A}' - \frac{1}{C^2} \frac{\partial^2 \mathbf{A}'}{\partial t'^2} + \frac{4\pi}{C} \rho' \mathbf{V}' &= 0, \\ \frac{1}{C^2} \frac{\partial \phi'}{\partial t'} - \frac{U}{C^2} \frac{\partial F'}{\partial t'} + \text{div. } \mathbf{A}' &= 0. \end{aligned} \right\} \dots (3)$$

All the operations are supposed to be performed in the system Σ' .

The values of ϕ' , $\mathbf{A}'$ are given by the usual integrals

$$\left. \begin{aligned} \phi' &= \iiint [\rho'] \frac{d\xi' d\eta' d\zeta'}{r'}, \\ \mathbf{A}' &= \iiint \left[\frac{1}{C} \rho' \mathbf{V}' \right] \frac{d\xi' d\eta' d\zeta'}{r'}, \end{aligned} \right\} \dots \dots (4)$$

where r' is the distance from the point (x', y', z') , at which the field is required for time t' , to the element $d\xi' d\eta' d\zeta'$; the square bracket denotes that the quantity inside is to be taken for the element $d\xi' d\eta' d\zeta'$ at the time $t' - \frac{r'}{C}$, which is the time of emission of the disturbance due to that element.

It should be noticed that the field is the same as if the system (Σ') were at rest as a whole, except that $\mathbf{E}'$ involves an additional term $\frac{U}{C} \nabla F'$. This term only vanishes when all the charges are at rest relative to one another. In our case it is present, because we are dealing with systems of electrons in orbital motion.

§ 6. We shall only require the field at a distance from the system of electrons large compared with its largest linear dimension. ϕ' , $\mathbf{A}'$ are the sums of periodic expressions of the form $\frac{f(t' - r'/C)}{r'}$, where r' is the distance measured from any convenient origin in the system. If we neglect quantities of relative order $\frac{\lambda}{r'}$, where λ is the wave-length, we may write

$$\frac{\partial}{\partial r'} = -\frac{1}{C} \frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial x'} = \frac{x'}{r'} \frac{\partial}{\partial r'} = -\frac{l}{C} \frac{\partial}{\partial t'}, \dots$$

where (l, m, n) are the direction cosines of r' . Hence

$$\text{div. } \mathbf{A}' = -\frac{1}{C} \frac{\partial}{\partial t'} (lF' + mG' + nH').$$

We shall find it convenient to transform to polar coordinates (r', θ, ϕ) in the system (Σ'). For polar axis choose any convenient line fixed in this system, and for initial plane the plane through this axis and Ox . Let the direction cosines of the axis be (a, b, c) , those of the radius vector $(r',$

the meridian (θ) and parallel (ϕ), (l, m, n), (l', m', n'), and (l'', m'', n''), these forming a right-handed system.

Let the components of $\mathbf{E}'$, $\mathbf{H}'$ in these directions be (R', T', P'), (P', Θ', Φ') respectively. In virtue of the third of equations (3) we find for a distant point

$$\left. \begin{aligned} R' &= P' = 0, \\ T' &= \Phi' = -\frac{1}{C} \frac{\partial}{\partial t'} (l' F' + m' G' + n' H'), \\ P' &= -\Theta' = -\frac{1}{C} \frac{\partial}{\partial t'} (l'' F' + m'' G' + n'' H'). \end{aligned} \right\} \quad (5)$$

These equations show that at a great distance from the system the electric and magnetic forces are perpendicular to the radius vector and to each other. Since equations (4) do not explicitly involve U , the distant field of (Σ') is calculated exactly in the same way as if U were zero. It differs from that of the system (Σ), when at rest, in so far as the motion changes the relative configuration and velocities of the electrons of the system from those existing in the system at rest.

§ 7. The electric and magnetic forces in the system (Σ) are given in terms of the forces T', P' by the following equations, which follow at once from (1).

$$\left. \begin{aligned} &= l' T' + l'' P', & L &= l'' T' - l' P'. \\ &= \left(m' + \frac{U}{C} n''\right) T' + \left(m'' - \frac{U}{C} n'\right) P', & \kappa M &= \left(m'' - \frac{U}{C} n'\right) T' - \left(m' + \frac{U}{C} n''\right) P'. \\ &= \left(n' - \frac{U}{C} m''\right) T' + \left(n'' + \frac{U}{C} m'\right) P', & \kappa N &= \left(n'' + \frac{U}{C} m'\right) T' - \left(n' - \frac{U}{C} m''\right) P'. \end{aligned} \right\} \quad (6)$$

Let the Poynting vector, $\frac{C}{4\pi} [\mathbf{E}\mathbf{H}]$, be denoted by $\mathbf{P}$, (P, Q, R).

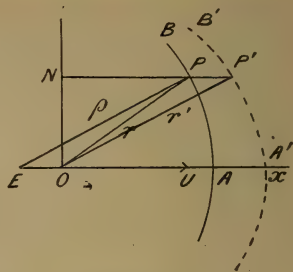
Equations (6) give

$$\left. \begin{aligned} P &= \frac{C}{4\pi\kappa^2} \left(l + \frac{U}{C}\right) \left(1 + l \frac{U}{C}\right) (T'^2 + P'^2), \\ Q &= \frac{C}{4\pi\kappa} m \left(1 + l \frac{U}{C}\right) (T'^2 + P'^2), \\ R &= \frac{C}{4\pi\kappa} n \left(1 + l \frac{U}{C}\right) (T'^2 + P'^2). \end{aligned} \right\} \quad (7)$$

The equations (6), (7) have a simple geometric interpretation.

Let P be the point (x, y, z) , O the position of the system at time t , the distance OP being supposed large compared with the largest dimension of the system and with the wave-length.

The disturbance (6), (7), which exists at P at time t , was emitted at a previous time τ , when the system was at E , such that



$$EP = C(t - \tau), \quad EO = U(t - \tau) = \frac{U}{C} EP.$$

Let $EP = \rho$, and let its direction cosines be (λ, μ, ν) .

Let P' be the point (x', y', z') , so that $NP' = \kappa NP$; then $OP' = r'$, and (l, m, n) are the direction cosines of OP' .

We easily find

$$\lambda = \frac{l + U/C}{1 + lU/C}, \quad \mu = \frac{\kappa m}{1 + lU/C}, \quad \nu = \frac{\kappa n}{1 + lU/C}, \quad \rho = (1 + lU/C) \frac{r'}{\kappa}.$$

Thus equations (7) express the fact that P is along the ray EP , and that its magnitude is

$$\frac{C}{4\pi\kappa^2} \left(1 + l \frac{U}{C}\right)^2 (T'^2 + P'^2).$$

Equations (6) give

$$XL + YM + ZN = \lambda X + \mu Y + \nu Z = \lambda L + \mu M + \nu N = 0,$$

$$X^2 + Y^2 + Z^2 = L^2 + M^2 + N^2 = \frac{1}{\kappa^2} \left(1 + l \frac{U}{C}\right)^2 (T'^2 + P'^2).$$

These express the facts that the electric and magnetic forces are perpendicular to each other and to the ray EP , and that they are equal to each other.

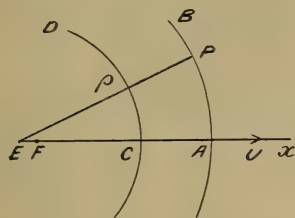
The wave surface, which at time t passes through P , is the sphere whose centre is E and radius EP .

§ 8. We require to solve three problems:

- (1) To find the rate at which the system of electrons loses energy by radiation.
- (2) To find the radiation pressure on it, that is, the reaction due to the æther on account of radiation.
- (3) To find the radiant energy received per second by a fixed surface exposed to the radiation.

It might be supposed that (1) was simply (3) treated for a surface completely enclosing the system; but this is not so,

because the energy of the æther inside this surface is continually changing on account of the motion of the system.



E is as before the position of the system at time τ , APB the wave emitted at E, for the time t .

F is the position of the system at time $\tau + d\tau$, CD the wave emitted at F, for the time t .

The energy radiated by the system during the interval $d\tau$ is the energy contained between the two waves AB, CD; since they are not concentric, this energy requires a different interval of time, dt , to pass each point P of AB. Since $t = \tau + \frac{\rho}{U}$, we have

$$dt = d\tau \left(1 + \frac{\rho}{C} \right) = d\tau \left(1 - \frac{U}{C} \cos AEP \right).$$

Now $\cos AEP = \lambda = \frac{l + U/C}{1 + lU/C}$; hence $dt = \frac{\kappa^2 d\tau}{1 + lU/C}$.

Let dS be an element of area of the wave AB at P. The energy which crosses dS during time dt is, by (7),

$$(\lambda P + \mu Q + \nu R) dS dt = \frac{Cd\tau}{4\pi} (1 + lU/C) (T'^2 + P'^2) dS.$$

Hence the rate at which the system radiates energy is

$$R = \frac{C}{4\pi} \iint (1 + lU/C) (T'^2 + P'^2) dS.$$

Since T' , P' are given for the system (Σ') it is convenient to transform the integral to this system. Let dS' correspond to dS , and let (λ', μ', ν') be its direction cosines.

(Σ') being derived from (Σ) by stretching every line parallel to Ox in the ratio $\kappa : 1$, every area parallel to the yz plane is unaltered, while every area parallel to Ox is increased in the ratio $\kappa : 1$. Hence we have

$$\lambda dS = \lambda' dS', \quad \mu dS = \kappa \mu' dS', \quad \nu dS = \kappa \nu' dS'.$$

These equations give at once

$$\lambda = \frac{l + U/C}{\sqrt{1 + 2lU/C + U^2/C^2}}, \quad \mu' = \frac{m}{\sqrt{1 + 2lU/C + U^2/C^2}}, \quad \nu' = \frac{n}{\sqrt{1 + 2lU/C + U^2/C^2}},$$

$$dS = \frac{1 + lU/C}{\sqrt{1 + 2lU/C + U^2/C^2}} dS'.$$

If α be the angle made by the normal to dS' with r' , we have

$$\cos \alpha = l\lambda' + m\mu' + n\nu' = \frac{1 + lU/C}{\sqrt{1 + 2lU/C + U^2/C^2}}.$$

If dS' subtend the solid angle $d\omega$ at O we have

$$dS' = \sec \alpha \cdot r'^2 d\omega.$$

Hence

$$dS = \cos \alpha dS' = r'^2 d\omega,$$

$$R = \frac{C}{4\pi} \iint (1 + lU/C)(T'^2 + P'^2) r'^2 d\omega. \quad \dots \quad (8)$$

This equation expresses the rate at which the moving system (Σ) loses energy by radiation at time τ , in terms of the field in the system of reference (Σ') at the time t' , corresponding to the time $t = \tau + \frac{r}{C}$ in the moving system. In this respect it differs from the expressions given by Abraham*, which express it in terms of the state of the moving system at the time τ .

§ 9. We shall now find the reaction of the æther on the moving system due to radiation.

According to the theory developed by Lorentz and by Abraham† the electromagnetic force on all the charges contained within the fixed surface S is the sum of (1) a surface integral

$$\frac{1}{8\pi} \iint [2\mathbf{E}(\lambda X + \mu Y + \nu Z) - \mathbf{U}n \cdot (X^2 + Y^2 + Z^2) + 2\mathbf{H}(\lambda L + \mu M + \nu N) - \mathbf{U}n \cdot (L^2 + M^2 + N^2)] dS,$$

where, as before, (λ, μ, ν) are the direction cosines of the outward normal to the element dS , and $\mathbf{U}n$ is a unit vector along that normal; and

(2) a volume integral

$$-\frac{1}{C^2} \iiint \dot{\mathbf{P}} \cdot d\mathbf{x} d\mathbf{y} d\mathbf{z},$$

extended over the whole space enclosed by the whole surface S.

The quantity $\mathbf{P}/C^2$ is the electromagnetic momentum per unit volume; the integral (2) is the rate of decrease of the

* *Loc. cit.* p. 278.

† Lorentz "Elektronentheorie," *Mathematische Encyclopädie*, vol. xiv. p. 161.

(2) Element dS along a transversal through E and perpendicular to it.

Let it be at Q. We have

$$\lambda=0, \quad l=-\frac{U}{C}, \quad \sqrt{m^2+n^2}=\kappa, \quad r'=\rho/\kappa.$$

The radiation is

$$S_2 = \frac{CdS}{4\pi\rho^2} \left(1 - \frac{U^2}{C^2}\right)^2 (T'^2 + P'^2) r'^2. \dots \dots (11)$$

The energy received per second by the whole of a fixed surface, which passes through P and at time t coincides with the wave-surface through P, is

$$S = \frac{C}{4\pi\kappa^2} \iint \left(1 + l\frac{U}{C}\right)^2 (T'^2 + P'^2) r'^2 d\omega. \dots (12)$$

It is instructive to compare this with (8).

When $U=0$, *i. e.* when the system (Σ) is at rest and identical with (Σ'), both expressions are identical and give the radiation from the system (Σ'): it is

$$R' = \frac{C}{4\pi} \iint (T'^2 + P'^2) r'^2 d\omega. \dots \dots (13)$$

R and **S** differ from **R'** by the presence of the factors $1 + l\frac{U}{C}$, $\left(1 + l\frac{U}{C}\right)^2$ respectively.

§ 11. It is necessary for further progress to make some assumptions as to the structure and orientation of the system (Σ), or of the system (Σ'). For our purpose it is little use considering systems other than those which will account for spectra; such systems satisfy the following conditions:—(1) they emit waves of determinate frequency and of intensity corresponding to that of a spectrum-line; (2) they are permanent or nearly so, so that their structure remains practically unchanged during long periods of time; and (3) they are stable, radioactive systems excepted. I have shown* that a circle of equidistant electrons in uniform rotation satisfies conditions (1) and (2); it follows from the example of J. J. Thomson's well-known model of the atom that (3) can be satisfied provided a suitable controlling field be chosen. But unfortunately a single ring does not emit a sufficient number of waves strong enough to account for all the lines of a spectrum. Therefore we must suppose our system built up of a number of rings.

* Schott, *loc. cit.*

It is probable that a group of electrons in periodic motion in any closed curve whatever can be permanent by itself, provided its electrons pass in succession through every point of the curve at regular and equal intervals; but it by no means follows that a system built up of such groups is permanent. Any two groups, on account of their permanent motion, disturb each other and emit waves, whose energy is a drain on the energy of the groups; hence the system is not permanent. But the drain of energy, and consequent change in structure, can proceed extremely slowly, provided all the most powerful waves due to perturbations are destroyed by interference between the several electrons of each group. This occurs for circles of equidistant electrons, provided they have a common axis of rotation and are not too close together, certain combinations of circles being excluded; but it does not occur for groups other than circles.

Therefore we make the following assumption (A):—The system (Σ) when at rest consists of a number of circular rings of equidistant electrons in rotation about a common axis, in a controlling field of such a kind as to make the system stable. The number of electrons, the angular velocity and radius of each ring, and the mutual distances of all pairs of rings, are such that the system is quasipermanent.

§ 12. At a great distance from the quasipermanent system at rest the electromagnetic forces T' , P' are the sums of terms

of the type $\frac{A}{r} \cos \left\{ p \left(t - \frac{r}{C} \right) - m\phi \right\}$, where p , m are constants depending on the type of vibration producing the forces. The polar axis can be chosen so that the amplitude A is a function of θ alone and at most changes sign when we put $\pi - \theta$ in place of θ .

We shall make a second assumption (B):—

The system of reference (Σ') has the property just mentioned.

In order to explain the negative Michelson result Lorentz, as is well known, makes the assumption that the system (Σ') is identical with the system (Σ) when at rest. In this case assumption (B) at once follows.

We can now average the expressions (8)..., (1) for the time, (2) for the coordinate ϕ .

(1) Let us average for an interval of time containing a large number of periods.

During this interval (Σ) moves, and the relative coordinates of a fixed point change by amounts of order $U\lambda/Cr'$. On account of the smallness of λ/r' these changes may be

neglected. Denote averages taken in this way by a bar over the symbol of the quantity averaged, *e. g.* $\bar{\mathbf{R}}$. During this process the coordinate ϕ disappears at the same time as t from the expressions for T'^2 , P'^2 , in virtue of assumption (B).

(2) Put $d\omega = \sin \theta d\theta d\phi$; note that

$$l = a \cos \theta + \sqrt{1-a^2} \sin \theta \cos \phi,$$

$$m = b \cos \theta - \frac{ab}{\sqrt{1-a^2}} \sin \theta \cos \phi + \frac{c}{\sqrt{1-a^2}} \sin \theta \sin \phi,$$

$$n = c \cos \theta - \frac{ac}{\sqrt{1-a^2}} \sin \theta \cos \phi - \frac{b}{\sqrt{1-a^2}} \sin \theta \sin \phi.$$

Integrate from $\phi=0$ to 2π and from $\theta=0$ to π .

(8), (13) give

$$\bar{\mathbf{R}} = \bar{\mathbf{R}}' = \frac{1}{2}C \int_0^\pi (\bar{T}'^2 + \bar{P}'^2)^{r'/2} \sin \theta d\theta. \quad . \quad . \quad (14)$$

For shortness write

$$\bar{\mathbf{R}}_i = \frac{1}{2}C \int_0^\pi (\bar{T}'^2 + \bar{P}'^2)^{r'/2} \cdot P_i(\cos \theta) \sin \theta d\theta \quad (15)$$

with the usual notation of zonal harmonics. We note that $\bar{\mathbf{R}}_i$ is identically zero for odd values of i , because $\bar{T}'^2$, $\bar{P}'^2$ do not change when $\pi-\theta$ is put for θ .

(9), (12) give

$$\left. \begin{aligned} \bar{X} &= -\frac{4U}{3(C^2-U^2)} \left\{ \bar{\mathbf{R}} + \frac{1}{2}P_2(a) \cdot \bar{\mathbf{R}}_2 \right\}, \\ \bar{Y} &= -\frac{U}{C\sqrt{C^2-U^2}} ab \cdot \bar{\mathbf{R}}_2, \\ \bar{Z} &= -\frac{U}{C\sqrt{C^2-U^2}} ac \cdot \bar{\mathbf{R}}_2. \end{aligned} \right\} \quad . \quad . \quad (16)$$

$$\bar{\mathbf{S}} = \frac{1 + \frac{1}{3}U^2/C^2}{1 - U^2/C^2} \bar{\mathbf{R}} + \frac{2U^2}{3(C^2-U^2)} P_2(a) \cdot \bar{\mathbf{R}}_2. \quad . \quad . \quad (17)$$

The radiation pressure $\mathbf{F}$ does work on the moving system (Σ) at the rate $\bar{\mathbf{W}} = -U\bar{X}$; hence

$$\bar{\mathbf{W}} = \frac{4U^2}{3(C^2-U^2)} \left\{ \bar{\mathbf{R}} + \frac{1}{2}P_2(a) \cdot \bar{\mathbf{R}}_2 \right\}. \quad . \quad . \quad (18)$$

With respect to equations (10), (11) of § 10 the integration

§ 13. The equations (14) . . . (22) give the solution of the problems relating to the radiation from a single system moving with uniform velocity U , on the assumptions (A) and (B), the values of $\bar{T}'^2$, $\bar{P}'^2$ being those of the system of reference (Σ') for the corresponding type of vibration. Since the process of averaging for t gets rid of products of forces due to different types of vibration, the total radiation from the system is simply the sum of a series of terms such as those considered, and each of these may be treated separately.

When the external field, in which the system is moving, contains no terms of the period of the vibration considered, the forces T' , P' are those due to the system alone. But when terms of the same period are present resonance effects are produced; in this case the values of X . . . in (6) of § 7 must be supplemented by the addition of terms X_0 . . . due to the external field; in addition T' , P' include terms produced from X_0 . . . by resonance, as well as terms independent of the external field. The values of P , Q , R given by (7) must be correspondingly supplemented.

When we wish to represent the action of a quantity of radiating gas, or of a bundle of canal rays, by means of a group of systems of electrons such as the one considered above, resonance effects necessarily arise in consequence of the interaction of the several systems. These effects are particularly large when two systems happen to approach very near to one another, that is, at collisions. Since we have supposed our system to be moving with uniform velocity, our analysis can only apply to the interval between successive collisions of the system, during which its distance from other systems is so large that resonance effects are comparatively unimportant. In extending our results to the case of a group of systems we shall therefore make a third assumption (C).

In the groups of systems to be considered, the density, *i. e.* the number of systems per unit volume, is so small, and the mean free path of a system so large, that the interval between successive collisions is a large multiple of the duration of a collision.

This is equivalent to assuming that at any one instant the number of systems in collision is a small fraction of the whole number.

It must be remembered that for ions the radius of appreciable action is very much larger than for a neutral system, such as an atom, and therefore the duration of a collision must be reckoned as longer in proportion. On the

other hand, in a radiating gas the number of ions is only a small fraction of the number of atoms present, and on this account collisions between ions are less frequent than between atoms. For this reason assumption (C) does not seem less probable for ions than for atoms or molecules.

We shall now extend the results of § 12 to the case of a group of a large number of similar systems, which we may treat as independent of each other, in accordance with assumption (C); in other words, we shall neglect the radiation during collision in comparison with the radiation during the intervals between collisions. The only rôle of collisions is assumed to be to produce the disturbances, wholly or in part, to which the radiation is due. We wish to decide how much of the radiation is due to the collisions, how much to radiation pressure.

§ 14. Let us consider the systems contained in any small element of volume. Just as in the kinetic theory of gases, so in our case we suppose this element to contain a very large number of systems, although its linear dimensions are small compared with distances usually employed in experimental work. We must first sum the expressions (14)...(22) for all systems having velocities between U and $U + dU$, but differently orientated; afterwards we must sum for all velocities and directions of velocities.

In the first place, we cannot generally assume that all directions of polar axes (a, b, c) are equally probable, because the direction of the velocity U is a singular direction. In fact, in the case of a bundle of canal rays, since the bundle as a whole is equivalent to an electric current, there is a magnetic field inside it, which causes all the systems of electrons to turn their axes towards the lines of magnetic force, that is, away from the direction of motion. Let us assume that the chance that the axis (a, b, c) of our system may lie within the elementary solid angle $d\omega$ is $F d\omega / 4\pi$. The distribution function F depends on (a, b, c) and on U ; write $\tan \chi = c/b$. We may expand F in a series of spherical surface harmonics of the form

$$F = \sum_{i=0}^{\infty} \sum_{s=0}^s (A_{is} \cos s\chi + B_{is} \sin s\chi) (1 - a^2)^{\frac{s}{2}} \cdot P_i^{(s)}(a), \quad (23)$$

where $P_i^{(s)}(a)$ stands for $d^s P_i(a) / da^s$.

In the particular case mentioned, of the canal rays, the part of F due to the magnetic force consists of harmonics of the first order. Other harmonics may very well occur in consequence of collisions.

Multiply by $\frac{1}{2}P_i(\cos \theta) \sin \theta d\theta$ and integrate from $\theta=0$ to $\theta=\pi$; we find

$$C_i = \sum_{s=0}^{s=i} (A_{is} \cos s\Omega + B_{is} \sin s\Omega) \kappa^s P_i^{(s)}(-U/C), \quad (29)$$

where, as before, $\kappa = \sqrt{1-U^2/C^2}$. Again write

$$\frac{1}{2\pi} \int_0^{2\pi} F \cos 2\psi d\psi = \sum_{i=2}^{i=\infty} D_i \sin^2 \theta P_i''(\cos \theta).$$

We find

$$D_i = \sum_{s=0}^{s=i} (A_{is} \cos s\Omega + B_{is} \sin s\Omega) \left\{ \frac{i-2}{i+2} + s \left[\frac{1+U^2/C^2}{4} \left\{ \frac{\kappa^{s-2} P_i^{(s-2)}(-U/C)}{i+s-4} \right. \right. \right. \\ \left. \left. + \frac{\kappa^{s+2} P_i^{(s+2)}(-U/C)}{i+s} \right\} + \kappa U/C \left\{ \frac{\kappa^{s-1} P_i^{(s-1)}(-U/C)}{i+s-3} \right. \right. \\ \left. \left. - \frac{\kappa^{s+1} P_i^{(s+1)}(-U/C)}{i+s-1} \right\} + \frac{3}{2} (1-U^2/C^2) \frac{\kappa^s P_i^{(s)}(-U/C)}{i+s-2} \right] \right\}. \quad (30)$$

Also write

$$\bar{Q}_i = \frac{1}{2} C \int_0^\pi (\bar{T}'^2 - \bar{P}'^2) r'^2 \cdot P_i''(\cos \theta) \sin^3 \theta d\theta. \quad (31)$$

We find

$$\bar{S}_2 = \frac{dS}{4\pi\rho^2} \left(1 - \frac{U^2}{C^2}\right)^2 \cdot \left\{ \bar{R} + C_2 \cdot \bar{R}_2 + C_4 \cdot \bar{R}_4 + \dots \right\}. \quad (32)$$

$$\bar{S}_2' = \frac{dS}{4\pi\rho^2} \left(1 - \frac{U^2}{C^2}\right)^2 \cdot \left\{ D_2 \cdot \bar{Q}_2 + D_4 \cdot \bar{Q}_4 + \dots \right\}. \quad (33)$$

We notice that $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_2'$ involve inequalities of even order only; in general the existence of such inequalities produces polarization of the normally emitted light, and *vice versa* polarization may be regarded as a test of the existence of such inequalities.

In the particular case when the distribution is symmetrical about the direction of motion, and thus does not involve Ω , the only coefficients involved in $\bar{S}_2$, $\bar{S}_2'$, are A_{20} , A_{40} ..., just as in $\bar{S}_1$.

§ 16. We must now average for all values and directions of the velocity U . Two cases are of special interest:

- (1) All the systems move in parallel lines.
- (2) The velocities of the systems are distributed equally in all directions.

(1) In this case the group of systems may serve as a model of a bundle of canal rays. The model is not a perfect one; for in the canal-ray bundle, owing to collisions, the velocities of the particles cannot all be in the same direction, though

the direction of transmission of the bundle is doubtless the preponderating direction of motion of its individual particles. Further, the velocity of a particle is not absolutely constant, even in the interval between collisions, on account of the retardation due to radiation pressure ; however, (24) shows that this retardation is small on account of the smallness of U/C , which is of order $1/1000$; therefore we shall neglect the retardation due to radiation pressure. Lastly, the velocity is not the same, on the average, throughout the bundle. On account of collisions it diminishes considerably as we pass along the bundle in the direction of transmission. We shall for simplicity treat the velocity as uniform on the average ; for normal emission this leads to no error, for the radiation practically all comes from the same section of the tube ; for emission parallel to the rays, there is an error, but it cannot be very large, for on account of absorption most of the radiation comes from the end section of the tube.

Let $f(U)dU$ be the chance that in the section of the tube considered the velocity of a system lies between U and $U+dU$. Multiply each of the equations (24)...(33) by $f(U)dU$ and integrate from $U=0$ to $U=\infty$, remembering that $A_{20}, \dots$ are functions of U . The functions $\bar{R}, \bar{R}_2, \dots \bar{Q}_2, \dots$ must also be treated as functions of U , for we cannot assume that the disturbance of a system is the same when in motion as it is when at rest. We may without appreciable error neglect U/C ; therefore we may treat the structure of the system as unaltered by the motion, but the amplitude of the disturbance may be appreciably altered. We get

$$\left. \begin{aligned} \bar{X} &= -\frac{4}{3C^2} \int \bar{R} U f(U) dU - \frac{2}{15C^2} \int \bar{R}_2 A_{20} U f(U) dU, \\ \bar{Y} &= -\frac{1}{5C^2} \int \bar{R}_2 \cdot A_{21} \cdot U f(U) dU, \quad . \quad . \quad . \quad . \quad . \\ \bar{Z} &= -\frac{1}{5C^2} \int \bar{R}_2 \cdot B_{21} \cdot U f(U) dU, \quad . \quad . \quad . \quad . \quad . \end{aligned} \right\} \quad (34)$$

$$\bar{S} = \int \bar{R} f(U) dU + \frac{2}{15C^2} \int \bar{R}_2 \cdot A_{20} U^2 f(U) dU \quad . \quad . \quad . \quad (35)$$

$$\bar{W} = \frac{4}{3C^2} \int \bar{R} U^2 f(U) dU + \frac{2}{15C^2} \int \bar{R}_2 \cdot A_{20} \cdot U^2 f(U) dU. \quad (36)$$

$$\bar{S}_1 = \frac{dS}{4\pi\rho^2} \int \{ \bar{R} f(U) dU + \bar{R}_2 A_{20} f(U) dU + \dots \} \quad . \quad . \quad (37)$$

$$\bar{S}_2 = \frac{dS}{4\pi\rho^2} \int \{ \bar{R} f(U) dU + \bar{R}_2 C_2 f(U) dU + \dots \} \quad . \quad . \quad (38)$$

$$\bar{S}_2' = \frac{dS}{4\pi\rho^2} \int \{ \bar{Q}_2 \cdot D_2 f(U) dU + \dots \} \quad . \quad . \quad . \quad . \quad (39)$$

§ 17. (2) In this case the group of systems serves as a model of a radiating gas, *e. g.* a hot flame, the negative glow of a vacuum-tube, the relatively stationary gas through which the canal-ray particles move. This produces the "irregular" radiation of Stark, seen in the series lines of a flame and the undisplaced lines of the canal-ray bundle.

We assume that the velocity is on the average equally in all directions, and, in accordance with this, that the axes of the systems are equally distributed, so that the inequalities $A_{20} \dots$ all vanish. We shall, as before, neglect U/C in comparison with unity. We get

$$\bar{X} = \bar{Y} = \bar{Z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$\bar{S} = \int \bar{R} f(U) dU \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$\bar{W} = \frac{4}{3C^2} \int \bar{R} U^2 f(U) dU. \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

The remaining equations are unnecessary.

We draw the following conclusions from the equations of this and the last section :—

(1) Whatever be the source of the radiation $\bar{R}$ from one of the moving systems, the average total radiation per system, $\bar{S}$, is of the order $\bar{R}$, but the average rate of working of the radiation pressure per system, $\bar{W}$, is only of order $\bar{R} \frac{U^2}{C^2}$.

(2) Hence only a very small fraction of the energy radiated can be ascribed to work done by radiation pressure, either for a radiating gas, or for a bundle of canal rays.

§ 18. We must now consider in what way the values of T' , P' are influenced by the motion, that is to say, we must enquire whether it is possible to satisfy assumption (B) continually by supposing the system (Σ) to be suitably disturbed. For this purpose we must calculate the mechanical force acting on any electron of (Σ).

Denote this force for the moment by $\mathbf{F}$, and its components by (X , Y , Z), the meanings of these symbols of course being different from those used in § 9. For an electron with charge e and velocity ($u + U$, v , w) we have

$$X/e = X + \frac{v}{C} N - \frac{w}{C} M,$$

$$Y/e = Y + \frac{w}{C} L - \frac{u+U}{C} N,$$

$$Z/e = Z + \frac{u+U}{C} M - \frac{v}{C} L.$$

By equations (1) of § 5 these become

$$\begin{aligned} X/e &= X' + \frac{v'}{C} N' - \frac{w'}{C} M' + \frac{U}{C} \left(\frac{v'}{C} Y' + \frac{w'}{C} Z' \right), \\ Y/\kappa e &= Y' + \frac{w'}{C} L' - \frac{u'}{C} N' - \frac{U}{C} \frac{u'}{C} Y', \\ Z/\kappa e &= Z' + \frac{u'}{C} M' - \frac{v'}{C} L' - \frac{U}{C} \frac{u'}{C} Z'. \end{aligned}$$

The equations of motion become, if m be the mass of the electron,

$$\begin{aligned} \frac{d}{dt} \left(\frac{mu}{e} \right) &= X' + \frac{v'}{C} N' - \frac{w'}{C} M' + \frac{U}{C} \left(\frac{v'}{C} Y' + \frac{w'}{C} Z' \right), \\ \frac{1}{\kappa} \frac{d}{dt} \left(\frac{mv}{e} \right) &= Y' + \frac{w'}{C} L' - \frac{u'}{C} N' - \frac{U}{C} \frac{u'}{C} Y', \\ \frac{1}{\kappa} \frac{d}{dt} \left(\frac{mw}{e} \right) &= Z' + \frac{u'}{C} M' - \frac{v'}{C} L' - \frac{U}{C} \frac{u'}{C} Z'. \end{aligned}$$

Equations (2) show that (L', M', N') are determined from the position and velocity of the corresponding electron in the system of reference (Σ') just as for a system of rest, but (X', Y', Z') have additional terms

$$\frac{U}{C} \left(\frac{\partial F'}{\partial x'}, \frac{\partial F'}{\partial y'}, \frac{\partial F'}{\partial z'} \right).$$

Denote by suffix 0 the values of the forces in the case of a system at rest, but with the actual relative velocities of its electrons. Then

$$L' = L'_0, \dots \quad X' = X'_0 + \frac{U}{C} \frac{\partial F'_0}{\partial x'}, \dots$$

Further, write

$$q = \sqrt{u^2 + v^2 + w^2}, \quad q' = \sqrt{u'^2 + v'^2 + w'^2};$$

since

$$u = \kappa^2 u', \quad v = \kappa v', \quad w = \kappa w', \quad q = \kappa \sqrt{q'^2 - \frac{U^2}{C^2} u'^2} = \kappa q' \Delta,$$

where

$$u' = q' \cos \psi, \quad \Delta = \sqrt{1 - \frac{U^2}{C^2} \cos^2 \psi}.$$

The mass m is a function $f(V)$, where V is the velocity of the electron, and is given by

$$V = \sqrt{(u + U)^2 + v^2 + w^2}.$$

Unless q is large m is practically constant; hence we may suppose U small compared with q , and write

$$V = q + \frac{Uu}{q} = \kappa \Delta q' + \frac{\kappa U u'}{\Delta q'} + \dots = \kappa \Delta q' + \frac{\kappa U \cos \psi}{\Delta} + \dots,$$

$$m = f(V) = f(\kappa \Delta q') + \frac{\kappa U \cos \psi}{\Delta} f'(\kappa \Delta q') + \dots$$

Lastly, in accordance with the equations of transformation from system (Σ) to system (Σ') , $\frac{d}{dt} = \kappa \frac{d}{dt'}$. Hence we get

$$\left. \begin{aligned} \kappa^3 \frac{d}{dt'} \left\{ \frac{u' f(V)}{e} \right\} &= X_0' + \frac{v'}{C} N_0' - \frac{w'}{C} M_0' + \frac{U}{C} \left(\frac{\partial F_0'}{\partial x'} + \frac{v'}{C} Y_0' + \frac{w'}{C} Z_0' \right) \\ &\quad + \frac{U^2}{C^2} \left(\frac{v'}{C} \frac{\partial F_0'}{\partial y'} - \frac{w'}{C} \frac{\partial F_0'}{\partial z'} \right), \dots \dots \dots \\ \kappa \frac{d}{dt'} \left\{ \frac{v' f(V)}{e} \right\} &= Y_0' + \frac{w'}{C} L_0' - \frac{u'}{C} N_0' + \frac{U}{C} \left(\frac{\partial F_0'}{\partial y'} - \frac{u'}{C} Y_0' \right) - \frac{U^2 u'}{C^3} \frac{\partial F_0'}{\partial y'}, \\ \kappa \frac{d}{dt'} \left\{ \frac{w' f(V)}{e} \right\} &= Z_0' + \frac{u'}{C} M_0' - \frac{v'}{C} L_0' + \frac{U}{C} \left(\frac{\partial F_0'}{\partial z'} - \frac{v'}{C} Z_0' \right) - \frac{U^2 v'}{C^3} \frac{\partial F_0'}{\partial z'}, \end{aligned} \right\} \quad (43)$$

These equations are general; U/C is involved explicitly, as well as implicitly, because (u', v', w') must obviously depend on U , and $X_0' \dots L_0' \dots F_0'$ depend on (u', v', w') , their *form* being the same as for a system at rest. On the left-hand side U occurs in the first power only in V explicitly, on the right-hand side also in the first power. It follows that in general (u', v', w') also involve first-power terms. On account of the smallness of U/C we shall neglect all powers above the first; hence we put $\kappa = \Delta = 1$,

$$f(V) = f(q') + U \cos \psi f'(q') = m' + \frac{U}{C} \cos \psi \cdot \frac{dm'}{d\beta'},$$

where $\beta' = \frac{q'}{C}, \quad m' = f(q').$

To the first approximation the systems (Σ) , (Σ') are identical, except for the Doppler effect; hence we may omit accents everywhere. We shall also omit the suffix $_0$ as no longer necessary. In all the small terms, the relative velocity, electric and magnetic forces and vector potential may be replaced by their values for steady motion, all disturbances being treated as of the first order. Collecting all small terms

on the right-hand side of the equations, these become

$$\left. \begin{aligned} \frac{d(mu)}{dt} - e \left(X + \frac{v}{C} N - \frac{w}{C} M \right) &= \frac{U}{C} e \left(\frac{uX + vY + wZ}{C} + \frac{\partial F}{\partial x} - \frac{u}{C} X \right) \\ &\quad - \frac{d}{dt} \left(\frac{U}{\beta} \frac{dm}{d\beta} \frac{u^2}{C^2} \right), \dots \dots \dots \\ \frac{d(mv)}{dt} - e \left(Y + \frac{w}{C} L - \frac{u}{C} N \right) &= \frac{U}{C} e \left(\frac{\partial F}{\partial y} - \frac{u}{C} Y \right) - \frac{d}{dt} \left(\frac{U}{\beta} \frac{dm}{d\beta} \frac{uv}{C^2} \right), \\ \frac{d(mw)}{dt} - e \left(Z + \frac{u}{C} M - \frac{v}{C} L \right) &= \frac{U}{C} e \left(\frac{\partial F}{\partial z} - \frac{u}{C} Z \right) - \frac{d}{dt} \left(\frac{U}{\beta} \frac{dm}{d\beta} \frac{uw}{C^2} \right). \end{aligned} \right\} \quad (44)$$

On the left-hand side the coordinates, velocities, and forces have their actual values, including disturbances of all kinds. The effect of the translation is represented by the disturbing force on the right-hand side, which causes the system to deviate from the configuration it would have had in the absence of translation. A knowledge of the equations of motion of the system thus enables us to calculate the deviation due to translation. To this we now proceed.

§ 19. In accordance with assumptions (A), (B) of §§ 11, 12, we shall suppose the system to consist of a number of circles of equidistant electrons, each revolving with its own uniform velocity about the common axis, and so adjusted as to be stable and quasipermanent. Perturbations of a ring due to its neighbours are neglected, so that only their steady field need be taken into account. The forces on any one electron due to the rest of its ring are obviously constant; thus by, what has been said, the total forces, due to the whole system, are also constant.

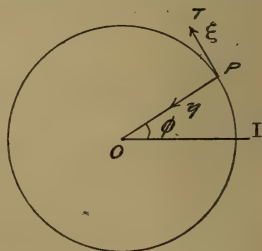
Consider any one ring of n electrons, of radius ρ and angular velocity ω , so that $\beta = \frac{\omega\rho}{C}$; let

O be its centre, P an electron. For initial line OI take the projection on the plane of the ring of the velocity of translation U. For the i th electron we write

$$\phi = \omega t + \delta + \frac{2\pi i}{n},$$

where δ is a constant.

Take moving axes P ξ , P η , P ζ along the tangent, towards the centre, and parallel to the axis. The projections of U in these directions are $-U \sin \alpha \sin \phi$, $-U \sin \alpha \cos \phi$, $U \cos \alpha$, where α is as before the angle between U and the axis.



Let (ξ, η, ζ) be the displacement of the i th electron from its position in steady motion; ξ, η, ζ may be expressed as sums of terms of the type $A \exp. i\left(pt - k\frac{2\pi i}{n}\right) \dots$ where p is generally complex, of the form $q + i\kappa$; q is the frequency relative to the ring, κ measures the damping, k is an integer between $\pm \frac{n}{2}$. The frequency to an outside observer is $q + k\omega$; for the vibration has k waves, which are carried round by the ring in its rotation.

Changing the notation let $u, v, w, X, Y, Z, L, M, N, F, G, H$ now refer to the new axes of ξ, η, ζ ; denote the disturbing force by F , and its components by X, Y, Z . We have

$$H=0,$$

$$u=C\beta, v=0, w=0,$$

$$\frac{du}{dt}=0, \frac{dv}{dt}=\frac{C^2\beta^2}{\rho}, \frac{dw}{dt}=0,$$

$$X = -\frac{eU}{C} \sin \alpha \left\{ \frac{\partial F}{\partial \xi} \sin \phi + \left(\frac{F}{\rho} + \frac{\partial G}{\partial \xi} \right) \cos \phi \right\} + \frac{UC\beta^2}{\rho} \frac{dm}{d\beta} \sin \alpha \cos \phi,$$

$$Y = -\frac{eU}{C} \sin \alpha \left(\frac{\partial F}{\partial \eta} \sin \phi + \frac{\partial G}{\partial \eta} \cos \phi \right) + \frac{eU\beta}{C} \sin \alpha (-X \cos \phi + Y \sin \phi) \\ + \frac{UC\beta^2}{\rho} \frac{dm}{d\beta} \sin \alpha \sin \phi,$$

$$Z = -\frac{eU}{C} \sin \alpha \left(\frac{\partial F}{\partial \zeta} \sin \phi + \frac{\partial G}{\partial \zeta} \cos \phi \right) + \frac{eU\beta}{C} (X \cos \alpha + Z \sin \alpha \sin \phi).$$

In the last three equations $X, F, \dots$ are all independent of t ; in the functions $\frac{\partial F}{\partial \xi} \dots \xi, \eta, \zeta$ must be put equal to zero *after* differentiation. Thus we see that in the disturbing force terms of two classes only occur:

(a) Class 0: the term $\frac{eU\beta}{C} X \cos \alpha$ in Z , a constant.

(b) Class -1: the remaining terms, all of which involve $\cos \phi$ or $\sin \phi$ linearly, where $\phi = \omega t + \delta + \frac{2\pi i}{n}$.

The corresponding relative frequencies are 0, ω ; the absolute frequencies, being $q + k\omega$, are both zero.

Hence neither set of terms gives rise to periodic waves in the surrounding medium.

It follows that a system of electrons in uniform translation to the present approximation emits only waves due to disturbances other than those produced by translation. These disturbances are modified by the translation to a certain extent, in accordance with the results of §§ 15–17. In addition their frequencies are altered by second-order amounts on account of changes in the structure of the system, and those of the corresponding emitted waves by first-order amounts representing the Doppler effect.

§ 20. The result of § 19 is a consequence of the fact that a disturbance of the type $A \cos\left(\omega t + \frac{2\pi i}{u}\right)$ consists of a wave which travels round the ring with velocity ω in the opposite direction to that of rotation; since by that rotation it is also carried forward with velocity ω , it remains at rest in space. Obviously a disturbance of class 0 of small frequency, and one of class -1 , of frequency nearly equal to ω , give rise to waves of small frequency in the surrounding medium. We find such disturbances amongst the free vibrations of each class. For stability β must generally exceed $1/100$, so that ω is large compared with the frequency to light-waves; but the small frequencies referred to may be of the right order of magnitude. Further, I have shown elsewhere* that vibrations of classes 0, -1 are amongst those which are able to produce spectrum lines of sufficient intensity to be observed. These particular disturbances are precisely those which are most strongly excited when the velocity and orientation of the system are changing. Hence, although a system moving with uniform velocity and with invariable orientation cannot of itself produce spectrum lines, we have in changes of velocity and orientation a possible source of such lines.

It is beyond the scope of this paper to discuss these vibrations fully, but it will be instructive to study the simplest of them, namely those parallel to the axis.

The equation of motion for axial vibrations is of the form

$$\ddot{\zeta} + P\dot{\zeta} + Q\zeta = Z/m,$$

where P , Q are functions of the frequency p with real coefficients, P^2 being small compared with Q . The corresponding frequency equation for the free vibrations,

$$p^2 - \imath pP - Q = 0,$$

has two complex roots, p_1 , p_2 , not very different from $\pm\omega$ respectively, and an infinite number of roots, p_3 , p_4 ,... very

* Schott, *loc. cit.*

much larger. For stability we must have $p = q + i\kappa$ with κ positive for each root.

§ 21. We shall solve the general case where $Z = a\epsilon^{ip t}$, the amplitude a being slowly variable, on the assumption that for $t = t_0$, $a = a_0$, a constant. This makes

$$\zeta = \frac{a_0 \epsilon^{ip t}}{(p - p_1)(p - p_2)} \text{ for } t \leq t_0,$$

terms of orders $p/p_3, p_1/p_3, \dots$ being neglected. We find in the usual way to the same approximation, for $t \geq t_0$,

$$\zeta = - \frac{a_0 \epsilon^{ip t_0}}{(p - p_1)(p - p_2)} - \frac{t}{p_1 - p_2} \left\{ \epsilon^{ip_1 t} \int_{t_0}^t a \epsilon^{i(p - p_1)t'} \cdot dt' - \epsilon^{ip_2 t} \int_{t_0}^t a \epsilon^{i(p - p_2)t'} \cdot dt' \right\}.$$

Integrating repeatedly by parts we find

$$\begin{aligned} \zeta = & - \frac{\epsilon^{ip t}}{(p - p_1)(p - p_2)} \left\{ a + i\dot{a} \left[\frac{1}{p - p_1} + \frac{1}{p - p_2} \right] \right. \\ & - \ddot{a} \left[\frac{1}{(p - p_1)^2} + \frac{1}{(p - p_1)(p - p_2)} + \frac{1}{(p - p_2)^2} \right] \dots \left. \right\} \\ & - \frac{a_0 \epsilon^{ip t_0}}{(p - p_1)(p - p_2)} \left\{ 1 - \frac{p - p_2}{p_1 - p_2} \epsilon^{ip_1(t - t_0)} + \frac{p - p_1}{p_1 - p_2} \epsilon^{ip_2(t - t_0)} \right\}. \quad (45) \end{aligned}$$

The first two lines obviously represent the forced vibration induced by the force of variable amplitude; the third represents the free vibration left over after the amplitude first began to vary; if the constant state be sufficiently far back the free vibration disappears owing to damping. The transformation is obviously legitimate only when the series in the first line is convergent, that is, when $\dot{a}/a, \dots$ are all small compared with $p - p_1$ and with $p - p_2$. This condition ceases to hold when there is resonance. For instance, when $p = p_1$ the corresponding terms in the first line must be replaced by the term $-\frac{i\epsilon^{ip_1 t}}{p_1 - p_2} \int_{t_0}^t a dt'$, which represents a free vibration of variable amplitude.

§ 22. When the disturbing force Z is of class k the waves emitted by the disturbance ζ are:—

(1) A forced wave of frequency $p + k\omega$ and class k , of variable amplitude.

(2) Two free waves of frequencies $p_1 + k\omega, p_2 + k\omega$, which vanish when the constant state is sufficiently far back.

In the cases under consideration we must consider

(a) A force $\frac{e\beta X}{e} U \cos \alpha$, of zero frequency and class, giving rise to no forced wave.

(b) Forces of the types $A \frac{\cos \phi}{\sin \phi} U \sin \alpha$, where $\phi = \omega t + \delta + \frac{2\pi i}{n}$. These are of frequency ω and class -1 , and therefore give rise to no forced waves.

This is clearly true of any disturbance of type $\frac{\cos}{\sin} k\phi$.

It follows that spectrum lines cannot generally be excited by the variable motion of a system, whether changing in velocity or orientation, or both.

§ 23. An exception occurs when the system is subject to a force of nearly the same frequency as one of its free vibrations, but only when, owing to the motion, the amplitude of the force varies at a rate comparable with the difference of the two frequencies.

For instance, two neighbouring systems, *e.g.* two ions, act on each other by resonance; the difference between the frequency of a free vibration of the first ion and that of the force of corresponding type due to the second is small, being merely the Doppler effect due to the relative motion of the ions. It is easy to see that in this case $\dot{a}/a$ is of order $\dot{r}/r$, where r is the distance between the ions, while the difference of frequency, δp , is of the order $p \frac{\dot{r}}{c}$; thus $\dot{a}/a\delta p$ is of order

$\lambda/2\pi r$. This is particularly large during a collision. Thus we conclude that the free vibrations of an ion, or of any other moving system of electrons, are excited especially in consequence of the variations of its motion due to collisions with other systems. Since the corresponding waves are sufficiently intense and of suitable frequency we can in this way get observable spectrum lines.

§ 24. The chief conclusions from our theoretical discussion are as follows:—

- (1) The radiations due to disturbances existing in a system of electrons are augmented in intensity by the motion of the system (§§ 16, 17).
- (2) The work done by radiation pressure on the system represents only a very small fraction of the energy radiated (§§ 16, 17).

- (3) Steady translation alters the configuration of the system by an amount proportional to the velocity (to a first approximation); the change of position of any electron may be represented as due to a disturbing force acting on the original system, but it is of such a type as to produce no periodic waves in the surrounding medium (§ 19).
- (4) Variable motion of the system gives rise to periodic waves only when there is resonance, and when the variations in velocity and orientation of the system are sufficiently rapid (§ 22). Waves are produced by vibrations excited more especially during collisions between different systems.

§ 25. These conclusions have been obtained on the three assumptions (A), (B), (C). The two first relate to the structure of the system of electrons; they are made for the sake of simplicity, but, as has been already pointed out, there is much to be said in their favour on account of the permanence of systems which are to serve as models of atoms or ions. It is hardly possible that other assumptions made in their place can yield qualitatively different results. The third assumption is one which is usually made in the kinetic theory of gases and, although possibly only approximate in the present case, must at any rate lead to results of the right order of magnitude.

The conclusions, such as they are, are entirely at variance with the hypothesis of Stark, that the energy emitted in the displaced lines of the canal rays is transformed from kinetic energy of translation by the direct action of radiation pressure. Obviously the only way in which radiation pressure can reasonably be said to act directly, is by resisting the motion of the canal-ray particles and so consuming work which is emitted as radiant energy. The argument against this hypothesis may briefly be stated thus:—

Consider a group of similar systems of electrons in motion, which represent a bundle of canal rays, or a quantity of radiating gas. Let $\bar{R}$ be the average radiation per second per system, of any type. It is ultimately derived from one or more of three sources:—(1) by direct transformation of kinetic energy of translation by radiation pressure, to the average amount $\bar{W}$; (2) by indirect transformation from the same source, and also from other types of vibrational energy by collisions, to the total average amount $\bar{K}$; (3) by absorption, due to resonance, from electromagnetic energy of waves

in the surrounding æther, to the average amount $\bar{\mathbf{A}}$. When a stationary state is reached we have $\bar{\mathbf{R}} = \bar{\mathbf{W}} + \bar{\mathbf{K}} + \bar{\mathbf{A}}$.

By the results of §§ 16, 19–22, we have $\bar{\mathbf{W}} = a\bar{\mathbf{R}} + b\mathbf{R}_2$, where a, b are of the order U^2/C^2 , while $\bar{\mathbf{R}}, \mathbf{R}_2$ are, to the first order, independent of U/C ; since each varies as the square of the amplitude of vibration, $\mathbf{R}_2/\bar{\mathbf{R}}$ and $\bar{\mathbf{W}}/\bar{\mathbf{R}}$ are independent of the amplitude and depend only on the structure of the system.

$\bar{\mathbf{A}}$ varies as the energy-density of the field around the system and is of the form $c\bar{\mathbf{R}} + \bar{\mathbf{E}}$, $c\bar{\mathbf{R}}$ denoting the part of the energy-density due to radiation from neighbouring systems, $\bar{\mathbf{E}}$ that due to external sources. Hence we may write $\bar{\mathbf{A}} = d(c\bar{\mathbf{R}} + \bar{\mathbf{E}})$, where c, d depend on the nature of the systems and their mean distance apart, but not on the amplitude of vibration.

With these values for $\bar{\mathbf{W}}, \bar{\mathbf{A}}$, we get

$$\bar{\mathbf{R}} = \frac{\bar{\mathbf{K}} + d\bar{\mathbf{E}}}{1 - a - b\mathbf{R}_2/\bar{\mathbf{R}} - cd}$$

This equation shows that, because $\bar{\mathbf{W}}$ is very nearly proportional to $\bar{\mathbf{R}}$, $\bar{\mathbf{R}}$ is determined by the external energy $\bar{\mathbf{E}}$, and by the kinetic energy $\bar{\mathbf{K}}$ transformed into energy of vibration at collisions. Resonance between the several systems and radiation pressure merely increase the value of $\bar{\mathbf{R}}$.

It is well to bear in mind that $\bar{\mathbf{R}}$ does not directly measure the radiation emitted by the group of systems constituting the canal-ray bundle or radiating gas as a whole; for on account of resonance the greater part of $\bar{\mathbf{R}}$ may be absorbed by neighbouring systems. In § 24 we saw that resonance will generally be diminished between systems moving with different velocities, on account of Doppler effect due to relative motion; it is hardly possible that for systems moving with the same velocity it can be appreciably greater than for systems at rest; for the structure of a system is only altered by motion by quantities of the second order in U/C . Thus it is unlikely that resonance can play a greater part in a bundle of canal rays than it does in a radiating gas with atoms moving with relatively small velocities.

§ 26. Thus Stark's hypothesis cannot be reconciled with an electron theory; it is therefore of importance to enquire how far his experimental results require that hypothesis. The fundamental results are given in §§ 11–12, 14–16 of his

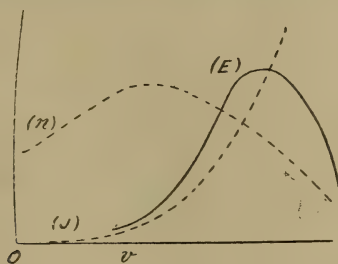
latest paper. It must be remembered that the intensities given for the various lines rest on eye estimations of the blackening of the negative, and therefore great accuracy is not claimed for them. Stark appears to base his hypothesis mainly on the following facts:—

- (1) As regards intensity of the undisplaced line, there is parallelism between it and that of the band spectrum; both are attributed to the residual gas, the undisplaced line to slowly moving positive ions produced by collisions between canal-ray ions and atoms, the band spectrum to the atom at recombination (§ 11).
- (2) As regards intensity of the displaced line, there is parallelism between the velocity of the canal-ray ions and the temperature of a radiating gas (§ 12).
- (3) Between the undisplaced and displaced lines there is a dark space, attributed by Stark to the supposed small emission at small velocities of the canal-ray ion, and not to a dearth of slowly moving canal-ray ions (§ 14).

In § 13 Stark assumes that the intensity E of the displaced line is given by an equation of the form

$$E = nJ(v^2, \lambda)$$

where n is the number and v the velocity of canal-ray ions

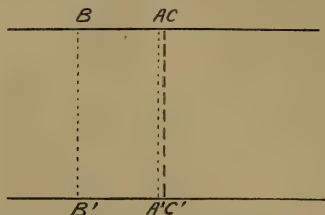


emitting the line of wave-length λ . He supposes (1) n to be given in terms of v by the curve (n) in the diagram; (2) J to be given by the curve J , from the parallelism between velocity and temperature. Then we get the curve (E) , which is of the form required by experiment.

The first supposition requires that canal-ray ions exist in appreciable numbers for all velocities between the small ones corresponding to the undisplaced line up to velocities of order 10^7 cm./sec. belonging to the displaced line. We must enquire how far this supposition is legitimate. For this purpose it is necessary to consider the generation of the canal-ray ions in the vacuum-tube.

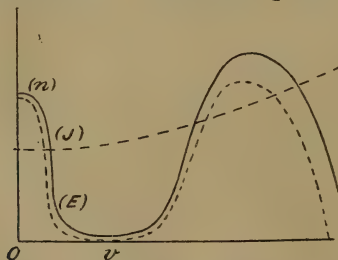
§ 27. There appears to be general agreement on the following points:—

- (1) In a vacuum-tube, near the cathode CC' , there is a thin layer $AA'CC'$ in which ionization takes place, due to the bombardment by positive ions.



- (2) In the negative glow, beyond BB' , ionization takes place, due to cathode rays.
- (3) In the intervening Crookes dark space, $BB'AA'$, there is no ionization, because neither positive ions nor negative electrons have yet acquired the necessary kinetic energy.
- (4) In the negative glow there is a very decided maximum of ionization (proved by H. A. Wilson*).
- (5) The canal-ray particles acquire their velocity in falling through the difference of potential between their place of generation and the cathode. Their velocity is however diminished, by the time they reach the space behind the perforated cathode, by something like 20 to 50 per cent., on account of collisions.

It follows from (3) and (5) that no canal-ray particles are produced at all with velocities between the limits corresponding to AA' , BB' . The existence of the dark space between the undisplaced and the displaced line seems to indicate that no ions with intermediate velocities are produced by collisions



of canal-ray particles with each other or with atoms. On this view the curve connecting the number of positive ions with the velocity is (n) , as shown in the adjoining figure,

* Wilson, Phil. Mag. [5] vol. xlix. pp. 515-6.

where the first maximum is for ions moving with small velocities in all directions on the average, and the second is for canal-ray particles. If the curve (J) gives the intensity of the radiation of an ion as a function of its velocity, the curve (E) will be that for the resulting radiation. It has the required form, the first maximum giving the undisplaced and the second the displaced line. Whether the present (n) curve, or that assumed by Stark, be the correct one, can only be decided by experiment; the present one has the advantage of leading to no contradiction with the electron theory.

§ 28. In §§ 15, 16 Stark shows that, as the average velocity of the canal-ray particles increases, the intensity of the displacement line increases; and further that the light emitted [normally] to the direction of motion is polarized, the electric vibration parallel to the motion predominating very slightly.

Equations (35), (37), (39) show that both these effects are to be expected from theory, but both are due to inequalities of distribution of axes; since the polarization is small these inequalities are small, and therefore the change in intensity to be expected on this account is only small.

Stark finds that the change is rapid, and all the more rapid the higher the frequency.

These changes are most probably due to the action of collisions. Great velocity implies collisions of great violence and of short duration, and therefore great disturbance of the ion and preponderance of high harmonics. In the present state of our knowledge of the details of collisions detailed explanations of Stark's results can hardly be expected. Without entering further into the matter I think there is sufficient prospect of accounting for them on the basis of collisions, and that it is premature at present to invoke a new hypothesis as to the action of radiation pressure, which moreover is difficult to reconcile with the electron theory.

Note I.—Since this paper was written Stark (*Phys. Zeitschr.* 8. p. 79) has himself admitted the possibility that collisions may play a part in canal-ray emission.

Note II.—It is incorrect to call (u' , v' , w') the velocity in the system (Σ'); the expression for this velocity has been given by Einstein (*Ann. Phys.* 322. p. 916). The results of this paper are, however, not appreciably affected thereby.

LX. *Recovery of Nickel and Carbon Steel from Overstrain.*
 By E. L. HANCOCK, Assistant Professor of Applied
 Mechanics, Purdue University*.

[Plate XVI.]

IT has been known for some time that wrought iron when stressed to its ultimate strength, but not fractured, will, when allowed to rest, show a higher ultimate strength and a higher limit of elasticity. This fact was discovered, independently, by Captain Beardslee and by Professor Thurston. Later experimenters on the subject of overstrain of iron and steel have confined their attention, for the most part, to the behaviour of these materials after they have been stressed to the yield-point. This yield-point is the elastic limit of the contractor, and is often so designated. The true elastic limit, however, or proportional limit has a value somewhat less than the yield-point.

Yield-Point Phenomena.

The yield-point might be defined as that load at which the piece elongates considerably, forming a flat place in the stress-strain diagram. The material under this load seems to be in a somewhat plastic condition, similar to that shown by ductile materials when they have been stressed to the ultimate strength, and rapid deformation takes place. The yield-point has always been a subject of interest to those interested in the elasticity of materials. The great deformation at this point without any increase of load indicates a peculiarity in the molecular condition or arrangement caused by the stress. The desire to know the effect of loads great enough to stress iron or steel to this yield-point, and the recovery of these materials after release from such loads, brought about the investigations reported in this paper.

Previous Investigations.

It has been known for some time that iron or steel stressed to the yield-point in tension or compression exhibits no elasticity when subjected to stress of the opposite kind. That is, a piece overstrained in tension shows no elasticity in compression and *vice versa*. (In this paper a piece is considered overstrained when the stress has reached the yield-point and the deformation has been allowed to take place.)

* Read before the Indiana Engineering Society, January 18, 1907. Communicated by the Author.

The question naturally arose in this connexion as to whether or not the elasticity returned, and if so, what were the most favourable conditions for early recovery. The most important investigation of the subject, carried on in Europe, is reported by Mr. J. Muir, in the Philosophical Transactions of the Royal Society of London, vol. A, 1900. The report deals with the recovery from overstrain in tension only, and includes the following :—

- | | | |
|-----|---------------------------|--|
| (a) | Recovery from overstrain— | Specimens allowed to rest. |
| (b) | “ “ “ | Specimens kept under stress. |
| (c) | “ “ “ | Specimens kept at moderate temperature. |
| (d) | “ “ “ | Specimens subjected to mechanical vibration. |
| (e) | “ “ “ | Specimens subjected to magnetic agitation. |

The conclusions reached as a result of these investigations are :—

- (a) Elastic properties gradually restored (see Pl. XVI. fig. 1).
- (b) Only a few tests made, but evidence seems to show that that treatment produces little effect.
- (c) Recovery accelerated to a considerable extent.
- (d) Recovery hindered.
- (e) No appreciable effect upon recovery.

The diagrams resulting from the tests of one of the specimens of the series (a), as reported by Muir, in the article referred to above, are shown in fig. 1 (Pl. XVI.). The material used for this test was steel, one inch solid round, having an ultimate strength of 64,000, a yield-point of 54,000, and a percentage of elongation in 8 inches of 23. As indicated in the drawing (see fig. 1), the specimen was first stressed to the yield-point (see curve i). Ten minutes later test No. 2 was made; the corresponding curve shows that the overstrain in the first test entirely destroyed the elastic limit and the modulus of elasticity of the specimen. Tests 3, 4, and 5 were made after intervals of 4 hours, 23 hours, and 2 days 22 hours respectively. The corresponding curves show that the elasticity is gradually restored by rest. Test No. 6, made 6 days 3 hours after No. 1, shows the elasticity restored, the elastic limit being even a little higher than originally. This test was continued until a new yield-point of the material was reached, that is, it was again overstrained. This second overstrain again destroyed the elastic properties, as shown by test No. 7 made 20 minutes

after No. 6 (see curve 7). The specimen was now allowed to rest 472 days, when it was again tested. This last test No. 8 shows the elasticity again restored, the elastic limit raised, and the modulus of elasticity the same as originally.

Tests made at Purdue University.

Some tension tests of nickel-steel similar to those outlined above, were made by Professor W. K. Hatt, and reported by him in the 'Proceedings' of the Indiana Academy of Science. The results obtained agree closely with those of Muir.

During the last three years further tests have been made in the Laboratory for Testing Materials of Purdue University, under the direction of Prof. E. J. Fermier and the writer. The work planned included an investigation of the recovery of structural carbon-steel and low nickel-steel from overstrain in both tension and torsion. Part of this work has been completed and the results are given in this paper. The work completed may be divided into three parts, viz. :—

- (1) Recovery from overstrain in tension.
- (2) " " " in direct torsion.
- (3) " " " in reverse torsion.

With the exception of the three and one half per cent. of nickel, the nickel-steel had the same chemical composition as the carbon-steel. The tension-test pieces were turned to a diameter of 0.75 inch for a length of 9 inches. The torsion specimens were 0.875 inch in diameter and about 20 inches long; they were left unturned, and were tested on a gauge length of 10 inches. Torsional deformations were measured with an Olsen troptometer.

Results of Tests.

Only a few typical curves are shown in this report. In fig. 2 is shown the results of tension tests on nickel-steel. The piece showed an elastic limit of 54,000 lbs. per sq. inch. The first test was continued until the piece was overstrained. Test No. 2, made 15 minutes later, shows that the elasticity was destroyed by the overstrain in No. 1. After a period of rest of 96 hours the piece was again tested. This test, No. 3, indicates a recovery of the elastic properties, the elastic limit now being 36,000 lbs. per sq. inch, and the modulus of elasticity being about the same as before the first test. Test No. 4, made 120 hours after No. 3, shows the elastic limit still higher, it now being 48,000. Test No. 5, made 792 hours after No. 4, shows the elasticity entirely restored. The

elastic limit—60,000—being somewhat higher than that shown by the first test. The slope of the curves 3, 4, and 5 shows that in the recovery from overstrain the piece regained the same modulus of elasticity as it had originally; this is indicated by the parallelism of the curves below the elastic limit. The curves show that overstrain destroyed the elastic properties of the piece and that these were restored by subsequent rest.

A summary of tension tests made on the carbon steel is shown in fig. 3. The test, No. 1, shows the elastic limit of the material to be 35,000 lbs. per sq. inch. This test was continued until the piece was overstrained. Test No. 2, made immediately after No. 1, shows that the elastic properties were entirely destroyed by the original overstrain. After a period of rest of 70 hours the piece was again tested, the curve, No. 3, shows a recovery of the elastic limit and modulus of elasticity. Tests 4 and 5, made after periods of rest, indicated in fig. 3, show a continued rise in the elastic limit, but no change in the modulus of elasticity. In test No. 5 the piece was again overstrained. Test No. 6, made immediately after this second overstrain, shows the elasticity again destroyed. The piece was now immersed in boiling water and subjected to its annealing effect for 15 min., when it was again tested. This treatment restored the elasticity of the material, shown by curve No. 7. The elastic limit, as shown by this last test, is higher than before, but the modulus of elasticity is the same.

Figure 4 shows the results of torsion tests on nickel-steel. The overstrain in the first test destroyed the elastic properties. This is shown by the second test made immediately. Tests 3, 4, and 5, made at intervals shown on the drawing, show a gradual recovery of the elastic limit and a return of the modulus of elasticity. Test No. 5 was continued until the piece was again overstrained. After 620 minutes the piece was again tested, but showed no elasticity. It was immersed in boiling-water 15 minutes and again tested (see curve No. 7), showing the elastic limit higher than before, and the modulus of elasticity restored.

The results of the direct torsion tests made on carbon-steel are shown in fig. 5. The original test shows an elastic limit of 28,000 inch-pounds moment. The test was continued until the piece was overstrained, as in the preceding case, and immediately retested showing an absence of elasticity. Tests 3, 4, 5, and 6 were made after different periods of rest. The elastic limit gradually returned and even exceeded the original amount by 1200 inch-pounds, and the modulus of elasticity

returned to its original value. The piece was again overstrained in test No. 6, destroying the elasticity. Immersion in boiling water restored the elasticity (see curve 8).

Figure 6 shows the results of tension tests made on carbon steel. The piece was first overstrained and then tested again immediately, showing no elasticity in this second test. It was desired to see what effect mechanical agitation of the piece would have on the recovery of its elastic properties. After subjecting it to 500 end blows of a twenty oz. hammer, the third test was made. This test, made 40 minutes after No. 2, shows little or no recovery due to the treatment.

The results of the reverse torsion tests are shown in figures 7 and 8. In the first test (curve marked direct) the piece was overstrained. The curve marked "reverse immediately" shows the result of an immediate torsion test in the opposite direction. The test shows the elasticity destroyed. After a period of rest of 7 days similar pieces were reversed. The curve in this case shows an elastic limit of 9000 lbs. per sq. inch at the outer fibre, and a modulus of elasticity the same as originally. Fig. 8 is self-explanatory and needs no comment.

The direct torsion may be regarded as a test that should bring out the same facts as those developed by the tension tests, since elements on a 45 degree helix were in simple tension. From this point of view the outer fibres may be considered overstrained in tension. A release of the load and subsequent treatment, similar to that given the tension specimens, should bring out the same facts. Similarly the tests in reverse torsion may be regarded as tests in which the material was first overstrained in tension and then tested in compression. This is seen to be the case when it is remembered that a helix under tension in the first test was under compression in the reverse test. Looking at the matter in this way, it is seen that the results of the reverse torsion tests are in exact agreement with facts previously known, viz., that when iron or steel is overstrained in either tension or compression it loses its elasticity for stress of the opposite kind.

General Conclusions.

It is seen from the curves exhibited :—(a) That an overstrain in either tension or torsion destroys the elasticity of the material, but that this elasticity gradually returns when the piece is allowed to rest, the elastic limit becoming, in some cases, greater than its original value, and the modulus of elasticity the same as its original value. (b) That the

elastic properties of the overstrained material are restored by immersion for a short time in boiling-water. (c) That recovery is not aided by repeated impact from a light hammer. (d) Materials overstrained in either tension or compression lose their elasticity for stress of the opposite kind, but the elasticity is restored by rest. (e) The carbon-steel seems to recover more quickly than the nickel-steel. This, however, has not been shown conclusively by these tests, but all evidence goes to show that it is true.

The writer wishes to acknowledge the efficient help of the following senior students who assisted in carrying out the testing work :—Mr. Henry Jacobson and Mr. Harry W. Steindorf, of the class of 1906 ; Mr. Clinton G. Reed, Mr. Lawson Stone, Mr. F. O. Blair, and Mr. Charles E. Shearer, of the class of 1905.

LXI. *Notes on Æther and Electrons*.

By C. V. BURTON, D.Sc.*

1. *APHENOMENAL Processes defined*.—There are some processes whose type is such that an observer with his surroundings may be the seat of them without any resulting *phenomena* being manifested to him; that is to say, the processes in question are without influence on the senses of the observer, or upon any instrumental test or measurement which he can make. We may exclude from present consideration such motions and other effects as remain undetected solely by reason of their minuteness, defining as *aphenomenal* those processes which elude observation through the absence of any standard of permanence by comparison with which they may be judged.

Thus absolute velocity in space, if admissible at all as a physical conception, appears to be rigorously aphenomenal.

2. *Absolute Acceleration*.—As a further example, consider a mass of perfectly incompressible substance of uniform density, completely filling an absolutely rigid envelope. We may suppose the incompressible substance to be a deformable solid, or a liquid either possessing viscosity or perfectly frictionless. Let the deformable substance be in motion in any definite manner, subject only to the limitations imposed by its incompressibility and by the rigid containing boundary. At any subsequent time, provided the envelope is without rotation, the relative motions affecting the deformable substance are completely determinate, and independent of any

* Communicated by the Author.

changes in the translational velocity of the envelope. The translation of the whole system may be accelerated, or even changed impulsively, without the production of any effects which could be detected by means of (ideal) observations confined to the interior of the envelope. On certain suppositions, this result has an interesting theoretical application. We are familiar with the view that the æther, whether free from or encumbered by ordinary matter, constitutes an incompressible *plenum* having everywhere the same density; from which it would follow that, if a translational acceleration of any magnitude could by suitable means be imparted to the whole of the explorable universe, we should be entirely without means of detecting it. Though changes of velocity, if actually occurring through so large a region, would almost certainly take place with extreme slowness, yet we are faced by the possibility that, even in regard to *constancy* of absolute translational motion, no true dynamical criterion may be available; experimental dynamics being more strictly concerned with motion relative to the æther.

3. Since the mere motion of a system through the æther does not cause that system to be the seat of a magnetic field, or of an electrostatic field, it follows that, *unless we attach a very real physical significance to absolute motion of the æther in space*, neither the magnetic nor the electric vector at any point is to be identified with translational velocity of the æther at that point*. Otherwise our motion through the æther, being equivalent to a contrary flow of æther with ourselves at rest, would be evidenced to us by the existence of a magnetic or electric field; and if the density of the æther is very great, the field-intensity arising from a moderate velocity of flow would be correspondingly great.

4. For a certain class of cases, the argument may be presented in a somewhat different form. Let us suppose, as in § 2 above, that the æther is incompressible, and that atomic matter is built up of strain-distributions, involving no departure from uniform density, which obtains everywhere. Now consider a limited region of the æther through which we can establish a uniform electrostatic field. Let a material system be present in the region considered, and let this system be entirely unconstrained, and initially at rest, subject to no external forces; then let a uniform electrostatic

* In the case of the electric vector, there is of course the further objection that electrons would, on the view referred to, be sources and sinks in the medium, involving continual creation and destruction of the stuff of which the medium is made. This, I believe, was first pointed out by von Helmholtz.

field be established throughout the region. If we suppose such a field to be constituted by a uniform flow of æther along the lines of force, the effect upon our material system will be simple. Precisely as in the case examined in § 2, the system will participate in the purely translational motion of the æther, without experiencing any of the effects which should be produced by an electric field.

If, on the other hand, with a like theory as to the constitution of atomic matter, we adopt the view that a magnetic field (in free æther) is constituted by a bodily flow of æther along the lines of magnetic force, we are led by precisely similar reasoning to a similar contradiction.

Thus it would seem that a fairly strong case may be made out for the proposition

That neither the electric nor the magnetic vector is to be identified with translational velocity of the æther . . . (A)

5. In view of this result, the question arises: Do the volume-elements of æther concerned in electromagnetic processes suffer any bodily displacement? For example, in a region through which radiation is being propagated, we have to consider elements which, though large enough to be treated as homogeneous portions of æther, are in every dimension very small compared with any effective wavelength. May we then conclude *that the resultant translational velocity of an æther-element never differs from zero*? Such a conclusion, though not necessarily implied by the admission of proposition (A) above, seems indicated as highly probable. It appears indeed evident that, if translational æther-velocity is an essential of an electric or of a magnetic field, it is only absolute velocity in space which can be involved*, since the choice of standards of position, relatively to which motion is to be measured, in no way affects our judgment as to whether or not an electric or magnetic field is present, or as to the direction and intensity of such a field. Let us write, then, tentatively,

$$0=u=v=w, \text{ universally, } \dots, \dots (1)$$

where u, v, w are the components of translational velocity of a volume-element of æther having its centroid at any given point.

The motion through the æther of electrons and atomic matter generally is not improbably a purely electromagnetic

* This is not to be taken as implying a belief in absolute motion as a valid physical conception; the qualification is added only to avoid passing by a conceivable alternative without mention.

phenomenon, reservation being made of small residual effects, evidenced to us by gravitation. Except for such residual effects, the tentative conclusion (1) is suggested as applying throughout the ætherial plenum, including those regions in which matter is present.

6. It should be remarked that, whereas the modern view of atomic matter reduces all motion to stationary motions of the æther, the suggestion now put forward reduces the motion of each æther-element to stationary motion on a scale still more minute. Though the indication thus afforded as to the nature of the electric and magnetic vectors is but slight, it may be pointed out that if an electromagnetic scheme can be formulated in conformity with (1), it will apparently be free from a difficulty which has embarrassed more than one hypothesis. For let Q be any quantity which varies from point to point of the ætherial plenum, and let $\partial Q/\partial t$ represent the time-rate of change of Q at a point fixed in space, dQ/dt being the rate of change corresponding to a point which moves with the æther; then, in virtue of the kinematical relation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z},$$

(1) would give

$$\frac{d}{dt} = \frac{\partial}{\partial t} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (2)$$

7. The processes concerned in the transmission of radiation can with some confidence be defined in terms of the electric and magnetic vectors, but so long as the nature of these vectors is undetermined, the actual character of the vibration necessarily remains unknown. Of all forms of transverse waves, the most readily conceived is that which consists in laminar motion of the medium, so that each infinitesimal element concerned in the propagation of a wave-train suffers bodily displacements in a plane transverse to the direction of propagation, and acquires thereby kinetic energy which is purely translational. So far as I am aware, this simple laminar form of motion, with bodily translation of æther-elements, has been assumed in nearly every attempt yet made to define dynamically the nature of light-waves. This is equally true of those hypotheses which are explicitly electromagnetic and of those which are not; the optical theories of Fresnel and MacCullagh, the labile (contractile) æther, vortex-sponge æther, and quasi-rigid (gyrostatic) æther of Lord Kelvin are among the most notable examples.

A wave-propagation involving no bodily motion for æther-elements, as suggested in § 5, might conceivably be realized in a medium incapable of experiencing a bodily shearing-stress, or of yielding to compressional stress: that is, in a medium constituted of frictionless incompressible fluid, which must necessarily be in motion. A further condition for such a medium appears to be a certain heterogeneity, as for example when coreless vortices are present. On this subject I hope to present some considerations in a later communication.

Strain-figures and Electrons.

8. In 1891 a paper was read before the Physical Society of London*, wherein an attempt was made to account for some of the properties of matter by means of a theory of "strain-figures," the equations of motion of a strain-figure of general specification being established. The intervening years having seen the initiation and development of the electronic theory of matter, the main feature of the view put forward in the paper referred to will be understood from the briefest restatement. It was suggested "that a given portion of matter consists, not of any individual portion of ætherial or other substance, but of modifications in the structure or energy or other qualities of the æther; and when matter moves, it is merely these modifications of structure or of energy or of other qualities which are transferred from one portion of the æther to another." No attempt was made to connect the inertia or other properties of atomic matter with any electromagnetic phenomena, and in fact, as it was assumed that the inertia of a given body was strictly invariable, the conclusion was drawn that the strain concerned was probably related closely to gravitation. I was not at that time aware of Professor J. J. Thomson's memoir "On the Electric and Magnetic Effects produced by the Motion of Electrified Bodies"†, which forms the starting-point as well as the permanent basis for an electronic theory of matter.

9. The especial object of this "strain-figure" theory was to reconcile the mobility of matter with the apparently perfect elastic properties of the æther, and from this point of view it may be interesting to compare the strain-figure of 1891 with subsequent specifications of the electron. The suggested

* C. V. Burton, "A Theory concerning the Constitution of Matter," *Phil. Mag.* February 1892.

† *Phil. Mag.* April 1881.

constitution for the strain-figure (not of course its supposed mode of formation) was thus explained * :—"Consider a region, either infinite or having very distant boundaries, and filled with a homogeneous isotropic elastic medium, whose condition throughout is one of stable equilibrium for small strains of any type. Let the medium now be strained, and held in its strained condition by some compelling agency : there will be a corresponding distribution of stress in the medium, and, provided the strain has at no point too great a value, the original condition will be completely regained after the compelling agency has been removed. But suppose that, instead, the medium is strained further and further from its initial state, and suppose that the restoring stresses do not always increase with the strain, but that beyond a certain point in the process they begin to fall off in value, until at last a point is reached at which the general tendency of the stress is to further increase the strain. If the compelling agency is now withdrawn, the medium will subside into a new condition of stable equilibrium, involving stress and strain at every point." It may not be superfluous to point out that nothing in the nature of elastic hysteresis is here contemplated, the stress being conceived as always determined by the strain alone, independently of the manner in which the strain may be changing.

Though the possibility of ideally creating a strain-figure by a process such as this is quite gratuitously assumed, it may be remarked that the constitution so specified, if admissible at all, ensures mobility. If this is not immediately evident, imagine a strain-figure occupying a position A remote from all disturbing influences, to be destroyed by the application of an ideal forcive, so that the medium is allowed to revert to its primitive unstrained state. Next suppose that a new forcive is so applied that a like strain-figure is formed at a new position B, which is only infinitesimally distant from the old position A. The energy of the new distribution of strain is precisely the same as that of the old, and since we suppose that the medium exhibits no defect akin to imperfect elasticity, it follows that the total work to be performed in the two operations is zero. But in the new position B of the strain-figure, the strain imposed upon any element of the medium differs only infinitesimally from the strain of the same element corresponding to the neighbouring position A. Therefore to effect a transference of the strain-figure from A to B it is only necessary for each strained element of the medium to suffer a definite infinitesimal strain, the total expenditure

* *Loc. cit.* p. 192.

of work demanded being precisely zero ; so that the strain-figure is free to move through the medium *.

10. It will now be convenient to classify strain-distributions under three heads. Any distribution which we conceive as

* So long as only an infinitely slow displacement of the strain-figure through the medium is in question, complete mobility appears to be assured ; but in order that the strain-figure may be able to move through the medium with finite velocity, and without involving dissipation of energy, a further limitation seems called for. The need for this was made evident by an objection expressed to me in general terms by Professor Larmor, whose view I have attempted to deal with in this note.

Let $D, D', \dots$, functions of x, y, z , serve to define the distribution of displacement, concerning which nothing is known, except perhaps that it is not bodily translational displacement at each point of the medium. Then, in accordance with the assumed continuity of the process by which a strain-figure is ideally producible, $D, D', \dots$ are continuous functions of x, y, z ; in other words,

$$(\partial/\partial x, \partial/\partial y, \partial/\partial z) (D, D', \dots)$$

are all finite. But when the strain-figure is moving with velocity-components U, V, W , which are small compared with the velocity of radiation, we readily find that

$$\frac{\partial D}{\partial t} = -U \frac{\partial D}{\partial x} - V \frac{\partial D}{\partial y} - W \frac{\partial D}{\partial z}; \text{ etc.} = \text{etc.};$$

while

$$\begin{aligned} \frac{\partial^2 D}{\partial t^2} = & -\frac{dU}{dt} \cdot \frac{\partial D}{\partial x} - \frac{dV}{dt} \cdot \frac{\partial D}{\partial y} - \frac{dW}{dt} \cdot \frac{\partial D}{\partial z} \\ & + \left(U \frac{\partial}{\partial x} + V \frac{\partial}{\partial y} + W \frac{\partial}{\partial z} \right)^2 D; \text{ etc.} = \text{etc.} \end{aligned}$$

Though, strictly speaking, the differentiation refers to a point fixed in space, $\partial D/\partial t, \dots$ are (in a generalized sense) of the nature of velocities, $\partial^2 D/\partial t^2, \dots$ being of the nature of accelerations; and while the assumptions already explained make $\partial D/\partial t$ everywhere finite for moderate values of U, V, W , we shall not be able to avoid infinite values of $\partial^2 D/\partial t^2, \dots$ without introducing the further assumption that $\partial^2 D/\partial x^2, \dots$ are everywhere finite. We must assume, in fact, that

$$(\partial/\partial x, \partial/\partial y, \partial/\partial z) (D, D', \dots)$$

are not only finite but continuous functions of the coordinates. If impulsive time-changes of the (generalized) velocities $\partial D/\partial t, \dots$ were to accompany a finitely rapid motion of the strain-figure, dissipation of energy would presumably be involved; but it does not appear to me that a (finite and) continuous distribution of the strain-magnitudes

$$(\partial/\partial x, \partial/\partial y, \partial/\partial z) (D, D', \dots)$$

need be incompatible with the above-postulated constitution of a strain-figure. Such continuity, in addition to the assumptions originally made, would ensure perfectly free mobility without any loss of energy except through radiation ; this latter being insignificant so long as the velocity (U, V, W) is very small compared with the velocity of radiation, with a corresponding restriction as to the acceleration $d/dt (U, V, W)$.

maintained by the application of an ideal force may be called a *forced* strain; a strain-figure specified as in the preceding paragraph, and requiring therefore no application of force to maintain it, being called *free*. It is proposed to restrict the designation "free" to a strain-distribution so constituted that it could be (ideally) brought into existence by a continuous process, involving no cutting or slitting of the medium, or any deviation from perfect elasticity (or quasi-elasticity). On the other hand, distributions of strain have been conceived which can only be brought into being by some such discontinuous operation as slitting the medium, and welding together or reuniting the slit surfaces after imposing on them a relative twist, or otherwise changing their presentation to one another. Such distributions, though self-sustaining, are not in our sense free, and in conformity with general usage they may be termed *locked*.

11. *Locked Strain-figures are not in general mobile.*—We have seen in § 9 that if a strain-figure is capable of existing in the "free" state, there will at least be no static resistance to displacing it with respect to the medium. In discussing the possible mobility of locked strain-figures, we have to remember that a determinate displacement of any strain-figure with respect to the medium is equivalent to the superposition of an additional strain-distribution. Free mobility thus implies that there are certain special strains which can be superimposed on the strain-figure without the expenditure of work; that is to say, in regard to such special strains the equilibrium of the medium is *labile*. Now it might be expected from general considerations that, to a first order, the stability of a medium is not affected by locally cutting it asunder and reuniting the cut surfaces after altering their presentation to one another: a conclusion which can in fact be verified by a simple analysis.

Before any cutting has taken place, let the state of strain of the medium be defined by the values of the coordinates $\phi, \psi, \dots$, all supposed independent. Now let the medium, originally unstrained, be cut asunder and the cut surfaces so reunited as to involve a condition of strain; the specification of the cut and of the displacements introduced being determinate, so that the strain remaining when the medium is subsequently left to itself is likewise determinate, and is definable by assigning a definite value to a suitably chosen coordinate θ , with simultaneous zero values for $\phi, \psi, \dots$. There are precisely the same kinematic possibilities for the uncut medium, in which θ is constrained to be zero, and

for the cut and reunited medium in which θ is constrained to have some different value. Thus we may legitimately define the meanings of $\phi, \psi, \dots$ in any specification of strain for which θ does not vanish by saying that the configuration $(\theta, \phi, \psi, \dots)$ is geometrically related to $(\bar{\theta}, 0, 0, \dots)$ precisely as the configuration $(\bar{\theta}, \phi, \psi, \dots)$ is related to $(0, 0, 0, \dots)$.

Consider now particularly the state of the medium when it has been cut and strained and reunited, and then left to itself. In this state θ is different from zero, but $\phi, \psi, \dots$ all vanish in virtue of our definition of θ . Again, the state of strain which is defined by $(\theta, 0, 0, \dots)$ and with which we have now to deal, is due to no forces except mutually equilibrating tractions at the reunited surfaces; such tractions giving rise to no virtual work in any infinitesimal displacement which does not involve relative motion of the surfaces in question. In other words, the strain $(\theta, 0, 0, \dots)$ is due to applied forces which give rise to no virtual work in any infinitesimal strain specifiable in terms of $\phi, \psi, \dots$; so that we have

$$\left. \begin{array}{l} 0 = \Phi = \Psi = \dots \\ \text{corresponding to} \\ \theta \neq 0; \quad 0 = \phi = \psi = \dots \end{array} \right\}, \quad \dots \dots \dots (3)$$

where $\Phi, \Psi, \dots$ are the (generalized) forces, corresponding respectively to $\phi, \psi, \dots$, which have to be applied to the medium in order to maintain equilibrium. In other words, if W is the potential energy of strain,

$$\Phi = \partial W / \partial \phi, \quad \Psi = \partial W / \partial \psi, \dots \dots \dots (4)$$

All that has been said so far is independent of any limitation regarding the magnitude of the strains involved. We shall now suppose the strains to be so small that W is adequately represented by a homogeneous quadratic function of the coordinates of the medium; that is to say, we can write

$$W = (\text{a homogeneous quadratic function of } \phi, \psi, \dots) \\ + \frac{1}{2} A \theta^2 + B_1 \theta \phi + B_2 \theta \psi + \dots \dots \dots (5)$$

Differentiating in turn with respect to $\phi, \psi, \dots$, we find

$$\left. \begin{array}{l} \text{when} \quad \theta \neq 0; \quad 0 = \phi = \psi = \dots \\ \Phi = B_1 \theta; \quad \Psi = B_2 \theta, \dots \end{array} \right\} \dots \dots \dots (6)$$

But in this case, by (3) $\Phi, \Psi, \dots$ all vanish; hence

$$0 = B_1 = B_2 = \dots,$$

and (5) becomes

$$W = (\text{a homogeneous quadratic function of } \phi, \psi, \dots) + \frac{1}{2} A \theta^2; \quad \dots \dots \dots (7)$$

so that, to our degree of approximation, the increment of potential energy due to a change of $\phi, \psi, \dots$ from one set of values to another, is independent of θ . We are thus led to deny the possibility, by means of a locked distribution of infinitesimal strain, of conferring on the medium any new lability, in virtue of which no increase of potential energy would be involved in the superposition of certain other types of strain compatible with the locked condition.

12. The proof here given that no change of elastic properties is caused by a locked distribution of infinitesimal strain is of very general application. Nothing need be assumed as to the nature of the coordinate θ , except that it involves some violation of the constraints which normally are imposed on the medium, and in virtue of which the values of the coordinates $\phi, \psi, \dots$ suffice to render the whole configuration determinate. When the discussion is not limited to small strains, so that stress-strain relations need no longer be linear, the result obtained in § 11 ceases to be applicable without modification. But this is evidently far from implying that mobility is assured to a locked strain-figure, provided merely that the strains are great enough to exceed the limits of linearity. Rather is it evident, from consideration of the simple case where linear relations hold good, that *in a thoroughly stable medium* locked strain-figures are not rightly constituted for mobility.

13. The general difficulty of conceiving a freely mobile locked strain-figure is perhaps more forcibly realized when we begin to trace the kinematical consequences of mobility. In relation to the distribution of displacement constituting the strain-figure, the surface which we regard as having been the seat of a cutting and reuniting operation is a surface of discontinuity; the displacement in general suffering a finite change of value as we pass from a point on one side of the surface to an infinitely near point on the other side. If, then, we suppose the locked strain-figure to be moving through the medium, and consider the motion of any point of the medium lying in the path of the surface of discontinuity, we realize that, just at the instant when the discontinuity passes, the point in question suffers impulsively a finite change of position. Thus the assumed motion of the locked strain-figure

involves infinite velocity for those points of the medium which at any given instant lie in the surface of discontinuity, and is in fact characterized by an infinitely rapid shearing motion of the medium just at the surface in question. It is seen to be a direct kinematical consequence of the assumed mobility that, as the surface of discontinuity is transferred to new positions, there is a continual shearing asunder and reuniting of the medium in violation of the postulated "locked" condition. Under special assumptions we may evade the difficulty of impulsive displacements, if we suppose the motion of the locked strain-figure to be such that the surface of discontinuity is moving purely along itself, but even in this restricted case the locked condition would be violated.

These are broadly the considerations advanced in support of the contention that locked strain-figures are not in general mobile; and that the only self-sustaining distributions of strain which are freely mobile are such as have been designated *free* strain-figures (see § 10).

14. We must, however, examine Professor Larmor's ætherial strain-figure, or model electron, more closely before attempting to decide as to its mobility. The æther is supposed to have quasi-rigidity in virtue of which each element experiences a restoring couple proportional to its small absolute angular displacement. "Let, then, in the rotationally elastic medium a narrow tubular channel be formed, say for simplicity a straight channel AB of uniform section: suppose the walls of this channel to be grasped, and rotated round the axis of the tube : this rotation will be distributed through the medium, and as the result there will be lines of rotational displacement all starting from A and terminating at B: and so long as the walls of the channel are held in this position by extraneous force, A will be a positive electron in the medium, and B will be the complementary negative one. They will both disappear together when the walls of the channel are released. But now suppose that before this release the channel is filled up (except small vacuous nuclei at A and B which will assume the spherical form) with a plug of æther so as to be continuous with the surrounding medium; the effort of release in this surrounding medium will rotate the plug slightly until it attains the state of equilibrium in which the rotational elasticity of the new part of the medium formed by the plug provides a balancing torque, and the conditions all round A or B will finally be symmetrical" *.

A difficulty arises when we attempt to apply to electrons

* 'Æther and Matter,' p. 326.

thus constituted the form of argument developed in § 13. For the electron is conceived as having a vacuous spherical nucleus, for which mobility is only postulated in general terms, the kinematics of the motion of the nucleus not being so far imaginable. If it be allowed that this difficulty of the nucleus may have some unforeseen solution, the whole problem of the mobility of the electron is left lacking in that precision which would be essential to a strict investigation. Whether or not a configuration of strain self-locked by the reuniting of sundered and relatively twisted surfaces is capable in this case of being moved freely through the medium without involving fresh breaches of continuity, must be judged rather from an instinctive standpoint. Having had the advantage of discussing this point with Professor Larmor, who has cleared my mind of a misapprehension, I express with some hesitation my outstanding difference of view regarding this question of mobility.

15. The method of regarding such problems outlined in § 11 above may be more helpful. If the medium in the unstrained state is stable for every possible type of displacement, no locked distribution of infinitesimal strain can possess that special lability of equilibrium which is an essential of free mobility; but if the medium itself is labile in regard to certain types of strain, the aspect of the problem is clearly modified. The form then assumed by the question is this: the supposed transference of a self-locked strain-distribution from one position to another in the medium is equivalent to the superposition of a determinate differential strain. Is this latter of such a type as could be imposed on the originally unstrained medium, without breach of continuity and without the expenditure of work? If so, the transference of the strain-figure through the medium can take place freely; otherwise it cannot. The hypothetical medium with which we have to deal is one in which small non-rotational strains (but no others) can be produced without any resistance being encountered; that is, without work being done. Consider, then, a pair of equal spherical vacuous cavities at any distance apart in the unstrained medium: we have to enquire what non-rotational strains could be imposed on the system; the differential aspect of the non-rotational condition being equivalent to irrotationality of the motion of the medium at each instant. Supposing all velocity-components to vanish at infinity, the only motion which could instantaneously be taking place, without building up opposing reactions from the elasticity of the (incompressible) medium, would be that arising from changes of shape, size, or position of the cavities.

Without some rotational motion, there could be nothing involving circulation around any closed circuit; the region filled by the medium being simply-connected. Thus there can be no (infinitesimal) displacement which involves circulations, and in regard to which the system is labile.

If now we introduce a locked distribution of infinitesimal strain in accordance with the specification quoted in § 14, the elastic properties of the system remain unaffected, in virtue of the result obtained in § 11, so that still no motion involving circulation in any closed circuit can take place without calling up elastic resistance. On these grounds the ætherial circulation which is specified by Professor Larmor as equivalent to the translation of his electron through the æther does not appear to me to be admissible as a mode of motion unopposed by the elasticity of the medium.

16. It is readily admitted that the possibility of a "free" strain-figure has yet to be demonstrated; but none the less it may be fairly contended that if mobile strain-figures must necessarily be of the free type, the development of theory on the basis of this result is deserving of some attention.

17. Continuing the comparison of free with locked strain-figures, we may now consider how they stand in relation to the balance of positive and negative electricity in the known universe. Though any large excess of positive or negative electrification would tend to spread itself through space, and not to become concentrated by gravitation like electrically neutral matter, yet it is hard to resist the conclusion that the ratio of total positive to total negative charges, taken throughout the universe, is extraordinarily near to unity. If our electrons are so constituted that they can only appear in pairs, a positive and an equal negative being always produced simultaneously* in the ideal process of genesis, this equivalence of total charges receives a ready explanation. On the other hand, an electron of the free strain-figure type is supposed to exist independently of any complementary companion, and we are led to postulate that the formation of positive electrons having a given total charge, and the formation of negative electrons having a complementary total charge, are *a priori* equally probable. Then a very exact balancing of charges, where enormously great numbers of electrons are concerned, becomes an overwhelming probability. We have no right, however, on the score of symmetry, to make the assumption just referred to; since experiment shows that the circumstances are very far from symmetrical. It is nothing more than a conjecture that a fuller knowledge

* Cf. Larmor, 'Æther and Matter,' p. 326.

of the nature of electrons might furnish the justification. Incidentally reference may be made to the sufficiently obvious point that any theory which postulates symmetrical contrariety of properties for positive and negative electrons, even though descriptive of a certain range of electromagnetic phenomena, must fail to be ultimately satisfactory.

18. It will probably be felt as a difficulty of a free strain-figure theory of electron-constitution, that it is made to appear at least kinematically possible to produce in free æther a distribution of electric displacement which violates the solenoidal or "circuital" condition: for example, in the early stages of ideal genesis of a strain-figure (§ 9). This difficulty, indeed, seems to arise in some form as soon as the possibility of an isolated electron is assumed. To evade it we may suppose the energy of non-circuital distributions of electric displacement to be relatively very great; or it may even be suggested that the æther, though intrinsically capable of transmitting waves of other than electromagnetic type, is free from such disturbances because these are not produced by the only available source of excitation, namely the motion of electrons. But these attempts to deal with the difficulty are clearly of the most tentative character.

19. It can hardly be doubted that the uniformity which characterizes all electrons of the same kind is an essential property, and fundamentally related to the constitution of the electron. From this point of view it does not seem easy to conceive of an alternative to the free type of strain-figure, which is imagined as existing in *unconstrained* equilibrium with the surrounding comparatively unstrained medium. As I have suggested previously*, if free strain-figures could exist in a uniform (infinitely fine-textured) medium, only a limited number of types of such strain-figures (say two for example) would in general be possible as permanent forms; the only feature remaining undetermined for each type being expressible as a single linear magnitude serving to define the scale of the strain-figure. To secure definiteness of scale we must suppose that there is some linear magnitude characteristic of the medium, and capable of influencing the equilibrium-conditions for a strain-figure; and apparently the only possible basis for the existence of such a magnitude is a certain coarseness of texture of the medium, or, in other words, constitution on a scale not infinitely minute.

20. As will be seen from the ideal genesis of a free strain-figure outlined in § 19 above, we assume in the case of the electron that a point is reached where the ordinary linear

* Phil Mag. Feb. 1892.

relation between electric displacement and electric force ceases to hold good; that, in fact, when a certain displacement is exceeded, increase of displacement may imply decrease of electric force, the electric force being even reversed for very high values of the displacement*. Now although it is usual, in the absence of definite knowledge, to assume linearity of the electromagnetic equations, even for the intense field in the immediate neighbourhood of an electron, this assumption is recognized as involving an extrapolation of no ordinary kind. Since it leads to something like definite dimensions for a negative electron whose charge and mass have been determined, we know at once what order of electric force is implied as existing in immediate proximity to the electron. With 10^{-13} cm. for the radius of the electron, and 5×10^{-10} for its charge in electrostatic measure, the field-intensity just adjacent to the electron would be $e/a^2 = 5 \times 10^{16}$ electrostatic units, or about 1.5×10^{19} volts per centimetre. Apparently, then, the assumption that electric force is proportional to electric displacement throughout the region surrounding an electron, implies that the proportionality holds good for field-intensities of this order of magnitude, which is many millions of millions of times greater than any intensity for which such proportionality has been experimentally established. It seems hardly probable that electric force would remain proportional to electric displacement if the value of either quantity could be increased without limit, and in supposing that a sensible deviation from linear relations occurs long before such an intensity as 1.5×10^{19} volts per centimetre is reached, no serious *a priori* difficulty appears to be encountered. It is even suggested that a free strain-figure theory, by postulating excessive strains and stresses as an essential of electronic constitution, gains rather than loses in plausibility. None the less it may be assumed that, for the relatively feeble values of electric force involved in radiation phenomena and in realizable laboratory experiments, linearity of relation holds good to an extremely high order of accuracy.

21. Moreover, it is to be observed, as a corollary to §§ 11, 12, that, to a first order, the elastic properties of a medium are not modified by the presence of assemblages of locked strain-figures; no property corresponding to dielectric

* When the strain has reached such a point that the electromagnetic relations are no longer linear, some further definition may be needed as to what precisely is meant by electric displacement and electric force. But even in the absence of such definition, the uncertainty in the meaning of the terms as here used is hardly a serious inconvenience.

capacity being thereby introduced. It appears, in fact, that for assemblages of electrons to be able to give rise to a dielectric capacity different from that of free æther, the constitution of the electron must be assumed to involve electric displacement so great that electric displacement and electric force are no longer linearly related. It is not to be supposed that this would involve (for example) non-linearity of the equations of optics: in radiation traversing free æther, as already explained, we are dealing with values of electric displacement which could still be well within the limits of sensibly linear relation; while the optical phenomena exhibited by (say) ordinary transparent media would fall within the category of small oscillations about a configuration of equilibrium, or about a state of quasi-steady motion, and would accordingly be deducible from linear equations. In this connexion it is not even necessary to take account of the circumstance that each individual electron, notwithstanding the non-linear strain-relations involved in its constitution, would obey (for moderate velocities) very simple laws of motion. As a rough general analogy it may be helpful to consider a thin spring laterally loaded to such an extent that the relation between the load and the resulting deformation is not even approximately linear. None the less the *small* oscillations of such a system about a position of equilibrium will be isochronous, and deducible from a system of linear equations.

22. Some progress has recently been made towards a static conception of atomic structure, and in relation to such theories especial significance attaches to questions regarding the possible non-linearity of electro-magnetic relations, with corresponding modification of the law of force between electrons placed in close proximity to one another.

LXII. *The Fatigue of Metals subjected to Röntgen Radiation.*

By LOUIS T. MORE, *Ph.D.*, *Professor of Physics in the University of Cincinnati* *.

SINCE the discovery of Hallwachs that metals grow less sensitive to ultra-violet light after long exposure to the rays, many attempts have been made to determine the cause and laws of this "photo-electric fatigue." Some discrepancies in the results still exist, but there is general agreement that it varies with the kind of metal used, the state of the surface, the gas in which the metal is placed, and the character of the rays. We know so little about the nature

* Communicated by the Author.

of a metallic surface and of the condition of adherent gaseous layers, that it is natural to expect discrepancies where these are controlling factors.

As a summary of the work is given in Professor Thomson's 'Conduction of Electricity through Gases,' page 289, it is necessary only to supplement this by reference to what has appeared since. Ramsay and Spencer* have attempted to prove that this fatigue is conditioned by the disintegration of the metallic atoms. Allen† has solved one difficulty by proving that fatigue occurs in a good vacuum, and has added the new and important fact that the fatigue of zinc can be expressed as a sum of two exponential terms. He also shows that the longer waves possess the property of producing a partial recovery in the activity, the exponential decay having been obtainable only when they were absorbed. It is to be hoped he will continue the work to include the details of this important property. In addition to this new matter, Hallwachs‡ has continued his researches and published them in an elaborate and comprehensive paper. In the first part he considers the various hypotheses in detail, and apparently on good evidence rejects the conclusions of Ladenburg§ that the fatigue is caused by oxidation or corrosion of the metal, and proves, contrary to Buisson|| and Kreisler¶, that darkness does not prevent fatigue. In the second part of the paper he investigates the cause of the effect. Here he confirms the theories of Lenard** and Buisson, that the fatigue is due in part to absorption of the radiation from the metal by electrified double-layers and films of gas on the surface, modified in some way by the ultra-violet light. But he assigns the dominant influence to the presence of ozone in these absorption-layers. However, it has yet to be proved that minute quantities of ozone possess such an abnormally large absorptive power on the cathode rays issuing from the metal. He also shows that the activity may increase to a maximum value before fatigue begins.

As Röntgen rays excite a secondary radiation in metals we may expect to find, in this case, effects of fatigue; and the experiments described in this paper prove that this type

* Ramsay and Spencer, *Phil. Mag.* xii. 1906, p. 397.

† H. S. Allen, *Proc. Roy. Soc. A*, lxxviii. 1907, p. 483.

‡ Hallwachs, *Sitzungsber. der Ges. der Wissensch. zu Leipzig*, lviii. 1906, p. 341.

§ Ladenburg, *Annal. der Phys.* xii. 1903, p. 558.

|| Buisson, *Ann. de Chim. et de Phys.* xxiv. 1901, p. 320.

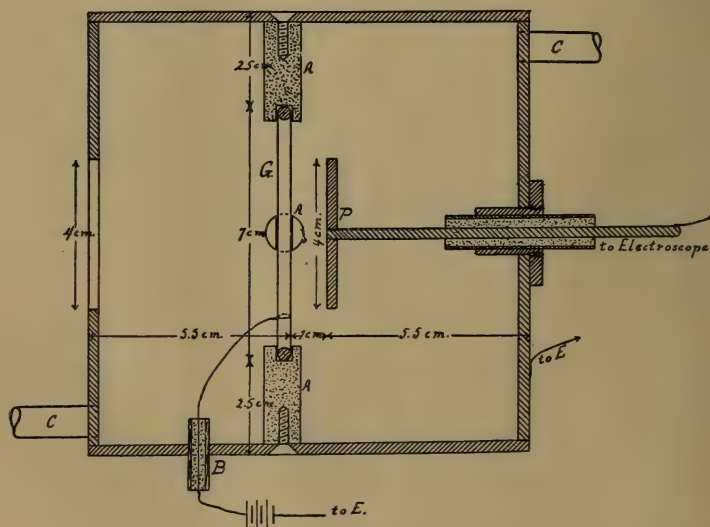
¶ Kreisler, *Annal. der Phys.* vi. 1901, p. 398.

** Lenard, *Annal. der Phys.* viii. 1902, p. 196; xii. 1903, p. 490.

of secondary radiation is variable; the changes, however, differ in many respects from the photo-electric effect.

Until we can devise a method of obtaining a constant source of Röntgen radiation, we are limited to a choice of comparative measurements. Another complication which makes accuracy difficult, is the impossibility of making an induction-coil work steadily for the length of time employed in photo-electric work. So the attempt was made to measure such small effects as might take place during a few hours of exposure. The method adopted to eliminate irregularities of coil and bulb was a comparative one. The secondary radiation excited by the Röntgen rays in two similar plates was measured. One of them, which I shall call the standard plate, was then screened from the rays that continued to fall upon the other, or trial plate. At intervals afterwards the secondary radiation from both plates was observed. Variations in the ratios of the radiation from the fresh plate and the other exposed continuously to the rays, are a relative measure of the effect of the Röntgen rays on the trial plate. To have values accurate as well as characteristic, a coil was chosen which worked with steadiness, and the bulb was provided with a regulator which, with gentle heating, kept the rays soft and as uniform as possible.

Fig. 1.

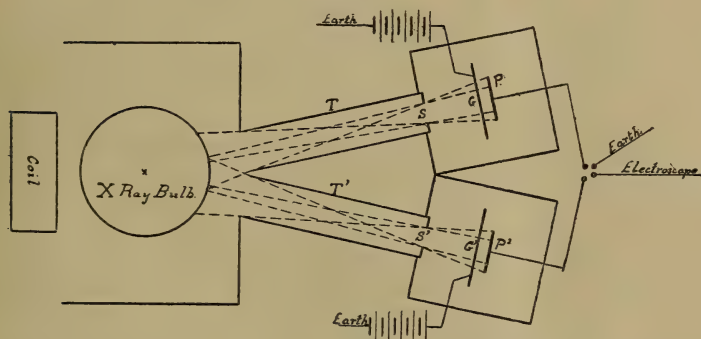


The apparatus adopted is shown in fig. 1. Two exactly similar brass boxes, 12 cms. in each direction, served as

ionization-chambers. The Röntgen rays entered each box by a circular window in the front face and, after passing through a disk of iron-wire gauze, impinged on a circular disk of the metal to be tested. The gauze disk, *G*, soldered to an iron ring, was held in place by three slotted pillars of ebonite, *A*. Wires, which entered the boxes through sulphur insulators, connected the gauze disks to the positive pole of the same battery of a hundred cells. The metal plate, *P*, placed a centimetre behind the gauze disk, was joined to a Wilson gold-leaf inclined electroscope by a rod which was insulated from the box by a sulphur plug. The tubes, *C*, served to admit various gases, and when air was not used, thin sheets of aluminium closed the windows in the boxes.

These boxes were mounted (fig. 2) so that horizontal and

Fig. 2.



approximately equal beams from a Röntgen-ray bulb, *X*, entered their windows. Spring-lever keys operated by cords were arranged to connect the experimental plates, *P* and *P'*, separately to the gold-leaf of the electroscope, to earth, or together. To measure the ionization-currents created by the secondary radiation between the condenser-plates, the electroscope was charged to 200 volts and was tilted at an angle to give a motion of the leaf of about forty divisions in the microscope-micrometer in half a minute.

As is usual, the bulb and coil were screened by a lead-sheathed box and all electrical connexions were encased in metal tubes, connected to earth. At *S* and *S'*, in front of the windows, lead shutters were hung, which could be raised or lowered by cords. When closed no appreciable ionization was produced in the boxes.

In the earlier part of the work, the rays were allowed to fall not only on the experimental plates but also on a guarding or the inner surfaces of the box. But it was found that

the secondary radiation excited in these latter caused serious errors. To eliminate them, the brass tubes T, T' were mounted in such a fashion as to stop all rays but those which struck the experimental plates.

The following preliminary steps show the reliability of the arrangements :—

1. When the electroscope was disconnected from the plates, rays from the bulb, within or outside the boxes, caused no appreciable motion of the leaf during a minute.

2. With the lead shutters closed and the electroscope connected to the plates, the rays produced no observable ionization in the boxes.

3. Since the Röntgen rays passed through the wire gauze and the air-space between it and the metal plate, a test was made to determine the secondary radiation from the gauze. First, the ratio of the ionization-currents in the two boxes was measured, using lead plates and iron gauze in both. One of these gauze disks was then replaced by a similar one of copper, and the ratio again found. As the two ratios remained the same, the action of the rays on the gauze can be neglected. As an additional proof, the ionization-current was measured, using a lead plate; this was then replaced by a disk of the same size made of very thin aluminium-foil mounted on a fine brass wire. The two observations showed that only about one per cent. of the first ionization remained, and as that was partly due to secondary radiation from the aluminium-foil, the direct ionization of the Röntgen rays and that due to the secondary radiation of the gauze must be less than one per cent. and so negligible.

The effect of the Röntgen rays was tried on iron, lead, nickel, zinc, copper, and aluminium, both with old and with polished surfaces in air. And, in addition to air, the gases hydrogen, coal-gas, and ozone were used. The results show that the secondary radiation excited by these rays depends on the length of exposure, the kind of metal, its surface, and the gas surrounding it; but the changes differ in many respects from the effects produced by ultra-violet light.

In the tables the column "Exposure" shows the period the trial plate was exposed continuously to the rays; the column for each metal gives the ratio of the ionization-current from the trial plate to that of the standard. In each case the initial reading of this ratio is expressed as 100; the per-cent. change in the amount of the secondary radiation from the trial plate can thus be found at once by subtracting any subsequent value from the initial reading. The ionization-

current was determined by noting with a stop-watch the time which the gold-leaf of the electroscope required to move over a given number of divisions of the microscope-micro-meter. The reciprocal of this time is a measure of the current produced by the secondary radiation of the metal. Alternative readings were made with the standard and trial plates and an average taken of the reciprocals for each plate. The ratios of these averages are the values given in the first column for each metal. A decrease in the number therefore means less radiation, and an increase, a greater radiation from the trial plate.

If the coil or tube worked irregularly while any one set of readings was taken, the values for the standard plate did not agree, and such a set could then be discarded. Fortunately this happened very seldom. Variations in the tube between two sets of readings did not affect the character of the change taking place in the trial plate, although they would alter the absolute value of the per-cent. change.

TABLE I.
Unpolished Metal Plates in Air.

Exposure.	IRON. Ratio to Standard Plate.	LEAD. Ratio.	NICKEL. Ratio.	ZINC. Ratio.	COPPER. Ratio.	ALUMINIUM. Ratio.
0 hours	100	100	100	100	100	100
1 "	91	...	97	97	100	99
2 "	...	93	95	94	99	96
3 "	90	94	92	91	98	98
4 "	...	95	...	...	...	...
5 "	...	...	...	...	...	...
6 "	...	94	...	...	...	...
Rest	...	...	...	...	...	...
30 min.	100	...	...	94	98	...
45 "	...	100	...	...	...	...
18 hours ...	...	...	97	99	...	...

The examples chosen are characteristic of the different metals when the surface has been allowed to age in air for some months, and they are not at all exceptional in uniformity of the readings. They show, with the possible exception of copper and aluminium, a decided falling-off in the power of giving off secondary radiation. This fatigue is, however, a

temporary effect, as the plates recover their original activity if the Röntgen rays are stopped for a time. Iron and lead required a rest of less than an hour, but the effect on nickel and zinc was more lasting, as they still showed some fatigue after resting until the following day.

Polished Metal Plates in Air.

The same plates were then polished with emery and with silk to a mirror surface and tested as soon as possible, allowing only time enough, before taking the first set of readings, for the plates to lose the heat which might have been developed by the friction. This delay, I think, is important, since if the fatigue is at all dependent on surface-films of gas and gas occluded in the metal, these would be strongly affected by changes of temperature.

TABLE II.

Exposure.	IRON ₁ . Ratio to Standard Plate.	IRON ₂ . Ratio.	LEAD. Ratio.	NICKEL. Ratio.
0 hours	100	100	100	100
1 "	105	106	103	98
2 "	103	108	105	101
3 "	102	109	101	98
4 "	...	109	101	...

The effect of the Röntgen rays is here just the reverse, as the secondary radiation for iron and lead rises to a maximum. Unfortunately, the experiments could not be continued long enough to see whether the ratios finally became less than the initial value. While nickel does not show this increase in radiation, it remains practically unaffected and does not fatigue as it did before the surface was polished.

In the following experiments the trial plates were immersed in gases, at atmospheric pressure, other than air.

Polished Plates in Coal-gas.

Both the trial and standard plates were well polished and placed in their respective boxes. The windows of both were closed with aluminium-foil free from holes, and the box containing the trial plate was sealed air-tight with wax. A slow

stream of coal-gas was then passed through it for at least a day and allowed to stand over night and started again. The gas entered by the lower tube, C, and issued from the higher level. During the test a very slow stream of the gas flowed through this box, as it seemed to be the best method of keeping the atmosphere uniform and pure, as occluded gas issuing from parts of the box would be removed.

TABLE III.

Exposure.	IRON. Ratio to Standard Plate.	NICKEL. Ratio.
0 hours	100	100
1 "	...	...
2 "	105	98
3 "	106	99
4 "	104	98
5 "	...	...
6 "	105	...

These results are similar in character to those for polished surfaces in air.

Polished Plates in Hydrogen.

The effect of immersing the plate in hydrogen was tried with iron, as this metal gives apparently the most characteristic results. The hydrogen was generated from pure zinc and sulphuric acid and passed through drying-tubes before entering the test-box. Considerable difficulty was experienced in getting rid of traces of air. The gas passed through the box for four hours in a slow stream, entering near the top and issuing from the bottom, and the plate was then allowed to stand all night in the atmosphere of hydrogen. In the morning, the stream of gas was again sent through the box continuously until two o'clock when a test was made. During the test also the stream, reduced in quantity, continued to flow.

It is essential for good results to drive out all traces of air from the trial plate and to keep the standard plate in air. These precautions are necessary because the ionization currents produced in hydrogen and air by the same secondary radiation are not at all equal. So, for example, we might suppose a

constant secondary radiation to be set up by the trial plate in hydrogen, and at the same time the quantity of air remaining in the box to change slightly and slowly. Then we should have all the appearance of a change in this constant secondary radiation. Or, if the quantity of air changed quickly, we should naturally attribute the consequent change in the ionization current to irregularity in the working of the Röntgen-ray bulb. But if we keep the standard plate in air, the latter error would be correctly interpreted by the constancy of the readings of its ionization current. The former difficulty cannot be certainly prevented. The best I could do was to make it strongly probable that slow changes in the gas did not occur. This was done by taking a set of readings for both the standard and the trial plates. Immediately afterwards, the lead shutters on both boxes were closed and remained so for half an hour. If another set of readings for both plates were then taken, the ratio should be the same as before. Previous to beginning a test for fatigue, this process was repeated two or three times at half-hour intervals, or until the ratios did remain constant within the limit of accuracy of reading. In some cases it was necessary to continue the flow of gas for two or three days before a steady state was reached.

TABLE IV.

Exposure.	IRON ₁ . Ratio to Standard Plate.	IRON ₂ . Ratio.
0 hours	100	100
1 „	94	103
2 „	95	100
3 „	93	101

The behaviour of the two samples of iron is different, as the first shows a slight fatigue and the second a small increase in activity. But I am led to believe that the effect of the hydrogen is practically to prevent change in the activity. The separate readings of the ionizations in the trial box, in spite of great care, were not so accurate as in the other experiments. And this was to be expected for the reasons given above. Even if the preliminary experiments remained constant and indicated that all the air was expelled, yet it is very possible that long continued use of the Röntgen rays may

affect surface-films of air or assist in liberating air occluded in the metal. This opinion is supported by the fact that in neither case was there a regularly increasing change with the time. The first reading for Iron₁ and the second value for Iron₂ are the only ones which show a decided change.

Effect of Ozone on Polished Plates.

Hallwachs* finds in his experiments on photo-electric fatigue that ozone in small quantities has an extraordinarily large influence. For example, a copper plate had its sensibility to these rays diminished as much in seven seconds when 0.01 of one per cent. of ozone was added to the air as would occur in a year if the ozone were absent from the air. We should expect that ozone would have also a comparatively large effect on the activity produced by the Röntgen rays. But no abnormally large effect could be found.

Ozone was generated by sparking between two coaxial cylinders, and air mixed with the ozone was sucked through the box by means of a water-pump. For fear the ozone might dissociate if allowed to stand, the stream of ozonized air was maintained during the entire experiment. A test for ozone was made every hour on the air issuing from the box, by passing it through a solution of potassium iodide and starch, and a strong reaction occurred each time. When the experiment was completed, the air in the box also smelled strongly of the ozone.

The first readings were taken with both plates in pure air. Fifteen minutes later the ozone was added to the air surrounding the test-plate.

TABLE V.

Exposure.	IRON. Ratio to Standard Plate.	COPPER. Ratio.
0 hour	100	100
$\frac{1}{2}$ „	97	95
1 „	97	97
2 hours	97	96
3 „	98	...

* Hallwachs, *loc. cit.* p. 381.

A comparison of these results with those for the same plates in pure air, shows that the presence of ozone in small quantities does increase the fatigue. The difficulties explained when hydrogen was employed do not apply here, and the separate readings were quite accurate. Fresh surfaces of iron in air experienced an increase in activity, and copper showed little or no change, but on the addition of ozone there results for both a small fatigue. But evidently no such abnormal effects occur as Hallwachs found for ultra-violet light.

Summary of Results.

The preceding experiments prove that all the metals tried, with the possible exception of copper and aluminium, experience a fatigue when subjected to Röntgen radiation for a long period of time, if their surfaces are not polished. The greatest diminution in activity was found in iron and zinc, and amounted to about 10 per cent. for a three-hour exposure to the rays. The fatigue is not a permanent effect, as the plates recovered their original activity after a comparatively short rest.

When iron and lead are freshly polished, the secondary radiation increases and reaches a maximum value after an exposure of an hour or two; the behaviour of nickel is different, as there was no regular change in its radiation during an interval of three hours.

The effect of immersing polished plates in coal-gas, which would probably form sulphides on the surface, is not noticeably different from the action in air. An inert gas, like hydrogen, apparently tends to prevent changes in activity. Lastly, when ozone is added to the air surrounding polished iron and copper plates, a considerable fatigue occurs, but not comparable to the effect noted by Hallwachs.

The explanation of these results is decidedly in favour of the fatigue being caused by changes of the gas in or on the plates, rather than to any change in the metals themselves. It is impossible as yet to decide definitely whether this effect on the adjacent gas-films is caused by an increased or diminished power of absorption of the secondary rays, or whether it is due to the formation or destruction of electrified double-layers.

But the experiments indicate that a modification of an electrified double-layer by the rays is the more probable of the two. A metal whose surface has stood untouched for months, diminishes in its secondary radiation from the beginning of the bombardment by the Röntgen rays. Now we must suppose that what condensed gas or double-layers may have formed on the plate during the preceding months, will be in such a sufficiently stable state as not to change appreciably

of themselves during the three hours or so of the experiment. Consequently the fatigue must be due to a modification in these films by the Röntgen rays. We do know that the contact potential of the metal changes during exposure to ultra-violet light and, presumably also, to Röntgen rays, and this change of potential is best explained as a modification of an electrified double-layer which has formed on the surface; but we have no evidence that the rays increase the density and absorbing-power of condensed gas-films. The fact that freshly polished plates increased in radiating power for an hour or so, points even more strongly to the above conclusion. The effect of polishing is undoubtedly to remove, at least partially, such double-layers and gas-films. During the first part of the experiment the films are forming and the radiation increases. The formation of an electrified double-layer may easily have such an effect, but it is extremely difficult to see how the absorptive power of a gas can diminish while it itself is growing denser. After the maximum value is reached and the radiation begins to decrease, we have the same conditions discussed above.

As these experiments are the first attempt to show the fatigue of metals due to Röntgen rays, they can be compared only to the similar action of ultra-violet light. Of the investigators of the latter effect, Lenard ascribes it to the formation of electrified double-layers, and Buisson to an absorption of the rays by occluded gas. Hallwachs, who finds these insufficient to account for all the phenomena, assigns the dominant action to an abnormal absorptive power of ozone existing as a thin film close to the metallic surface. He claims that even so minute a quantity as $1/130,000$ of one per cent. of ozone in the air would fully account for the normal fatigue.

In a general way, my results agree with these views. And in particular I have observed at times an increased sensibility to the radiation corresponding to what Hallwachs found for certain freshly prepared plates previously kept in closed vessels for a considerable time. It is rather surprising there should be such an agreement. The ultra-violet radiation consists entirely of easily absorbed rays, while the secondary radiation excited by Röntgen rays contains not only cathode rays but also a proportion of penetrating rays. The presence of these latter tends to complicate the results, as they would not be absorbed easily by surface-films of gas. Since Sagnac* found that the secondary radiation emitted by a light substance, such as aluminium, contains a large proportion of penetrating rays, we should expect its radiation to be the most unvarying

* Sagnac, *Ann. de Chim. et de Phys.* xxii. p. 493 (1901).

if the fatigue were due to absorption by gas. Also the presence of these penetrating rays may be one reason why ozone has a less influence.

The other theory advanced to explain the phenomena is that a change occurs in the metal itself. Ladenburg suggested chemical corrosion of the surface to be the cause, but the recent work of Hallwachs makes this hypothesis doubtful. Then we have the theories of Allen and of Ramsay and Spencer, that the radiation of metals is due to an actual disintegration of the substance. In some way, the ultra-violet light releases the atomic energy and causes the atom to explode. So if we begin with zinc (A) we soon get a disintegration product zinc (B), then zinc (C), &c. It is rather unfortunate that so fundamental an assumption as the transmutation of the elements should be advanced except as a last resort and with very convincing proofs. And certainly in the present case such proof is lacking. In the first place, the fatigue is not a permanent change and metals recover their primitive efficiency. Then, too, Allen finds that the change from zinc (A) to zinc (B) occurs in a few minutes. If such a change takes place so quickly, we should expect to find some evidence of this new product on the surface of metals which have been exposed to sunlight for years. But the facts discovered by him, that the fatigue progresses rapidly at first and then at a slower rate and that the action may be expressed as the sum of two exponentials, are important and interesting. But a less radical assumption would satisfy his exponential equation.

Ramsay and Spencer go much further. They suppose the ultra-violet light to produce a disintegration of the atom accompanied by the ejection of a corpuscle, and that this process is repeated until the number of corpuscles emitted is equal to the valence of the element. The proof of this circumstantial disintegration rests solely on the fact that their curves of radiation and time show breaks, or all their measurements do not lie on a smooth curve. When we consider the difficulty of keeping a source of ultra-violet light constant for hours at a time and the errors liable to occur when measuring the radiations from the plates, it is not surprising there are irregularities in the curves. An inspection of them does not carry conviction that successive disintegrations of matter under normal conditions are involved. Nor do the authors prove any extraordinary precautions to prevent errors were taken. In fact they do not explain why other points which lie at some considerable distance from the curves drawn (see particularly curves III. and IV., plate vi., and curve VIII., plate vii.) do not constitute points of disintegration.

After the exhaustive and rigorous treatment of the subject

by Hallwachs, in addition to the work of his predecessors, there seems little doubt that the simpler idea of changes in the absorptive power of electrified double-layers and other films of gases of great density near the surface of the metals, is the probable cause of fatigue. And the experiments described in this paper show no reason for not accepting the same hypothesis to explain the changes occurring in this type of secondary radiation.

I take great pleasure in thanking Professor Thomson for his advice and for his many courtesies to me during my visit to the Cavendish Laboratory.

Cavendish Laboratory, Cambridge,
March 1907.

LXIII. *The Action of Uniform Electric and Magnetic Fields on Moving Electrons.* By A. H. BUCHERER, D.Sc., Bonn University*.

IN my paper on a new principle of relativity in electromagnetism (Phil. Mag., April), in which I have calculated the forces acting on moving electrons in the uniform fields of electromagnets and of condensers, I have omitted to call special attention to the circumstance that the expressions obtained do not apply to all kinds of uniform fields. In fact, as will be evident at once from the reasoning adopted by me, *i.e.*, the forces on the electrons primarily depend on the distribution of electric and magnetic masses, and *indirectly* only on the intensities of the fields in which they are moving. Thus electrons moving in various uniform fields of the same intensity will in general experience different forces if these fields are due to different distributions of electric and magnetic masses, special care being requisite with solenoids which *must* be replaced by equivalent magnetic shells.

Closely connected with this is another important consequence. Let an electron move in the uniform superimposed fields H_0' and H_0'' of two electromagnets. Then, to find the resultant force, it is not permissible to substitute the resultant field H_0''' in the expression given by me. For evidently

$$\frac{VH_0'u}{1 - \frac{u^2}{v^2} \cos^2 (H_0'u)} + \frac{VH_0''u}{1 - \frac{u^2}{v^2} \cos^2 (H_0''u)} \neq \frac{VH_0'''u}{1 - \frac{u^2}{v^2} \cos^2 (H_0'''u)}.$$

I am preparing experiments with Becquerel rays in order to test the formulæ derived by me from the principle of relativity.

Bonn University, April 1907.

* Communicated by the Author.

LXIV. *Temperature of Inversion of the Joule-Kelvin Effect for Air and Nitrogen. Preliminary Communication.* By K. OLSZEWSKI*.

IN a paper published by me five years ago†, I had determined the inversion temperature of the Joule-Kelvin effect to be $-80^{\circ}.5$ for hydrogen; this figure has proved to be of fundamental importance in the design of liquefying apparatus‡ for this gas. The paper mentioned has also attracted the attention of physicists, and A. W. Porter§ has made use of it in a theoretical investigation of the accuracy of van der Waals's and Dieterici's state equations, based on my determination of the inversion temperature. On account of the great theoretical importance of such determinations, I have decided to carry out similar measurements for other gases, and above all for air and its more important components. Up to the present, I have only completed the experiments on the inversion temperatures for air and atmospheric nitrogen, and take the opportunity of presenting the results in a brief notice, reserving a detailed description of the experimental arrangements and the apparatus for a later communication. I only wish to remark that the apparatus used did not differ in principle from that employed by me five years ago, but that in view of the high temperature (up to 300°) at which the experiments on air and nitrogen had to be carried out, considerable modifications in the details as well as in the dimensions became necessary.

As Witkowski, as far back as 1898 ||, and, last year, Porter§, on theoretical grounds came to the conclusion that the inversion temperature of the Joule-Kelvin effect for gases was a function of the pressure, I have in the present experiments devoted special attention to the initial pressures of the gases which were made to undergo an irreversible expansion.

The apparatus was heated in an oil-bath. Temperatures were measured with a high-reading mercury thermometer; but for the purpose of determining the small temperature changes which occur during expansion, an iron-constantan thermo-couple was used, whose sensitiveness was about $0^{\circ}.2$ per mm. of galvanometer scale.

The gas, which had been compressed to the initial pressure p , was allowed to expand down to one atmosphere. The experiment was repeated a number of times under these conditions, the temperature of the gas being gradually reduced

* Communicated by the Author. (*Bull. Acad. Crac.* Nov. 1906.)

† K. Olszewski, *Bull. Ac. Crac.* 1901, p. 453; *Phil. Mag.* [3] p. 535 (1902).

‡ K. Olszewski, *Bull. Acad. Crac.* 1902, p. 625, and 1903, p. 241.

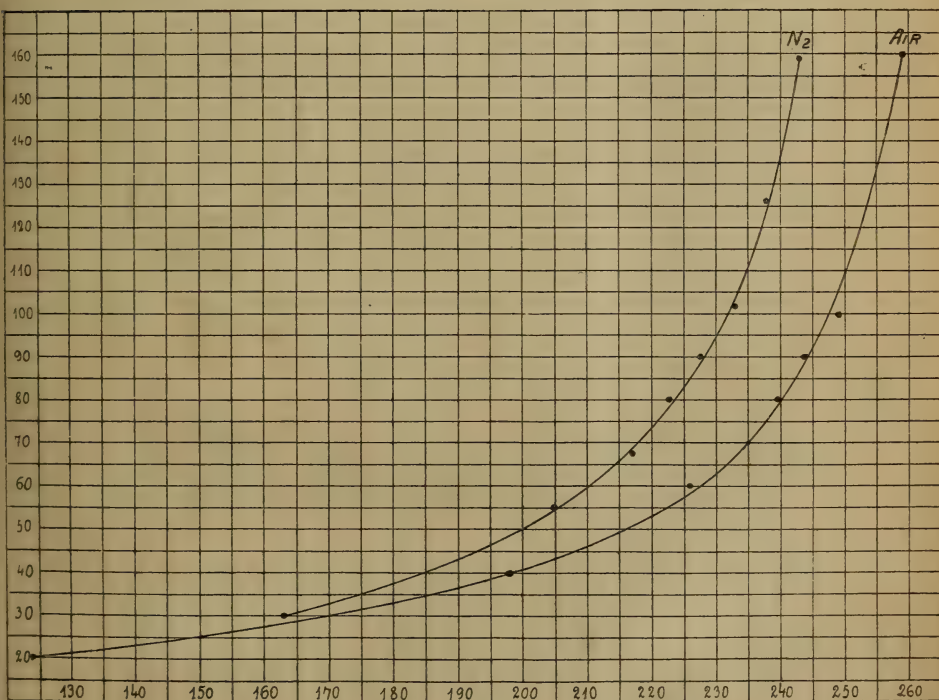
§ A. W. Porter, *Phil. Mag.* [6] xi. p. 554 (1906).

|| A. W. Witkowski, *Bull. Acad. Crac.* 1898, p. 282.

from $+300^{\circ}$. Above a certain temperature t_i the thermocouple showed a heating of the gas, below this temperature an unmistakable cooling, and at the temperature t itself the integral result of the Joule-Kelvin effect was zero.

In the following table are given the initial pressures p (expressed in kg. per sq. cm.) and the corresponding temperatures of inversion t_i .

AIR.		NITROGEN.	
p .	t_i .	p .	t_i .
160	$+259^{\circ}$	159	$+243^{\circ}$
100	249	126	238
90	244	102	233
80	240	90	228
70	235	80	223
60	226	68	217
40	193	55	205
20	124	30	163



In the accompanying figure the experimental results are represented graphically, the initial pressures p (kg./sq. cm.)

being used as ordinates and the corresponding inversion temperatures t_i as abscissæ. The curve thus represents the relation connecting the inversion temperature of the gas under consideration with its initial pressure. The curves fully confirm the conclusion of Witkowski and Porter that the inversion temperature is a function of the pressure. The values of the inversion temperature for air calculated by Witkowski by the aid of the empirical formula given by Rose-Innes ($+360^\circ$), and by means of that of van der Waals ($+500^\circ$) are, it is true, somewhat high when compared with those obtained by myself; but it must be remembered that the second value ($+500^\circ$) was calculated on the assumption of a small pressure-difference (1 atmo.), whereas my figures refer to the integral value of the Joule-Kelvin effect when a gas expands from a high pressure to 1 atmo.

In conclusion, I would refer to the connexion which appears to exist between the shape of the curve for air and the behaviour of this gas in a liquefying apparatus. By means of the apparatus described by me in 1902*, which I am in the habit of using at lectures for the demonstration of the liquefaction of air, it is easy to show that liquefaction takes place only so long as the initial pressure does not fall below 80 atmos.; a further expansion from pressures which are below 80 atmos. is entirely without result. From the accompanying figure it is seen that precisely at the point corresponding to a pressure of 80 atmos. the curve takes a sharp bend, and that at this point a sudden drop takes place in the temperature of inversion, whereby also the cooling effect rapidly decreases, the behaviour of air beyond this point approaching in this respect more and more nearly that of hydrogen, whose inversion temperature is very low.

Chemical Institute of the Jagellon University, Cracow.

LXV. *On the Isopiestic Temperatures of Saturated Vapours of Different Substances.* By ALFRED W. PORTER, *B.Sc., Fellow of, and Assistant Professor of Physics in, University College, London* †.

IN the *Philosophical Magazine* for Jan. 1886 Ramsay and Young published a law (known by their names) connecting the temperatures at which different saturated vapours have the same vapour-pressure. The law has been employed in two somewhat different forms. One only of these forms

* K. Olszewski, *Bull. Acad. Crac.* 1902, p. 623.

† Communicated by the Author.

is consistent with a general relation connecting vapour-pressure and temperature. This form only will be referred to here, and it may be stated thus :—

If T and T_0 be the temperatures at which two substances have the same vapour-pressure, and T' , T_0' be two temperatures at which the same substances have another value of pressure the same for both, then

$$\frac{T_0}{T} - \frac{T_0'}{T'} = c(T_0 - T_0'), \text{ where } c \text{ is a constant.} \quad (1)$$

Hence if $\frac{T_0}{T}$ be plotted against T_0 a straight line will be obtained.

In the *Phil. Mag.* for September 1902 (p. 337) the late Prof. J. D. Everett subjects this law to the criticism that it is equivalent to neglecting the term $\frac{\gamma}{T^2}$ in Rankine's vapour-pressure equation

$$\log p = \alpha - \frac{\beta}{T} - \frac{\gamma}{T^2}.$$

This is shown by writing equation (1) in a form equivalent to

$$\frac{b}{T} - \frac{b_0}{T_0} = 1,$$

and then pointing out that Rankine's shortened formula leads to this equation: for the shortened formula gives

$$\log p = \alpha - \frac{\beta}{T} = \alpha_0 - \frac{\beta_0}{T_0},$$

whence

$$\frac{\beta}{T} - \frac{\beta_0}{T_0} = \alpha - \alpha_0.$$

It would seem, however, that the criticism that the law will hold only when squares of the reciprocal temperatures can be neglected, is not justifiable; for it rests on the assumption that we are restricted to Rankine's form of equation.

In order that Ramsay and Young's law shall be accurately true, all that is necessary is that

$$f(p) = A - \frac{B}{T},$$

where the left-hand side may be any function of the pressure which is identical for the substances in question.

Now in his Treatise on Thermodynamics Bertrand showed that the equation

$$p = G \left(\frac{T-a}{T} \right)^n; \quad \text{or} \quad p^{\frac{1}{n}} = g \left(1 - \frac{a}{T} \right)$$

is a very accurate expression for representing the vapour-pressures of most substances. In the case of water vapour the best values of the constants are

$$n = 42.855 \quad \text{and} \quad a = 88.2;$$

the error in the temperature corresponding to any given pressure is less than $\cdot 3^\circ \text{C.}$ for all temperatures between -30°C. and $+230^\circ \text{C.}$ A formula with only $\cdot 5^\circ \text{C.}$ temperature error is obtained by taking $n = 50$ and $a = 78.3$, and this value of n gives practically the same accuracy for numerous other substances tested.

Such a formula is not as accurate as Rankine's complete formula, but it is much more so than the shortened formula as the following table shows. In column 3 are shown values of the pressure of water vapour calculated from the formula $\log_{10} p = 8.49375 - \frac{2093.6}{T}$; in column 4 the differences from Regnault's experimental values; while columns 5 and 6 contain similar values for Bertrand's formula ($n = 50$).

T.	p observed.	Rankine's Formula.		Bertrand.	
		p calculated.	$\Delta_{\text{R.}}$	Calculated.	$\Delta_{\text{B.}}$
273	4.6	6.7	+ 2.1	4.54	- .06
323	91.98	102.7	+10.7	93.07	+1.09
373	760	760	0	760	0
433	4651.60	4556	-95.6	4633.55	-18
473	11689	11682	- 7	11682.84	- 6
503	20926.4	21454	+528	21036.00	+110

Now Bertrand's formula carries with it the complete accuracy of Ramsay and Young's Law, for it is of the general form mentioned above. Since, moreover, it is far more accurate than Rankine's shortened formula, it follows that the Ramsay and Young law is also in general more accurate than Professor Everett's criticism would seem to imply.

LXVI. *Note on the Æther and on the Hypothetical Magnetic Flow.* By Sir OLIVER LODGE*.

IN connexion with my paper, on the Inertia of Æther per unit volume, in the April number of the *Phil. Mag.*, Professor Hicks, of the University of Sheffield, has kindly called my attention to his Address to Section A of the British Association in 1895 at Ipswich, wherein he gives some reasons for conjecturing an ætherial density not much less than that of water; and he likewise postulates for the æther an intrinsic mean square velocity about twice the speed with which it transmits light. I would have referred to these confirmatory speculations, in the Appendix to my paper, if I had remembered them.

Concerning a hypothetical ætherial flow along magnetic lines of force, suspected by some mathematical physicists—whence we get the term “permeability,”—the deduction from my series of experiments demonstrating the lack of all perceptible influence on the speed of light, as recorded in the paper above referred to, was to the effect either that the magnetic flow was very slow or else that it was actually zero.

My conception of the æther involves the supposition that as regards locomotion it is quite stationary, although full of constitutional rotational energy; and the hypothetical magnetic flow, though auxiliary, was not an essential part of my doctrine concerning the substantiality and extraordinary massiveness of every portion of the æther of space. So I am rather pleased to find in Professor Hicks’s Address a footnote containing a simple and ingenious proof that whatever be the kinetic nature of a magnetic field, it cannot consist of mere irrotational ætherial streaming. The argument may be paraphrased thus:—A conductor moving across a magnetic field generates an E.M.F. If the magnetic field itself were nothing but locomotion of æther along the lines of induction, then that motion could be compounded with the motion of the conductor, so as to be equivalent to a stationary conductor in a field inclined at some different angle. But in a stationary conductor no E.M.F. is generated. Therefore when a conductor moves so as to generate E.M.F. it must cut a *structure* of some kind, not a mere flow. The suspected magnetic streaming therefore—though not negatived—must be regarded as a superfluity, and becomes improbable.

University of Birmingham, May 1, 1907.

* Communicated by the Author.

LXVII. *Electrical Conductivity produced by Heating Salts.*

By A. E. GARRETT*.

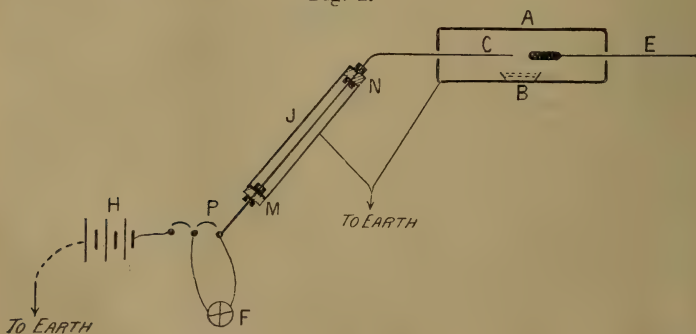
THE experiments described in the following paper form a continuation of those conducted by Dr. R. S. Willows and myself which were published in the Phil. Mag. for October 1904.

The experiments there described showed that increased electrical conductivity took place when the halogen compounds of zinc were heated to a temperature of about 360°C . This led us to think that other inorganic chemical compounds might possibly behave in a similar manner, so, in the first series of the present experiments, I tested a number of compounds to see if they increased the electrical conductivity when their temperatures were raised to about 360°C . In the second series of experiments I have confined my attention to a few special cases in which increased conductivity had been found to occur, in order to find out what are the causes of increased conductivity (more particularly so in the cases where the presence of the positive ions could be detected), and to find out the nature of the positive ions.

FIRST SERIES.

For these preliminary experiments the apparatus used is shown in figure 1.

Fig. 1.



A is an iron cylinder about 12 cms. in length and 5 cms. in diameter; caps of the same material partially close its ends. C is a platinum wire, connected to a wire passing through the centre of the metal tube J, and carefully insulated by means of ebonite plugs passed through the corks N and M. F is a quadrant electrometer; both pairs

* Communicated by the Physical Society: read April 26, 1907.

of its quadrants were first connected together and to the insulated pole of the battery H, as in figure 1. The quadrants were then disconnected, one pair being kept on the battery, the other pair to C. A leak of electricity from C is then shown by a constantly increasing deflexion of the electrometer-needle. The material to be tested was placed in the platinum dish B, which was usually placed in an iron dish so as to prevent those compounds which became liquid at the higher temperatures escaping on to the tube A.

The temperatures were ascertained by means of a nitrogen-filled mercury thermometer E reading to 360° C. The tube A and its contents were heated by a current passed round the tube through wires insulated from the tube and wound non-inductively.

In these first experiments the conductivity in cylinder A was measured at atmospheric pressure.

The following compounds when heated to temperatures not higher than 360° C. increased the rate of leak of both negative and positive charges from C except in the four cases noted.

- (a) *Chlorides* of iron (Fe_2Cl_6), aluminium (Al_2Cl_6), ammonium, magnesium, tin ($\text{SnCl}_2 + 2\text{H}_2\text{O}$), manganese, and cadmium (C negatively charged).
- (b) *Fluorides* of calcium, and aluminium (C negatively charged).
- (c) *Iodide* of cadmium.
- (d) *Nitrates* of ammonium, cadmium (C negatively charged), and cobalt (C negatively charged).

The following substances were tested, but no increased conductivity could be detected when using them:—

- (a) *Metals with low melting-point*.—Tin, lead, powdered bismuth.
- (b) *Chlorides* of copper, calcium, barium, strontium, lithium, potassium, and antimony.
- (c) *Iodides* of potassium, lead, and silver.
- (d) *Bromide* of potassium.
- (e) *Fluoride* of sodium.
- (f) *Oxides* of copper, zinc, tin, iron, calcium, and magnesium.
- (g) *Sulphates* of zinc, iron, copper, and magnesium.
- (h) *Carbonates* of zinc, magnesium, potassium, and sodium.
- (i) *Bicarbonate* of soda.
- (j) *Nitrates* of lead and barium.

In these experiments the conductivity was measured by means of a quadrant electrometer which gave a deflexion of

50-60 divisions for one volt. Hence it is quite possible that some of those cases in which no increased conductivity could be detected by such means, did produce an increase in the conductivity, but it was too small to be detected. *E.g.*, in some later experiments, when using a Dolezalek electrometer giving a deflexion of 600 scale-divisions for one volt, a considerable leak was obtained by heating lead iodide. It will be seen from the preceding list that the halogen salts of some metals when heated make the surrounding gas a conductor of electricity.

As no attempt at this stage was made to obtain anhydrous salts, the presence of water may have altered their ability to undergo chemical change, and in so doing may have altered their powers of producing conductivity.

Many of the chlorides which were found to be able to produce a leak of electricity are known to undergo chemical change when heated. It is probable that, at the moment chemical change takes place, an emission of an electrified particle occurs. In the case of the nitrates, the clean iron of the tube in which these compounds were heated was oxidized, and Beattie* and Schuster have both shown that this process causes the surrounding air to be positively electrified. As, however, in the case of ammonium nitrate an increased conductivity was observed when the electrode C was positively charged, this does not account for all the results obtained with those compounds.

As a general rule, the leak was larger when due to positive ions, and that leak could be detected in most cases at lower temperatures than when due to negative ions.

Later, some zinc phosphate was tested by heating up to 290° C. at a pressure of 10 mm. of mercury. Although the electrometer gave a deflexion of 700 divisions for one volt, no leak was detected. Potassium iodide was also tested up to temperature 260° C. at a pressure of 10 mm. of mercury. The electrometer gave a deflexion of 600 for one volt, but no leak was detected.

Thallium iodide was heated up to 210° C. at a pressure of about 15 mm. of mercury. The electrometer gave a deflexion of 350 for one volt, but no leak was detected.

The negative result with potassium iodide is of interest, as H. A. Wilson in his papers† on conduction by salt vapours when sprayed in solution into a hot tube, found conduction, even when measuring currents with a galvanometer, at temperatures near 270° C. (It is shown below that the leaks

* South African Association, vol. i. April 1903, p. 4.

† Phil. Trans. A. p. 296 (1901).

are not obtained in the absence of water vapour in some cases. If this is always so then dry KI would not give any leak.)

We hence conclude that in most cases definite chemical changes take place when increased conductivity is brought about.

SECOND SERIES.

I. *The Effect of Low Temperature on the Conductivity produced by Zinc Chloride.*

The zinc chloride was heated to 360° C. and the tube in which it was heated could be put into communication with the testing apparatus in two distinct ways, by means of glass tubes.

One of these glass tubes A passed direct from the part containing the heated zinc chloride into the testing vessel, the other B was bent, in between the two parts of the apparatus, to form a flat spiral. This spiral was immersed in solid CO_2 . The air was drawn over the zinc chloride and through A or B into the testing apparatus by means of a large glass aspirator so arranged as to produce an air current of about 10 cms. per second in the tube containing the electrode C (fig. 1).

(a) *Effect upon the positive ions.*—The rate of leak was first tested without aspirating, then when aspirating through A, and lastly when aspirating through B which was roughly equal in length to A. When the air was drawn through the flat spiral of glass immersed in the solid CO_2 contained in a beaker, the rate of leak was reduced to one-eighth the value it had when drawn through A. When drawn through A, a fog was produced in the aspirator like that previously noted by Dr. Willows and myself*, but when the air was drawn from the heating tube through B hardly any fog could be noted.

(b) *Effect upon negative ions.*—In this case the leak was reduced to $\frac{2}{3}$ the value when aspirating through B that it had when aspirating through A. The fog was much reduced.

The electrode C (fig. 1) was charged to 190 volts in both cases.

It is probable that the reason why more effect is apparently produced by the low temperatures on the positive ions is that the rate of leak obtained depended largely upon the velocity of the ions, as it was not possible under the conditions of the experiment to obtain a saturation-current.

* Phil. Mag. Oct. 1904, pp. 452-3.

The production of fogs would lead us to think that the ions are bodies of considerable size; in the case of the positive ions other evidence points to the same conclusion.

II. *Variation of the Leak with Changes of Temperature.*

Richardson has found * that the formula $I = a\theta^{\frac{1}{2}}e^{-\frac{b}{\theta}}$ can be used to express the relation between the saturation-current I and the absolute temperature θ in the case of a hot platinum wire and a metal cylinder surrounding it in a high vacuum.

I find in all the cases tried that a similar formula represents the relation between I and θ .

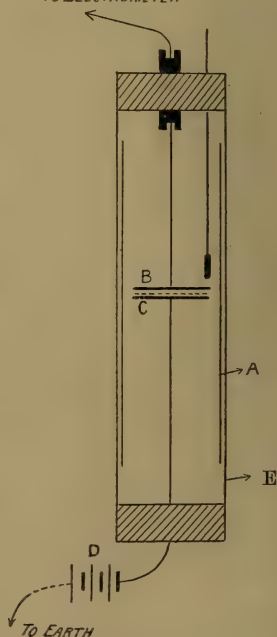
If logs. are taken of both sides of the equation, since a and b are constants it will be seen that if $\frac{1}{\theta}$ is plotted for abscissæ and $\frac{1}{2}\log_e\theta - \log_e I$ for ordinates, the result obtained should be represented by a straight line.

These earlier results were obtained with voltages which were too small to produce saturation-currents, although they were sufficiently large for the rate of leak to depend rather upon the total number of ions present than their velocity. In order therefore to ascertain whether Richardson's formula held for results obtained with sufficient voltage to produce a saturation-current, some zinc iodide was heated up and tested in apparatus in which saturation could be obtained. This is shown in fig. 2.

A is a brass tube earthed. B and C are brass plates, the material to be tested being placed on the lower of the two plates; the plates were separated by a few millimetres. D is a battery of storage-cells. E a glass tube fitted with corks and made air-tight.

The following results were obtained by this means for negative carriers :—

Fig. 2.
TO ELECTROMETER



* Proc. Camb. Phil. Soc. xi. p. 286 (1902).

Temperature in degrees Centigrade.	Leak with voltage sufficient to produce saturation.
180	25
190	70
202	130
204	170
214	400
222	500

These are found when plotted with $\frac{1}{\theta}$ for abscissæ and $\frac{1}{2} \log_e \theta - \log_e I$ for ordinates to give points which lie practically along a straight line. Zinc iodide was tested for negative ions up to temperature 255°C. with similar result.

From the formula $C = A\theta^{\frac{1}{2}}e^{-\frac{Q}{2\theta}}$ in which Q represents the amount of energy in calories associated with the production of one gramme molecular weight of the ions, it readily follows that $\log_e \frac{C_1}{C_2} = \frac{Q}{2} \left(\frac{1}{\theta_2} - \frac{1}{\theta_1} \right)$ where $\theta_1 - \theta_2$ is only a small interval of temperature.

Making use of this relation I obtained the following values of Q for ions obtained from the substances named. The average temperatures are stated in each case.

For positive ions.

Calcium fluoride (297°C.)	2.6×10^4 ,
Aluminium fluoride (330°C.)	2.9×10^4 ,
Ammonium nitrate (312°C.)	3.3×10^4 .

For negative ions.

Zinc iodide (241°C.)	2.9×10^4 ,
Iron chloride (355°C.)	6.1×10^4 ,
Ammonium chloride (352°C.)	5.0×10^4 ,
Calcium fluoride (346°C.)	6.0×10^4 ,
Ammonium nitrate (343°C.)	4.3×10^4 ,
Magnesium chloride (326°C.)	2.4×10^4 .

It will be seen that the values of Q obtained for the negative ions are in general larger than those obtained for the positive ions. A similar result for the ions produced by hot platinum has been noted by Richardson *, and the values here obtained are of the same order of magnitude as those obtained by him.

I found also on heating zinc iodide at a pressure of 2.5 mm. of mercury that the escape of positive ions could be represented by a similar formula, a saturation current being obtained.

* Phil. Trans. A. Nov. 1906, p. 61.

The following were the results obtained:—

Temperature in degrees Centigrade.	Leak with voltage sufficient to produce saturation current.
230	22
242	35
248	52
252	67
254	80
256	110
258	150
259	180

When these are plotted with $\frac{1}{\theta}$ for abscissæ and $\frac{1}{2} \log_e \theta - \log_e I$ for ordinates, it is seen that a decided break occurs in the straight line in the neighbourhood of temperature 250° C.

The probable reason for this is that some fresh source of ionization is tapped at this temperature. It was subsequently found that at reduced pressures, negative ions as well as positive could be detected at that temperature.

III. *Experiments on Halogen Salts of Zinc at reduced pressures.*

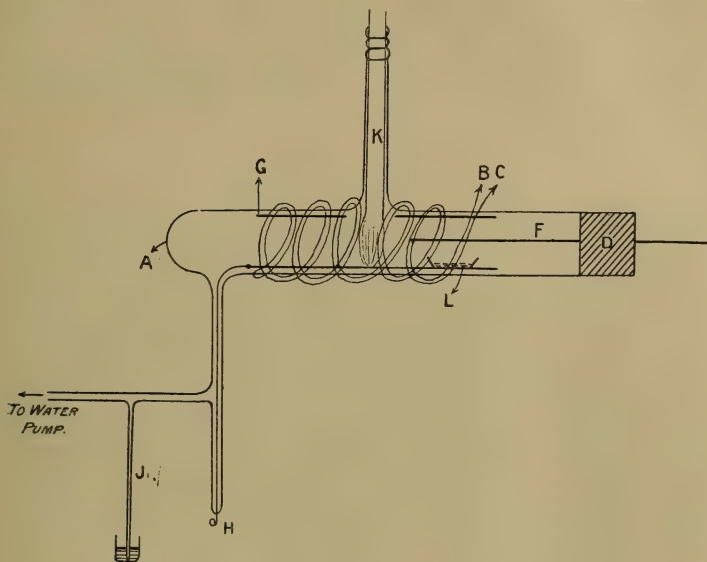
The apparatus now used is shown in fig. 3. A is a glass tube; B, C, wires for heating; D, ebonite stopper made airtight; F is the electrode; G, iron tube earthed by means of H; J is a gauge; K, thermometer; L, iron dish containing a small platinum dish in which materials to be tested were placed. Experiments with zinc bromide were tried with this apparatus. The pressure was reduced to 200 mm. of mercury, and it was then found that the leak with positive ions could be detected at somewhat lower temperatures than when the substance was heated at atmospheric pressures. The following is a typical set of readings obtained when the electrode F was charged negatively to a potential of 80 volts:—

Temperature, in degrees Centigrade.	Leak (scale-divisions).
195	7
207	17
216	33

From this it appears that extra leak can already be detected before 200° C. is reached.

" With a pressure of 220 mm. of mercury and an average temperature of $190^{\circ}\text{C}.$, a saturation-current was obtained.

Fig. 3.



The readings were all taken over an interval of one minute, and the temperature was nearly constant. Saturation was obtained with F charged to 160 volts when testing for positive ions. At still higher voltages a large increase in the leak took place. This is probably due to fresh ions being formed by collisions, since the voltages then used were nearly sufficient to produce sparking.

The apparatus was next arranged so that the temperature could be kept at a constant value during the observations, and the pressure was still further reduced to 20 mm. of mercury. The temperature was now $198^{\circ}\text{C}.$, and varied less than $1^{\circ}\text{C}.$ A saturation-current was now obtained in the case of the positive ions with a difference of potential of 110 volts. Hence at lower pressures a saturation-current is obtained with lower voltages.

It was found that the voltage required to produce saturation was greater in the case of the positive ions than of the negative ions—a result which can be explained if the positive ions move at a slower rate in the same electric field than the negative ions. This was afterwards proved to be the case.

In some later experiments with pure zinc iodide, it was found that at temperature 248°C ., at a pressure of 2.5 mm. of mercury a saturation-current could be obtained with a difference of potential of about 260 volts in the case of the positive ions, and with about 180 volts for the negative ions. The brass tube used in place of iron tube F (fig. 3) had an internal diameter of 1.4 cm., and the central electrode a diameter of .25 cm.

IV. *The Effect of Moisture.*

On one or two occasions, when a series of readings were required and the time at disposal was limited, some fresh material was placed in the apparatus while the latter was at a temperature between 250°C . and 260°C . On each occasion when using zinc iodide, it was noted that the rate of leak obtained was extremely small compared with previous results. From this it appeared that either the method of heating or the presence of water-vapour had an important effect on the rate of leak. Some zinc iodide was therefore prepared synthetically by heating pure zinc and iodine in distilled water until the colour of the solution disappeared, then evaporating to dryness, and leaving the residue in a desiccator for a week.

An addition was now made to the apparatus shown in fig. 3, in the form of a trap of concentrated H_2SO_4 between the tube A and the water-pump.

When the zinc iodide was placed in the apparatus and heated up slowly at a pressure of 30 mm. of mercury, the leak obtained was similar to that when commercial iodide of zinc was used under similar conditions.

When, however, some of this zinc iodide was placed in the apparatus while the latter was at 256°C ., no extra leak could be detected with pressures as low as 30 mm. of mercury.

The material was left in the apparatus for a week, care being taken to exclude moisture as far as possible. At the end of that time, with the temperature raised to 263°C . and the pressure reduced to 30 mm. of mercury, no extra leak could be detected. Hence moisture is necessary to make these salt vapours conductors of electricity.

Other effects of moisture were noted in the later experiments made to determine the velocities of the ions, and also in the experiments on the rate of decay of zinc iodide. These will be referred to in connexion with the parts named.

V. *Effect of the nature of the surrounding Gas upon the rate of Leak.*

The air was pumped out of the apparatus until the pressure was reduced to 20–30 mm. of mercury. The gas was then introduced, and the apparatus again pumped down to 30 mm. of mercury. More gas was then admitted, and the pressure again reduced; the process being repeated so as to get rid of as much of the air as possible.

The gases experimented with were hydrogen, oxygen, and carbon dioxide.

In the case of hydrogen the results obtained were somewhat irregular, although in the majority of cases a decreased rate of leak occurred in the presence of this gas.

No certain change in the previous rate of leak was noted in the case of either of the other gases tried.

VI. *Effect of strongly heating the Zinc Iodide previously.*

Some zinc iodide was placed in a piece of platinum-foil, and was then heated in the Bunsen flame until it was just melted.

After this, it was placed in the apparatus and tested as in the other cases. The rate of leak which was so produced was found to be of the same order as that produced by zinc iodide which had not been previously heated.

VII. *Experiments carried out with the Zinc Iodide at the ordinary temperature of the laboratory.*

These experiments required the use of an instrument of small capacity. A Wilson* sulphur-bead gold-leaf electroscope was therefore used. The gold-leaf system was enclosed in a brass tube, which was itself placed inside a glass vessel which could be evacuated at will. The pressure could be ascertained by means of a Macleod gauge.

(a) *At ordinary atmospheric pressure* an extra leak due to the positive ions was noted when zinc iodide was placed inside the electroscope. A small leak was also noted when the leaves were charged positively. On the average, it was found that with positively charged leaf the rate of leak was increased 30 per cent. when zinc iodide was placed in the electroscope.

(b) *At reduced pressures. Effect of a strong magnetic field.*—A series of observations was taken with zinc iodide in the apparatus, the leaves being positively charged and the pressures ranging from 70 mm. of mercury to 15 mm. of

* Proc. Roy. Soc. vol. lxviii. p. 152 (1901).

mercury. In all cases it was found that the time taken for the leaf to move over one scale-division could be increased in the ratio 8:10 by the application of a strong magnetic field, showing the emission of negatively charged particles from the zinc iodide to be taking place.

When the leaf was negatively charged, no alteration in the rate of leak could be detected (for pressures as low as 15 mm. of mercury) when the magnetic field was applied.

In these experiments with the magnetic field the part below the sulphur bead was shaped like **L**. The leaden box containing the zinc iodide was made long and narrow, and was placed with its long axis parallel to the lower limb of the **L** and at a distance of 1-2 mm. from it.

At pressures 2.54 mm. and .1 mm., when the leaves were positively charged, the time taken for the leaf to move over one division with the magnet off, to the time taken with the magnet on, was in the ratio 7:10. Practically the same results were obtained for both directions of the magnetic field; therefore they were not due to any dissymmetry in the apparatus.

It was noticed on some occasions that putting on a magnetic field caused a sudden discharge, such as would be caused by a spark. A similar effect with a Wehnelt cathode was shown by Prof. J. J. Thomson in a lecture at the Royal Institution a year ago.

E. g., when the pressure was reduced to .015 mm., and the magnetic field was put on, the leaf was instantly discharged if it were positively charged above a certain potential.

When the leaf was negatively charged under the same conditions, a partial discharge takes place when the magnetic field is first put on, and then no further discharge takes place when the magnetic field is again put on.

There also appears to be a distinct lag when the leaf is negatively charged between the putting on of the field and the partial discharge of the leaf.

In the case of both positively and negatively charged leaf, there was found to be a critical potential to which the leaf must be charged in order that a discharge may take place when the magnetic field is put on. This critical potential was much lower in the case of the negatively charged leaf.

The discharge effect with the magnetic field became apparent with pressures as high as .05 mm. of mercury. It was found that when the leaf was charged below the critical potential [so that discharge did not take place when the field was put on] and the pressure was .01 mm. of mercury, the magnetic field had not such a marked effect as

at higher pressures and higher potentials. Under these conditions, with leaf positively charged, the time with magnet off to the time with magnet on was in ratio 17:20. With negatively charged leaf, under similar conditions, the magnet produced no effects.

In the case of the sudden discharge, which was brought about by the application of the magnetic field, it is possible that the field produces a concentration of the paths of the ions; and hence a much more marked effect, owing to the formation of ions by collision, is the result. The effect on the positive ions may be a secondary one, since it is known that very strong magnetic fields are required to deflect positive ions.

(c) *Leak stopped by aluminium-foil.*—Some zinc iodide was placed in a small copper dish, and covered with two layers of aluminium foil, each .0004 cm. in thickness. Some wax was placed round the edges of the dish, and the foil was placed on the wax so as to reduce the amount of diffusion to a minimum. This was placed in the electroscope, and the pressure reduced to 2.8 mm. of mercury. No leak could now be detected, when the leaf was either positively or negatively charged. Hence it appears that the ions producing the leak must be very easily absorbed, as their action could be stopped by .0008 cm. of aluminium.

(d) It was repeatedly found that if zinc iodide was left in the apparatus (fig. 3) for two or three days, a very large leak due to negative ions occurred when the apparatus was first connected with the battery, but that the leak quickly died away. In four minutes the leak was less than one-half its original value. In order to see if the presence of the zinc iodide was the actual cause of this increased conductivity of the enclosed air, the tube was left empty for five days under similar conditions. At the end of that time no large leak was noted.

Hence it appears that the zinc iodide has the power of emitting negatively charged particles at ordinary temperatures.

VIII. *Experiments on the Velocity of the Ions.*

In the conditions in which the ions are produced, the choice of the method used to determine their velocity was limited.

Prof. J. J. Thomson has shown, in his 'Conduction of Electricity through Gases,' first edition, pp. 74-78, that if the ionization is confined to a very thin layer compared

with the total distance between two parallel plates, the current is given by the following formula :

$$V^2 = \frac{32 \pi}{9 R} i l^3,$$

where V = difference of potential between the plates in electrostatic units,

R = velocity of ion in cms. per second for unit difference of potential per cm.

i = the current obtained due to the ions per unit of surface measured in electrostatic units,

l = distance between the plates, measured in cms.

In deducing this equation it is assumed

(a) That the layer in which ionization occurs is very small compared with l .

(b) That the current i is only a very small fraction of that obtained with voltages large enough to produce a saturation-current ; *i. e.*, so that for small changes of voltage, the current varies as the square of the voltage.

In my investigations of the velocities of the ions from the iodides which I tested, I arranged my apparatus so as to be able to make use of the above formula. The apparatus was that shown in fig. 2. The glass tube E was about 50 cms. in length and was heated in the centre. The whole was made air-tight.

The ions were driven from the lower electrode C, which was connected to one pole of a battery of small storage-cells, the other pole of which was earthed, on to the insulated electrode B, which was connected with one pair of the quadrants of a Dolezalek electrometer, the other pair of quadrants being earthed.

The capacity of the whole system was measured so that the value of the current could be obtained in absolute units.

It will be seen from the equation that the value of the current i does not depend upon the number of ions present but only upon their velocity, so long as the conditions specified above are fulfilled. In order to do this a very thin layer of the salt to be tested was placed on C, and the distance between B and C was made equal to 2 cms. The variation of the current with the potential was also tested from time to time, in order to see whether the voltage applied was of the right order of magnitude.

The following results were obtained. They refer to the positive ions obtained from the various materials named ; the salts in each case being dry.

Bismuth iodide, lead iodide, barium iodide, and calcium

iodide were anhydrous as supplied by Kahlbaum ; the others were kept in a desiccator for some time.

The velocities are given in cms. per second for one volt per cm. No great accuracy for the absolute values of the velocities given is claimed. Their relative values are nearer the truth, since as far as possible the conditions for each set of observations were the same.

Temp. degree Cent.	Pressure in mm. of mercury.	Zinc iodide.	Bismuth iodide.	Lead iodide.	Cadmium iodide.	Pressure $\times$ Velocity for Bismuth iodide.
215	10	·055	·06	·061	·12	·600
"	15		·055	·054		·825
"	20	·044				
"	25		·041	·036	·087	1·025
"	30	·035			·073	
"	35		·032	·032		1·120
"	40	·031	·03			1·200
"	45		·027	·025	·06	1·215
"	50	·024	·023			1·150
"	55			·021		
"	60	·022				
"	70		·014		·05	·980
"	90		·009			·810

A number of determinations at other temperatures and pressures were also made with the same compounds. In all cases I found that the positive ions produced by heating zinc iodide, lead iodide, and bismuth iodide move with practically equal velocities under the same conditions ; while the positive ions from cadmium iodide move with just about twice the velocity of the positive ions from the other three compounds.

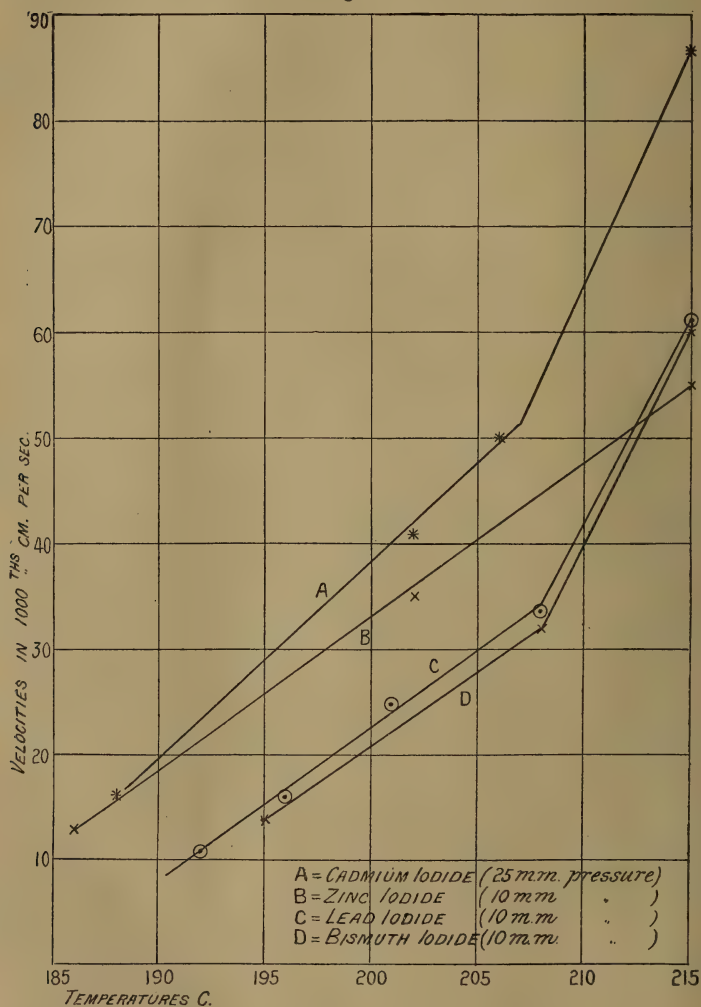
Quite recently Gehrocke and Reichenheim* have found that when various salts, chiefly chlorides, are made the anode in a vacuum-tube, and when this anode is heated, positively charged particles, which they call anode rays, are emitted. These effects seem to be the same as those studied here, except that the temperatures in their case are higher. They find the anode rays give the metal lines in the spectrum, from which it seems that the positive ions are metallic. They may, however, have attached to them, combined or otherwise, products of the chemical change. If this were not so, we should expect their velocities to be more unequal and to be roughly inversely proportional to their atomic weights. Zinc iodide very quickly and readily absorbs water, and it was found that the velocities obtained for positive ions from this salt were made much smaller by moisture. Bismuth

* *Deutsch. Phys. Gesell. Verh.* viii. 21. pp. 559-566, Nov. 1906.

iodide and lead iodide, on the other hand, do not become loaded in this manner.

The cadmium iodide, which does not absorb moisture, gives off positive ions which move with a velocity just about double that of the positive ions from lead iodide or bismuth iodide.

Fig. 4.



The influence of pressure upon the velocity of the ions can be seen by comparing the products obtained of velocity and pressure as given in the last column of the table. These are

for the ions from bismuth iodide. It will be seen that between pressures 50 and 25 mm. of mercury the product does not vary much. There is evidence of simplification taking place between pressures 90 mm. and 50 mm. Langevin* has shown that $P \times V$ remains very nearly constant for positive ions produced by Röntgen rays between pressures 75 mm. and 1435 mm. of mercury, thus showing that between those pressures the constitution of those positive ions remains the same. The change in $P.V$ at the lower pressures I cannot account for. The manner in which the velocity changes with the temperature at low pressures when the pressure remains constant is shown in fig. 4, in which the temperatures are plotted for abscissæ and the velocities for ordinates.

It will be seen that in all cases, except that of zinc iodide at a pressure of 10 mm. of mercury, there is a very rapid rise in the velocity at a temperature between 205°C. and 210°C. , indicating that a simplification of the ion has taken place at that temperature.

Phillips † has found a linear relation to hold between the velocity and temperature for ions produced by Röntgen rays between -64°C. and 138°C.

Velocities of the negative ions.—The velocity of the negative ions was measured in the case of those from barium, calcium, and zinc iodides; and the results obtained from all these sources under the same conditions of experiment were practically identical. These are shown in the following tabulated list. The velocities are all for temperature 215°C. , and are given in cms. per second for one volt per cm.

Pressure in mm. of mercury.	Zinc iodide.	Calcium iodide.	Barium iodide.	Pressure $\times$ Velocity in case of Zinc iodide.
10	1.05	1.10	1.11	11.05
20			.74	
30	56	.56		16.80
35			.45	
40	.42	.33		16.80
50	.35	.22		17.50
60		.2	.29	
80	.21			16.80

In this case also it will be seen that, with the exception of the velocity at 10 mm. pressure, the values of (Pressure $\times$ Velocity) are fairly constant.

* *Thèses Université de Paris*, p. 190.

† *Proc. Roy. Soc.* vol. lxxviii., 1906.

The emission of the negative ions from the salts of calcium and barium is interesting, since this has been made a special study in the case of calcium oxide by Wehnelt* and also by Dr. Horton†. The latter finds that the decreasing resistance of calcium oxide and barium oxide to the passage of electricity with rising temperatures is partly due to the increasing number of free corpuscles contained by those oxides. Under the conditions of experiment, no negative ions were detected in the cases of cadmium, bismuth, and lead iodides.

In the cases of the iodides of calcium and barium, no positive ions could be detected while the salts were dry; but if the salts were left in the apparatus and then tested, it was found that the velocity of the negative ions had been reduced to about one-third of their original value, and the presence of positive ions was detected in the case of calcium iodide.

It will be seen that the velocity of the negative ions is many times greater than that of the positive. This being the case, the nature of the ions must be different, since the conditions under which their velocities were obtained were similar. It was noticed during the experiments that in every case in which the positive ions were detected a deposit (apparently due to sublimation) could be seen on the upper electrode B, and the relatively slow movements of the positive ions made it probable that the ions in this case might be charged centres surrounded by particles of the salt. Bloch‡ came to the conclusion that dust particles were intimately connected with the conductivity of recently-prepared gases, and the conductivity produced by phosphorus.

In order to ascertain whether the conductivity could be due to such a cause, I heated some of the salts in a large tube which was fitted with two corks, through which passed two equal-sized, insulated, thick copper wires, which were connected to the opposite poles of a battery of 200 storage-cells.

The material was heated slowly up to about 250° C. by means of a Bunsen flame, which was kept moving so as to ensure more regular heating. The electrodes were first carefully balanced on a pair of scales.

With iodides of bismuth and cadmium it was found that, after heating up, the negative electrode weighed one or two milligrams more than the positive each time the experiment was tried.

With lead iodide the negative electrode was always heavier

* *Annalen der Physik*, Band xiv. (1904).

† *Phil. Mag.* April 1906.

‡ *Ann. de Chimie et de Physique*, Tome iv. 1905, pp. 127 *et seq.*

than the positive, although the difference was not so marked as in the above cases.

With zinc iodide no difference in the weight of the electrodes was noticed.

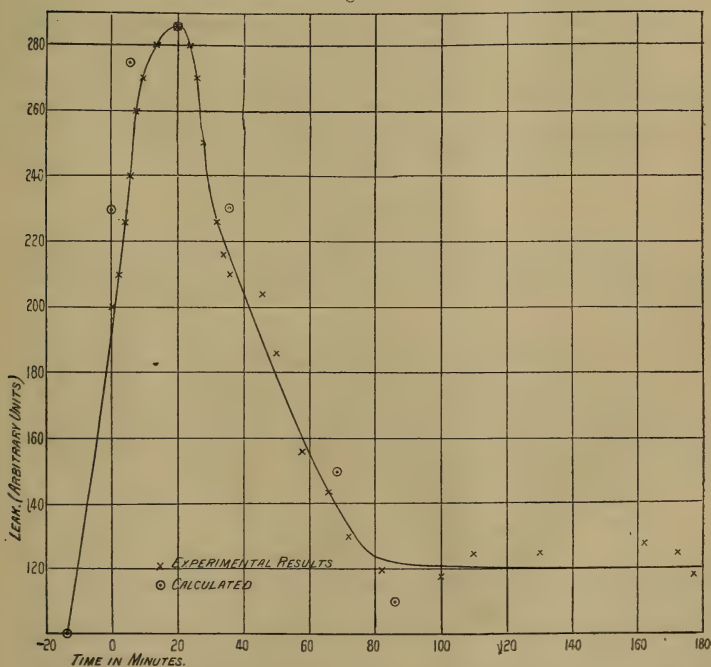
Hence it appears that some of the products in the first three cases become positively charged, and so tend to accumulate on the negative electrode; and it is probable that a good deal of the salts sublime without undergoing chemical decomposition, the particles being positively charged.

IX. Rate of Decay of Activity with the Time.

In our first experiments with zinc iodide (*loc. cit.*) Dr. Willows and I found that the activity first increases to a maximum, then gradually decays with the time, following an exponential law if the temperature is kept constant. I have made some further experiments on the decay of zinc iodide under different conditions of temperature and pressure.

A typical decay-curve is shown in fig. 5.

Fig. 5.



In this the times are plotted as abscissæ and the rates of leak as ordinates.

If the ascending branch is produced backward and the point where it cuts the axis is taken as the zero of time, it is found that the curve can be represented by an equation of the form

$$I = A(e^{-\lambda_2 t} - e^{-\lambda_1 t});$$

where I = leak at time t , A is constant, and λ_2 and λ_1 are constants ($\lambda_1 = 25 \times 10^{-5}$, $\lambda_2 = 157 \times 10^{-5}$).

The $\odot$'s represent points calculated from this formula on the assumption that the curve produced backward meets the axis 14 minutes behind the origin.

If we may reason from the analogy of radioactive change, such a curve would mean (see Rutherford, 'Radioactivity,' Ch. ix.) that two successive changes are produced. The first of these produces no ions, but the second does. The presence of water is necessary to start these changes. These successive changes are of interest when taken in conjunction with the fact observed by Allen*, that the decay of photo-electric activity of zinc can be represented as the sum of two exponential terms.

Fig. 5 also shows that after the material has been heated for some time the rate of decay becomes very much slower; in fact, for the last 90 minutes the rate of leak undergoes very little decrease. For the part of the curve which can be represented by the formula, the activity declines to one-half its value in about 46 minutes.

Conclusions.

(1) Many inorganic compounds when heated up to temperatures below 360° C. have the power to produce electrical conductivity.

(2) That in the case of some of these compounds (*e. g.*, halogen salts of zinc) chemical changes take place, and the conductivity produced has its origin in these changes. The ions so produced are similar to those which have been observed in the cases of recently prepared gases, and the conductivity produced by phosphorus.

(3) That in all cases the ions are of a large size, as is shown by their relatively small velocities. They probably consist of a charged centre surrounded by particles produced from the heated compound, and in some cases water-vapour.

Their nature could be best demonstrated by determining $\frac{e}{m}$, but means for producing a strong enough magnetic field were not available.

(4) The presence of water is in some cases intimately

* Proc. Roy. Soc., Feb. 1906.

connected with the conductivity produced, as is shown by (a) the difference in behaviour of ordinary and carefully dried compounds, (b) the change in the rate of decay after heating for some time.

(5) In some of the more active substances, *e. g.* halogen compounds of zinc, increased conductivity can be readily detected in air at ordinary temperatures and pressures.

(6) The ions, both negative and positive, move with a very slow velocity, similar to those which have been observed in the cases of phosphorus, recently prepared gases, and gases from flames. This velocity is greatly increased by a rise in temperature. For certain ranges a straight line relation holds between velocity and temperature, but sudden changes in the direction of the line occur probably owing to simplification of the ions. Such a change occurs in the compounds tested at about 210° C. and the pressure reduced to 25–10 mm. of mercury.

The product (Pressure $\times$ Velocity) is constant for certain ranges of pressure.

(7) The rate of leak at different temperatures can be represented in all the cases tried by a formula of the form

$$I = a\theta^{\frac{1}{2}}e^{-b/\theta},$$

in which a and b are constants, θ is the absolute temperature, and I is the saturation-current.

(8) The ions produced are not able to penetrate a layer of aluminium .0008 cm. in thickness.

In conclusion I should like to thank Dr. R. S. Willows, in whose laboratory these experiments have been carried out, for much valuable advice throughout the course of the research, and Mr. F. C. G. Bratt for help in the construction of the apparatus used.

Cass Technical Institute,
Jewry Street, E.C.

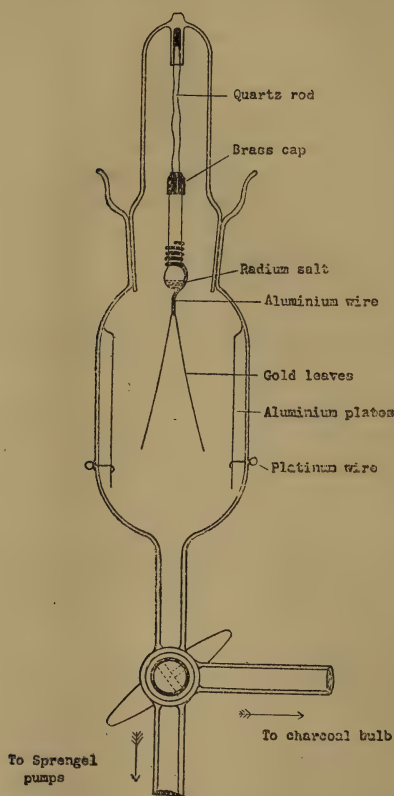
LXVIII. *On Experiments with Vacuum Gold-leaf Electroscopes on the Mechanical Temperature Effects in Rarefied Gases.* By J. T. BOTTOMLEY, M.A., LL.D., D.Sc., F.R.S., and F. A. KING*.

[Plate XVII.]

THE following is intended to be a preliminary notice of some striking mechanical effects which we have observed in the gold leaves of vacuum-electroscopes, and which, at first sight, might be taken to simulate electrification of bodies by light and radiant heat.

* Communicated by Lord Kelvin. (Read at the Royal Society on March 7th, 1907. We are indebted to the Royal Society for kindly furnishing us with clichés of the cuts illustrating this paper.)

Experiment 1.—The first indications which presented themselves to us, of the existence of the peculiar effects which we are about to describe, were obtained while we were experimenting with a “radium clock,” constructed much as described by the Hon. R. Strutt, F.R.S.* (Pl. XVII.) We have set up this apparatus for the purpose of carrying out secular observations, and in order to find, if possible, whether the activity of the radium seems to diminish with time.



The radium clock, as is well known, is practically a gold-leaf electroscope, contained in an outer vessel highly exhausted, in which a small glass tube containing some radium, and with its outer surface smeared with an aqueous solution of phosphoric anhydride, to render it conducting, is connected metallically to the gold leaves.

Two metallic plates, with connexions passing to the out-

* Strutt, Phil. Mag. Nov. 1903; Rutherford's 'Radio-activity,' 2nd edition, p. 122

side through the glass cover of the electroscope, are placed on the two sides of the gold leaves; and the leaves are long enough to touch these plates when their divergence reaches a certain amount. As the β -rays of the radium emanation escape the α -rays electrify the gold leaves, which then diverge; and, diverging, they come in contact with the side-plates. They are then discharged, and fall together, but again become electrified as before. A periodic action thus goes on, the gold leaves alternately diverging and collapsing. In the radium clock which we have constructed, and with a vacuum of $\frac{1}{20}$ M.*, the period of the gold leaves is about $3\frac{1}{4}$ minutes.

While observing the radium clock, and endeavouring to improve the vacuum of the apparatus, a Bunsen burner which was being used was placed in rather close proximity to the clock. It had a long powerful flame, and it was observed, after a short time, that the gold leaves of the clock were apparently repelled by the flame.

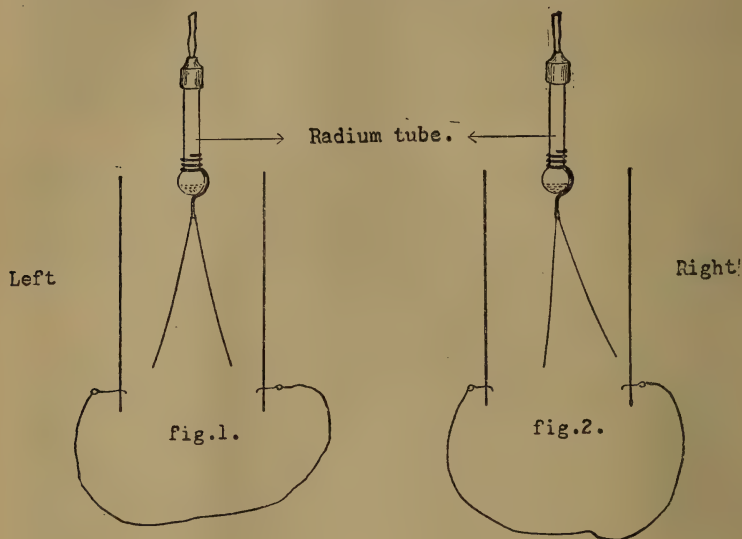
It is impossible to make the gold leaves so symmetrical that they shall both strike their respective fixed side-plates at the same instant. One must strike first, and then both are discharged. In our case, that which we call the "left-hand leaf," looking at the apparatus from what we will call the "front," had been striking first. The Bunsen burner was, by chance, standing on the left. It was noticed that the leaves did not hang symmetrically, as shown in fig. 1, but were repelled and hung as shown in fig. 2 (p. 750); and as their divergence increased the "right-hand" leaf was the first to strike on its side-plate, and it was found at the same time that the period of the clock was greatly increased.

Experiment 2.—A common soldering-bolt was next heated, sometimes to red heat, or almost to white heat, and, later on, to temperatures far below red heat but higher than that of the surrounding atmosphere; and it was found that, when this was brought near, the leaves of the radium clock behaved much as they did in the presence of the flame. The same effects were obtained when a spirit-flame, or a candle-flame, or the light from a Nernst lamp, was used instead of a Bunsen burner.

Experiment 3.—Even the heat of the hand was found to be quite sufficient to cause repulsion of the gold leaves of the clock. By bringing up the warm hand close to the outside cover of the clock the leaves could be driven slowly, first in one direction and then in the other; and they could in this way be prevented from striking on the side-plates; thus the

* M denotes one-millionth of standard atmospheric pressure.

period of the clock could be greatly augmented. In one case which we have noted, the striking on the side-plates was delayed, by alternately placing the hand to the right and to the left of the gold leaves, to such an extent as to make an interval of $6\frac{1}{2}$ minutes, the normal period of charge and discharge, at the time, being 3 minutes 4 seconds.



Experiment 4.—To try the effect of cold, a test-tube was filled with water, and with the help of a vessel filled with liquid air, the water was first frozen, and was then cooled down nearly to the temperature of the liquid air. We had thus a bar of solid matter, having a great capacity for cold, and chilled to a very low temperature ; and this was brought near to the clock, just as had been done with the flame and with the bar of heated copper. When the bar of chilled ice was brought near to the leaves they were markedly attracted ; and, in fact, the action of the chilled body appeared to be exactly the inverse of what we had observed in the case of the heated bar of metal.

Experiment 5.—When the clock was covered entirely with a cage made of fine brass gauze, and the gauze cage connected to a neighbouring gas-pipe, the peculiar attraction and repulsion of the gold leaves was still observed, with but little diminution in intensity, when hot and cold bodies were presented, as described above.

The leaves, however, under these circumstances, were unaffected when electrified rods of glass and sealing-wax were

brought near, being protected by the presence of the wire-gauze cover, as was shown by Faraday, in his Experimental Researches.

A great variety of experiments were also tried, but need not be described in detail here, on the effect of screens of metal, glass, mica, cardboard, paper, &c., also of liquids contained in a parallel-sided cell, and, lastly, of vacuum; the vacuum between the walls of a Dewar vessel for holding liquid air being used for this purpose.

To ascertain how far the phenomena observed might be attributed to heating and cooling effect, the region of air surrounding the radium clock was carefully explored with the help of a very sensitive thermo-electric junction. The result of this experimenting was to show that the thermal radiations, which were almost too slight to be detected with the thermo-junction, could produce very marked effects in causing disturbance of the gold leaves.

It was now determined to construct a special gold-leaf electroscope for the purpose of investigating the phenomena which these preliminary experiments pointed to; and, after several small electroscopes had been made, a convenient instrument was obtained, which may be described as follows, and of which fig. 3 (p. 752) is a drawing. The essentials aimed at are: (1) convenient dimensions; (2) perfect dryness of the interior; (3) high exhaustion.

The electroscope is contained in an enclosure, made from a piece of glass tubing, $1\frac{1}{2}$ inches in diameter, drawn out at both ends to form tubes e , e' , fig. 3. A piece of thermometer tubing t is sealed on to the neck e' , and the aluminium wire m , n , supporting the gold leaves, passes down the centre, and is held in position by the tube t , the joint between the glass and aluminium wire being made vacuum-tight by filling the cup c with "siegelwachs."

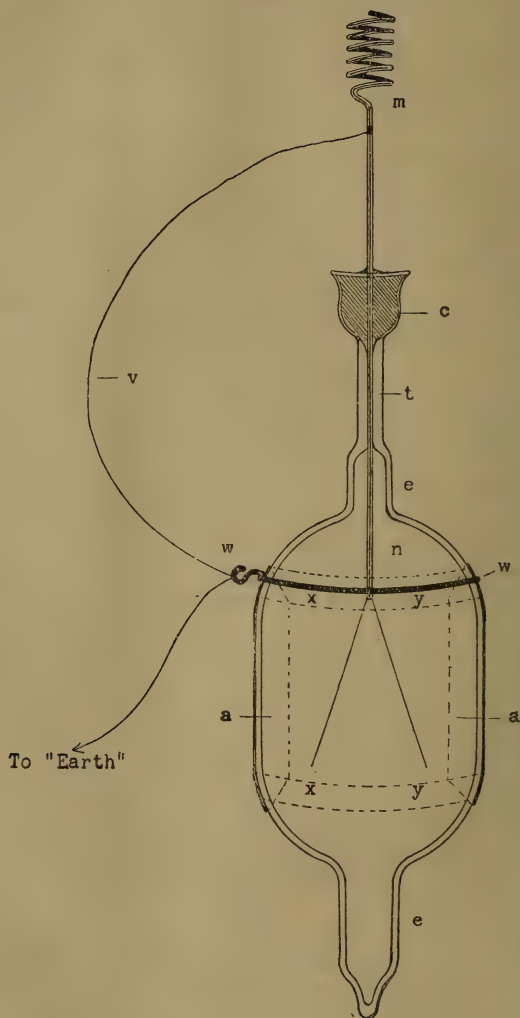
Two pieces of lead-foil a , a , about $1\frac{1}{2}$ inches long and $1\frac{1}{4}$ inches wide, are cemented on to the outside of the glass enclosure opposite the faces of the gold leaves, and are joined together by strips of lead-foil x , y , which completely encircle the glass enclosure.

A copper wire w is fastened over the upper strip of lead-foil to allow a connexion to the metallic coating to be easily made. The tube e is connected to a series of drying-tubes, and to a pair of five-fall Sprengel pumps, to maintain a high degree of exhaustion in the enclosure.

With the instrument thus described, the following experiments were carried out, both while the electroscope was still connected to the pumps and after it was sealed-off.

Experiment 6.—A copper wire *w* was passed round the outside of the electroscope so as to connect together the two outside metallic coatings *a, a*. A fine wire *v* was then attached

Fig. 3.



to this surrounding wire, and was connected to the aluminium wire *m, n*, from which the gold leaves are hung. Thus *the gold leaves are put in metallic connexion with the outside coating*

of the electroscope. A wire was also taken from the surrounding bands of lead-foil x, y to the water-pipes &c., and thus all question was avoided as to the condition, as to insulation, of the gold leaves and the surrounding coatings, all the parts being put in metallic connexion with each other, with the earth, and with the general surroundings.

It is to be remembered also, in this connexion, that the outside of the glass enclosure of the electroscope *is not an insulator*; it was not dried, nor specially cleaned, and no part of it could be charged to the smallest degree by means of rubbed glass or rubbed sealing-wax. To test this a second independent electroscope was charged; and when the knob was brought into contact with any part of the glass of the first, it was instantly discharged; and this was the case even when all the wires, described in the last paragraph, were cleared away. .

With the arrangements which have just been described, it will readily be understood that if a rod of glass be rubbed with silk (vitreously electrified) or a rod of sealing-wax with flannel (resinously electrified), and brought near to the electroscope, no effect is produced. The gold leaves being connected to the outside cover and to "earth," and being protected by a conducting envelope, cannot be electrified in the ordinary way. This was verified by experiment.

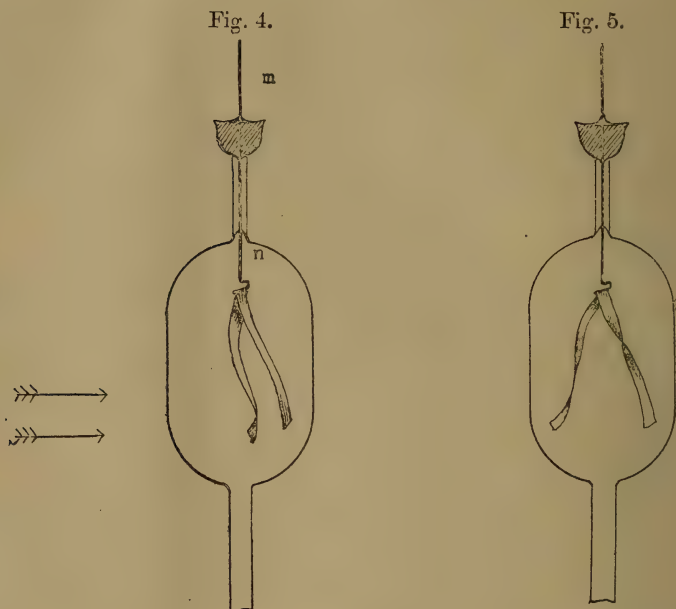
Experiment 7.—In these circumstances, when the flame of a spirit-lamp, or a light of any description, was suddenly brought up towards the electroscope, the leaves were instantly seen to diverge, and when the flame was withdrawn suddenly the leaves fell together again, but they did not respond instantly, as in the case of the divergence. In this experiment, the flame is not brought into contact with the glass; the effect is produced at a distance of many centimetres from it. The glass is not heated, or quite infinitesimally.

Experiment 8.—When the flame is suddenly brought up and then withdrawn, and this operation performed three or four times in succession, taking care never to heat the glass to the slightest degree, the gold leaves obtain a large permanent repulsion for each other, and stand permanently apart. The leaves could even be torn from their support by operating thus with the spirit-flame, if care was not exercised to avoid such violent action.

By suitably manipulating the spirit-flame, the gold leaves can be deflected in very curious ways. The flame being suddenly pushed up towards them, sometimes from one direction, sometimes from another, the leaves can be gradually

contorted into a sort of crumpled form. They can both be driven in the same direction relatively to the centre; and they remain in this contorted condition, only slowly resuming their natural state.

It is difficult to make a drawing to illustrate the crooked forms into which the leaves are sometimes twisted, but an attempt to do this is made in figs. 4 and 5. This twisting



makes it evident that the leaves are subjected to considerable local forces, with rapid variations as to direction and amount, from point to point within the enclosure. Putting the hand to the aluminium wire *m*, *n*, or catching hold of it between the fingers, makes no difference to the gold leaves; nor does the putting of a spirit-flame (Gilbert and Faraday's method of discharging electrification) in contact with the outside coil at the end of the aluminium wire *m*, *n* have any effect. The deflexion remains unaltered and the leaves are not relieved of their charge, as would be the case with an ordinary charged electroscope.

Experiment 9.—To find whether the gold leaves were affected from a distance, by a source of light, a Nernst lamp was set up about 6 metres distant from the electroscope, and the image of the filament was projected upon the glass enclosure. As soon as the light from the Nernst lamp fell

upon the gold leaves, they instantly diverged, although not to the extent of the divergence obtained in previous experiments, when the lower ends of the leaves were often two or more centimetres apart. When an opaque screen was interposed between the source of light and the electroscope, the leaves quickly collapsed. If the screen was passed backwards and forwards across the path of the rays, the leaves alternately diverged and fell together again. Even the movements of a string, of a pendulum swinging across the beam of light could be followed, by observing the slight kicks of the leaves, which were produced when the light was momentarily cut off by the string. Screens of glass, or other transparent substances, have practically no effect in cutting off the radiations which affect the gold leaves. The leaves responded to the movements of a shutter in front of the Nernst lamp when the light was made to pass through a plate glass door, just as if the door were not there.

Screens of various materials were interposed between the electroscope and the Nernst lamp. When the light was transmitted through coloured glasses the divergence of the leaves was diminished, ruby glass almost halving the deflexion. The effective radiation appears to pass with the greatest freedom through several sheets of white, or coloured paper, so long as the screen is at all transparent to light. A parallel-sided plate of ice was made, by sawing a slice off a large block of ice and then rubbing its surfaces down on a sheet of plate glass until they were quite parallel and true. The sheet of ice allowed the radiations which affect the gold leaves to pass through with scarcely any perceptible diminution in intensity. No effect, however, could be detected when the Nernst lamp was replaced by a spirit-flame, the sheet of ice still being present; although the light from a candle or match would produce the effects observed with the Nernst lamp. This was also found to be the case, when another vacuum electroscope was completely immersed in a glass vessel of alum solution, and the Nernst lamp, and the various flames, brought near to the gold leaves. The Nernst lamp* and the flames of a candle or match produce their full effects; but the influence of the spirit-flame, as far as can be detected by visual observation of the gold leaves, is completely stopped. Screens of lead, copper, zinc, &c.,

* Care must be taken when the Nernst lamp is used as a source of light in these experiments. The metallic holder of the lamp is frequently electrified, and it ought to be "earthed" by means of a suitably connected wire. It also seems as if the Nernst filament were perpetually shedding off an emanation, which will electrify an ordinary electroscope placed at a distance of several inches from it. This electrification is *resinous*.

appear to stop, or, at any rate, to reduce greatly the radiations which affect the gold leaves. Some curious observations, however, which may modify this last statement, were made when the radium clock was the subject of experiment. It was found that metal screens allowed radiations to pass through, which repelled the gold leaves of the radium clock, and thus caused its period of charge and discharge to be lengthened. Tables showing this numerically will be given in a future communication.

Experiment 10.—It was found that divergence of the leaves only took place when the light was shining directly upon them; and that no effect was perceptible when the light was projected through the glass envelope, without illuminating the gold leaves. In testing this, the image of an incandescent electric lamp was focussed, by means of a lens, first upon the gold leaves and then upon the glass cover, the gold leaves being avoided. The result mentioned above was obtained; but it was also found that if only one leaf was illuminated the opposite leaf was repelled away from the illuminated one, which remained hanging vertically, as shown in fig. 6. If the

Fig. 6.

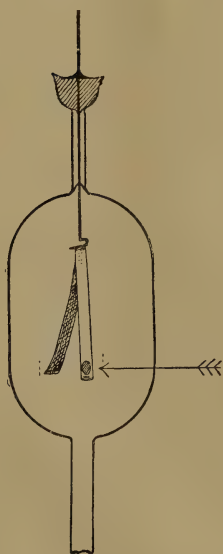
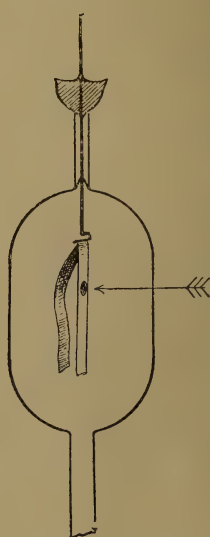


Fig. 7.



spot of light was so arranged that it illuminated a little patch in the centre of one of the gold leaves, the opposite

leaf took the form shown in fig. 7, very different from any that could be due to electrification.

The gold leaves of the vacuum electroscope show a distinct deflexion when bright daylight falls upon them, and a large divergence is obtained if the gold leaves are illuminated by direct sunlight. A black card interposed, so as to cut off the light, causes the leaves to collapse at once; but they again attain their maximum deflexion the instant the card is removed.

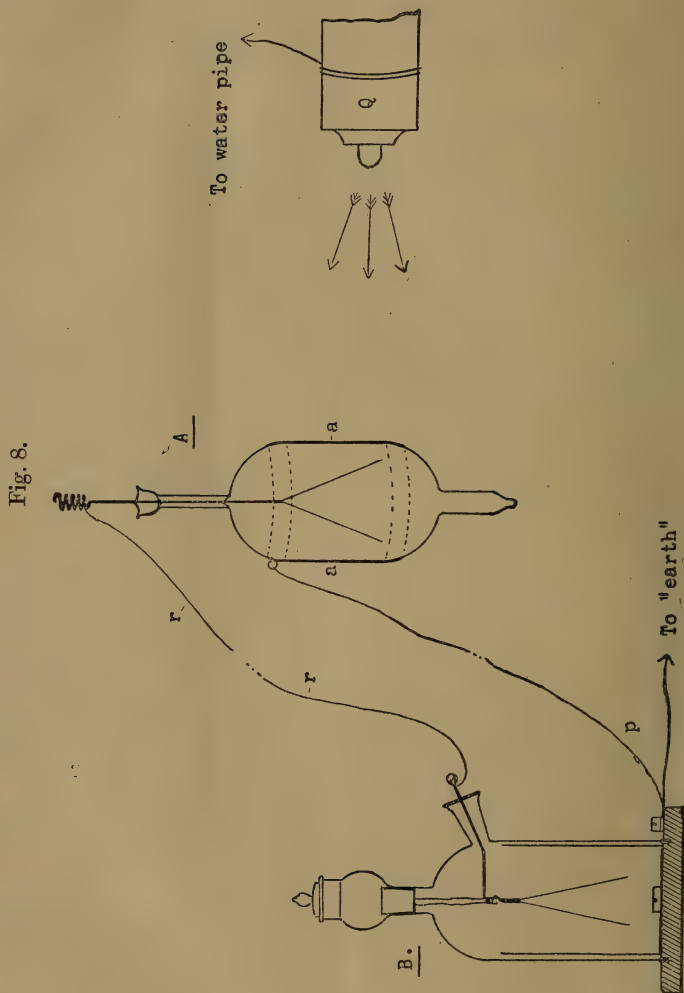
Turning the electroscope round its own vertical axis, so that the rays of light, from the sun or any other source, may fall upon the gold leaves at different angles, increases or diminishes the amount of the repulsion of the leaves.

Experiment 11.—To put in evidence further the difference between these phenomena and such as could arise from electrification of the vacuum electroscope, the following arrangements were made, of which fig. 8 is a diagram.

The gold leaves and the metallic coating of the vacuum electroscope A were connected up to the corresponding gold leaves and the metal plates of an independent electroscope B. A wire *p* made connexion between the metallic coatings and "earth." The two electroscopes were exactly similar in construction; the main difference between them was that one had a high vacuum, and the other had no vacuum. The gold leaves of both were of the same length and width, and the glass enclosures were made from the same length of glass tubing; thus, if a common charge were given to both electroscopes, the lower ends of the gold leaves would stand apart at approximately the same distance in each. A Nernst lamp Q, with its metal cover connected to a water-pipe, was brought near to the electroscope A, the gold leaves at once diverged, and those of the electroscope B also diverged; but the distance apart of the leaves of B was not more than half the distance by which those of A stood apart. If now the wire *r* connecting together the gold leaves of A and B was touched with the hand, electroscope B was discharged, but electroscope A was only partially discharged.

When electrified rods of glass and sealing-wax were brought up to the electroscope B, the charge on the leaves was found to be *resinous*, but the charge left on the gold leaves of A, after the electroscope B had been discharged by touching the wire *r*, was not electricity at all. To confirm this, both electroscopes were charged *vitreously*, and when the Nernst lamp was brought up to electroscope A, the (*vitreous*) charge of electroscope B was neutralized, but the gold leaves of

A remained apart so long as the light from the Nernst lamp fell upon them. If a spirit-flame was used instead of the Nernst lamp no charge could be got on electroscope B, although the gold leaves of electroscope A stood wide apart.



Other vacuum electroscopes, having metallic coatings of fine brass wire gauze and aluminium foil on the *inside of the glass enclosure*, were set up, and all the foregoing experiments were repeated. The results obtained correspond

with those already described in connexion with the electro-scope with the metallic coating on the outside of the glass envelope.

The experiments described above are to be considered as preliminary. The phenomena are still under observation, and we hope in a future communication to be able to give further details with regard to the magnitude of the forces concerned and the laws which govern their action.

Since the paper was communicated to the Royal Society, our attention has been called to a short paper by Sir William Crookes, of 1879, published in the 'Proceedings of the Royal Society,' in which certain movements of gold leaves in vacuum, similar to those which we have observed, are described.

LXIX. *Notices respecting New Books.*

Electrons, or the Nature and Properties of Negative Electricity. By Sir OLIVER LODGE, F.R.S., D.Sc., LL.D. London: George Bell & Sons. 1906. Pp. xvi+230.

THE discoveries of the past decade in physics will probably rank among the most important ever chronicled in the history of science. To the mind of the average cultured person as well as to the philosophic mind these discoveries appeal with an intensity which cannot be rivalled by that of other scientific discoveries, however important these may appear to the specialist. For the theories which have resulted from a detailed study of the phenomena of cathode and Röntgen rays, and of radio-activity, have enabled us to catch glimpses of the solution of that mysterious yet intensely fascinating problem, which for so long a time has entirely baffled human enquiry—the ultimate nature and constitution of matter. Quite apart from the comparatively small class who have made a special study of the subject, there are hundreds of educated persons who take a keen interest in the results of scientific investigations directed towards it, even though they may be unable to follow in detail the researches and arguments by which specialists have been led to form their opinions. To all such we can cordially recommend a perusal of Sir Oliver Lodge's latest contribution to the literature of pure electrical theory. It will prove a most useful guide to the serious student; and will be found to be full of interest to the general reader who has an elementary knowledge of physics. Few writers on scientific subjects can arouse the interest and enthusiasm of their readers in the way which is characteristic of Sir Oliver Lodge: and the work under notice will enable any intelligent reader to form a clear idea of the present state of a problem which has courted yet defied human enquiry ever since the days of the Greek philosophers.

Memoir and Scientific Correspondence of the late Sir GEORGE GABRIEL STOKES, Bart., Sc.D., LL.D., D.C.L., Past Pres. R. S., Kt. Prussian Order pour le Mérite, For. Assoc. Institute of France, etc., Master of Pembroke College and Lucasian Professor of Mathematics in the University of Cambridge. Selected and arranged by JOSEPH LARMOR, D.Sc., LL.D., Sec. R. S., Fellow of St. John's College and Lucasian Professor of Mathematics. 2 vols. Cambridge: at the University Press. 1907. Vol. I., pp. xii + 475. Vol. II. pp. vi + 507.

THE two volumes now before us form a fitting sequel to the collection of Sir G. G. Stokes's *Mathematical and Physical Papers*, the last two volumes of which had been prepared for publication by Dr. Larmor. They contain all that, after careful scrutiny, appeared to be of sufficient interest in the voluminous correspondence of Sir G. G. Stokes to justify publication. The question of an authoritative memoir of his life having arisen, his daughter—Mrs. Laurence Humphry—undertook the by no means easy or simple task of preparing one, and the interesting account which forms the opening part of the first volume is from her pen. Mrs. Humphry's contribution is followed by a selection of letters to Lady Stokes, letters on Science and Religion to A. H. Tabrum, and Appreciations by Colleagues. The remainder of the work contains the scientific correspondence, and covers—as might have been expected from the versatility of Sir G. Stokes's genius—a vast amount of ground. Nothing, perhaps, illustrates better the high esteem in which he was held by his contemporaries, and the implicit faith which was placed by workers in the most varied departments of science in the soundness of his judgment, than the constant stream of enquiries addressed to him, as revealed in the two volumes before us. The readiness with which he answered all such enquiries, and the interest which he was always capable of taking in the work of others, cannot but excite admiration. The letters relating to scientific subjects, though not strictly classified, appear under two headings, "General Scientific Career" and "Special Scientific Correspondence." No letters to or from Lord Kelvin are included in the present collection, and we are informed by the editor that they are to form a special volume.

The first volume is embellished with two portraits of Sir G. Stokes, and one of Dr. Romney Robinson, while the presentation bust of 1899 to Pembroke College forms the subject of a frontispiece to the second volume. From a typographical point of view, the two volumes are all that could be desired.

Index of Spectra. Appendix R. By W. MARSHALL WATTS, D.Sc. London: W. Wesley & Son. 1907. Pp. 67.

THIS appendix to the author's *Index of Spectra* contains the spectra of radium emanation, chromium (spark), palladium (arc and spark), and polonium.

Fig. 1.

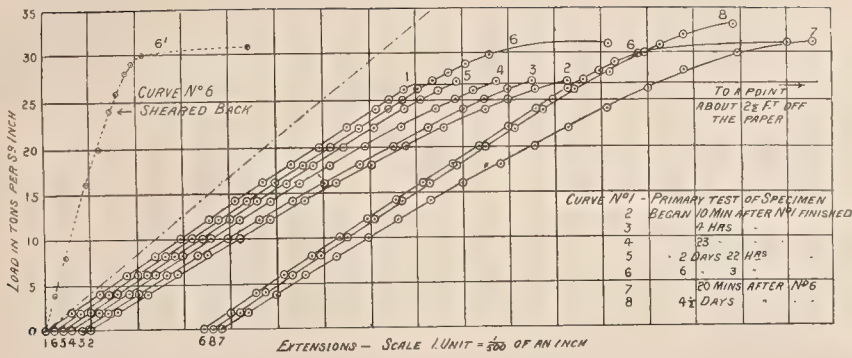


Fig. 2.

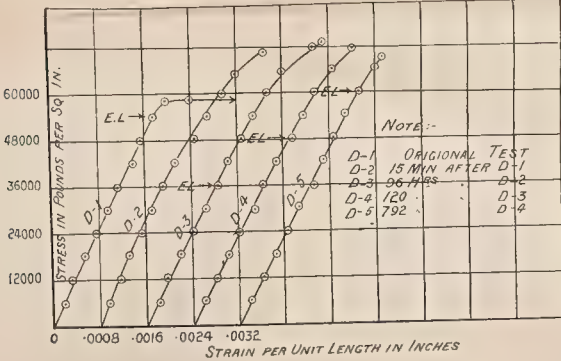


Fig. 3.

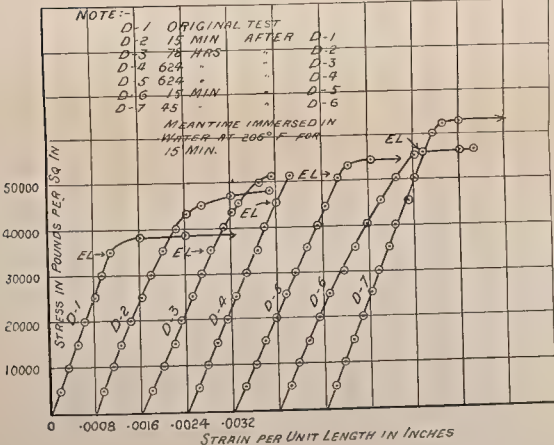


Fig. 4.

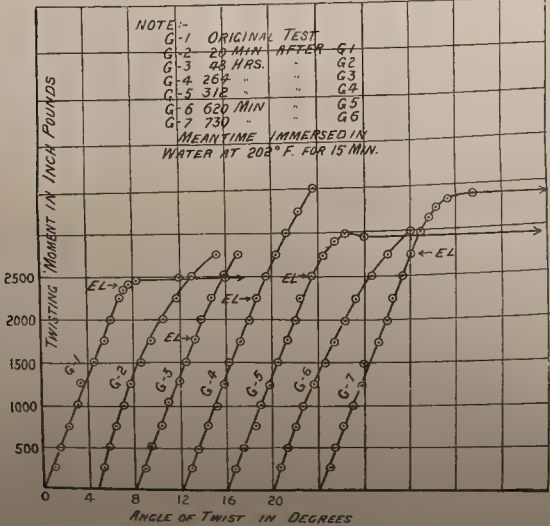


Fig. 5.

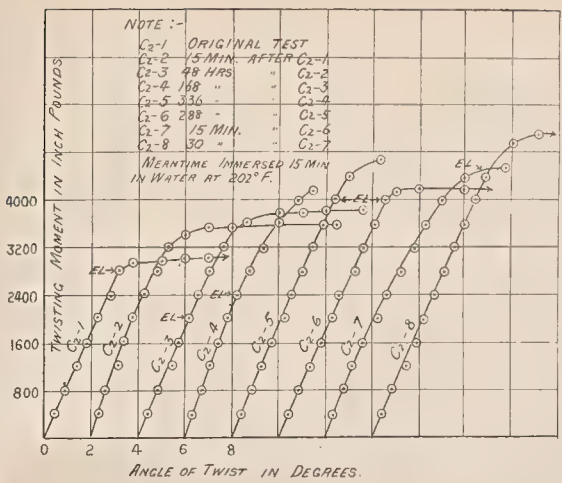


Fig. 6.

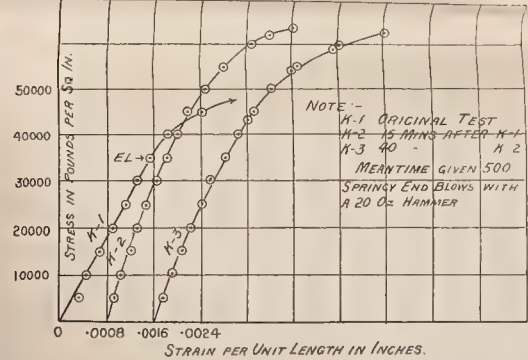


Fig. 7.

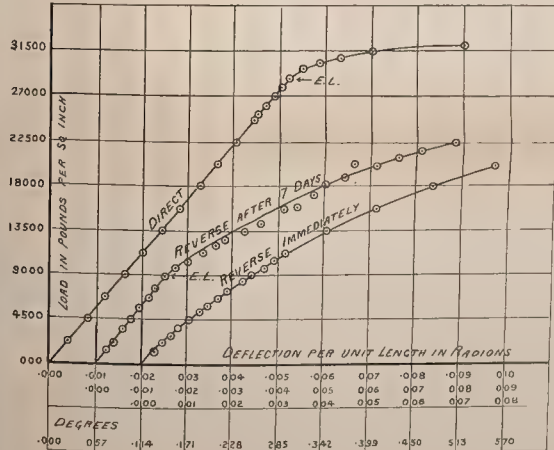
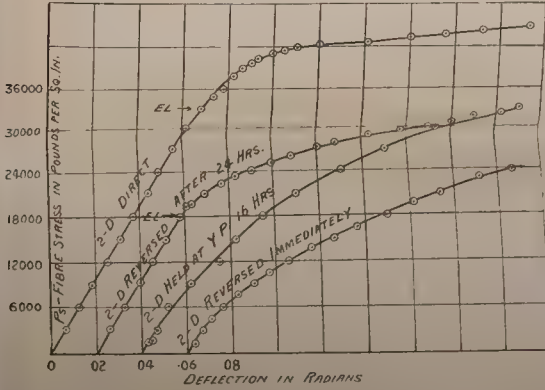
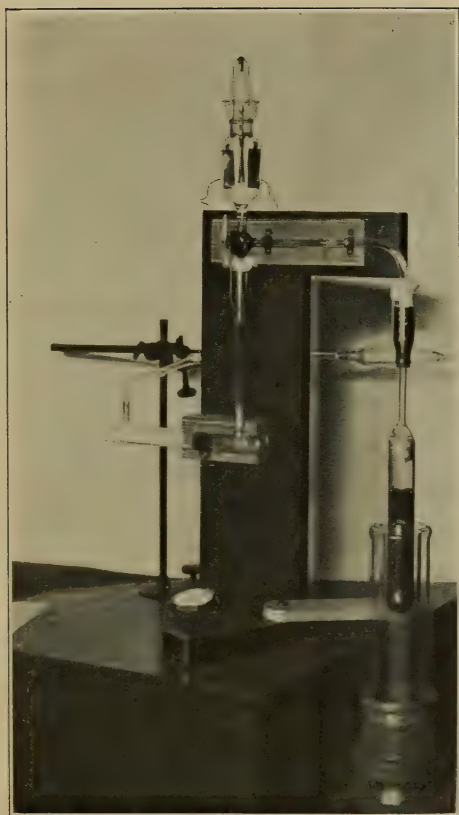


Fig. 8.



and spark, and poison



LXX. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 288.]

March 13th, 1907.—Aubrey Strahan, Sc.D., F.R.S., Vice-President, in the Chair.

THE following communications were read:—

1. ‘A Silurian Inlier in the Eastern Mendips.’ By Prof. Sidney Hugh Reynolds, M.A., F.G.S.

An account of the rocks with which the paper mainly deals was brought before the Society on February 7th, 1906; but, owing to the discovery just before the meeting of Silurian fossils in material thrown out by moles and rabbits, the paper was withdrawn and further work, including the digging of a series of seven trenches, was carried out. This proved that the fragmental igneous rock is of two types:—(1) Normal fine-grained tuff, from which in three localities over thirty species of Silurian (probably Llandovery) fossils were obtained and have been identified by Mr. F. R. Cowper Reed: the tuffs are seen at Sunnyhill to underlie the trap. (2) A remarkable coarse ashy conglomerate, the nature and relation of which to the other rocks are both obscure. The following four possibilities with regard to the nature of this rock are discussed in some detail:—(a) That it may be the basement-conglomerate of the Old Red Sandstone; (b) that it may be an aqueous deposit of the same general age, and belonging to the same igneous series as the associated trap and normal tuff; (c) that it may be an old river-gravel, deposited during a terrestrial period subsequent to the fossiliferous Silurian and prior to the Old Red of the district; and (d) that it may represent the necks of the volcanoes from which the trap and the normal tuff were ejected. The author is of opinion that the fourth of these possibilities agrees best with the observed facts. The paper concludes with a comparison between the Mendip Silurian rocks and those of Gower and Tortworth, in both of which localities it appears that the upper beds of the Silurian System are unrepresented.

2. ‘On Changes of Physical Constants which take place in certain Minerals and Igneous Rocks, on the Passage from the Crystalline to the Glassy State; with a short Note on Eutectic Mixtures.’ By James Archibald Douglas, B.A., F.G.S.

After giving an account of previous experiments in the direction of determining the physical changes accompanying the fusion of rocks and minerals, and their re-solidification into a glassy condition—experiments which are in many cases open to sources of considerable error,—the author describes the electrical apparatus employed by himself. Powdered rock of known specific gravity is fused as often as required in a loop of platinum-ribbon. The

fused product is powdered, examined with the microscope, and then placed in a diffusion-column, by the use of which errors due to the few small bubbles or unfused mineral-particles, on the few occasions when these exist, are eliminated. The diffusion-column is sealed in a glass tube. Acid rocks were found to increase 6 to 10 per cent. in volume, intermediate rocks 5 to 7 per cent., and basic rocks chiefly less than 6 per cent. Of the minerals tested, pargasite underwent the greatest expansion, albite gained 10 per cent., while in anorthite and leucite the increase was less than 4 per cent. It is pointed out that this expansion would be a very effective force in the case of the melting of rock-magmas kept solid by pressure. Experiments were also made in order to ascertain the melting-points of the rocks and minerals experimented upon: these were found to range from 1260° C. for rhyolite to 1070° C. for Cleve-Hill dolerite. The refractive indices of the glasses were determined in dense fluids. In determining the melting-points of individual minerals, the author's results agree fairly closely with those of Day & Allan, but differ from those of Joly & Cusack. An attempt was made to find experimentally the eutectic proportions of quartz and feldspar. It was found impossible to reproduce in the laboratory the conditions which, in nature, seem to have existed during the formation of pegmatites. On the other hand, a mixture of orthoclase and albite gave a melting-point lower than those of either mineral taken separately.

March 27th.—Sir Archibald Geikie, D.C.L., Sc.D., Sec. R.S.,
President, in the Chair.

The following communications were read :—

1. 'On the Southern Origin attributed to the Northern Zone in the Savoy and Swiss Alps.' By Prof. T. G. Bonney, Sc.D., LL.D., F.R.S., F.G.S.

Prof. Lugeon, with some other eminent Continental geologists, explains certain peculiar flat folds, the higher of which sometimes project considerably beyond the lower, in the more northern sedimentary zone of the Swiss and Savoy Alps, by supposing that to no small extent the strata have been thrust forward from an original position south of the watershed of the Pennine-Leptontine Alps; overriding, as they advanced, their crest and that of the Oberland (neither having then attained its present altitude). This pressure was produced by the greater thickness of deposits of Mid-Tertiary age, speaking in general terms. Prof. Sollas, in concluding a very interesting and suggestive paper on some experiments with cobbler's wax, published in the last volume of the Quarterly Journal of the Society, p. 716, suggests that the results are favourable to the views of the Lausanne professor.

The author takes exception to some of the cases, especially two to the east of the Simplon Pass, which are adduced by Prof. Lugeon in support of his hypothesis, pointing out that he has confused an ordinary with a crystalline limestone, and merely schistose

slates with true crystalline schists. But, without going into the details of sections, he shows from general reasoning that both the kind and the amount of transport, postulated by Prof. Lugeon, introduce serious mechanical difficulties. Sediments are assumed to have been transported from (say) near Ivrea to the northern margin of the Alps on either side of the Rhone valley below Martigny; that is, to have travelled something like 75 miles. As the average thickness of the Secondary and older Tertiary strata in this part of the Alps can hardly exceed 2·5 miles, we can form a general idea of the problem by drawing an oblong 30 times as long as high. Then, material at one end (southern) has to be squeezed across obstacles to near the other, by piling a wedge-like mass of sediment on a southern prolongation of the oblong, the thin end of which mass may perhaps trespass a little on the northward side of the original southern boundary.

After discussing the problem in some detail, the author shows that in Prof. Sollas's experiments there was, at the beginning of loading, a wedge instead of an oblong; the distance to be traversed was 6 times the maximum thickness of the (supposed) sedimentaries, instead of about 30 times; and the slope of the added mass was about double that which, in the Alpine case, is supposed to be the sole cause of the movement. Lastly, in the case which presents by far the greater difficulties, the material is limestone and shale instead of cobbler's wax. The author maintains that Prof. Lugeon's hypothesis receives no real support from Prof. Sollas's experiments, and involves mechanical difficulties which are practically insuperable.

2. 'The Coral-Rocks of Barbados.' By Prof. John Burchmore Harrison, C.M.G., M.A., F.I.C., F.G.S.

The results of the author's extended, and in many places detailed, re-examination of the coral-rocks in the southern half of Barbados, give no support to Dr. J. W. Spencer's theory of the existence of strata of the 'Antigua Formation' in that island. It is now shown that a certain knoll, whence Dr. Spencer collected corals which in his estimation proved that it and other parts of the coral-rocks were of Oligocene age, is in part made up of corals which, as stated by Prof. J. W. Gregory, 'certainly show no evidence of any age greater than the Pleistocene.' The author has failed to find any signs of the widespread formation, described in Dr. Spencer's paper as extending from Mount Misery to near Ragged Point, a distance of about 11 miles, and dipping south-eastward at from 12° to 20°. Such a formation would be about 15,000 feet thick; while the facts that nowhere in the island does the combined thickness of the limestone and of its basal or Bissex Beds exceed 280 feet, and that the limestone is not traversed by faults, are fairly-conclusive evidence of the non-existence of such a formation. This dip is referred to the action of landslips in some cases, and in others to current-bedding. The author's recent investigations have confirmed the statements made and the views expressed by Mr. Jukes-Browne and himself, in the notes published in the 'Geological Magazine' for December 1902, p. 550.

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